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IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY

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ABSTRACT. We provide in this paper, sufficient conditions for the existence of solutions for a class of initial value problem for impulsive fractional differential equations with state-dependent delay involving the Caputo fractional derivative.

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1. INTRODUCTION

This paper deals with the existence of solutions to the initial value problems (IVP for short) for the impulsive differential equations of the form,

$$ {}^cD^\alpha y(t) = f(t, y_{\rho(t,y_t)}), \text{ a.e. } t \in J = [0,b],\ t \neq t_k,\ k = 1,\dots,m,\ 0 < \alpha \le 1, \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1,\dots,m, \tag{1.2}$$

$$y(t) = \phi(t),\ t \in (-\infty, 0], \tag{1.3}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathcal{B} \to \mathbb{R}$, $\rho : J \times \mathcal{B} \to \mathbb{R}$, $\phi \in \mathcal{B}$ are given functions, $I_k : \mathbb{R} \to \mathbb{R}$, $k = 1,\dots,m$ are continuous functions, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left hand limits of $y(t)$ at $t = t_k$, $k = 1,\dots,m$, and $\mathcal{B}$ is an abstract *phase space* to be specified later.

For any function y and any $t \in [0,b]$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in (-\infty, 0]$. We assume that the histories y_t belong to $\mathcal{B}$.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media,

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electromagnetic, etc.; see the monographs of Kilbas *et al.* [19], Lakshmikantham *et al.* [22], Podlubny [24], and the papers [2, 3, 7, 9, 10, 31, 32] and the references therein.

Differential delay equations, or functional differential equations with or without impulse, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books [8, 14, 18, 20, 21, 23, 27, 29], and the papers [11, 13].

However, complicated situations in which the delay depends on the unknown functions have been proposed in modeling in recent years (see for instance [6, 25, 28] and the references therein). These equations are frequently called equations with state-dependent delay. Existence results, among other things, were derived recently for functional differential equations when the solution is depending on the delay on a bounded interval for impulsive problems. We refer the reader to the papers by Abada *et al.* [1], Ait Dads and Ezzinbi [4], Anguraj *et al.* [5], and Hernandez *et al.* [16, 17]. In [12], the authors considered a class of semilinear functional fractional order differential equations with state-dependent delay. As far as we know, no papers exist in the literature related to fractional order functional differential equations with state-dependent delay and impulses. The aim of this paper is to initiate this study.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

Definition 2.1 ([19, 24]). The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.2 ([19, 24]). For a function h given on the interval $[a,b]$, the αth Riemann-Liouville fractional-order derivative of h, is defined by

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3 ([19])**.** For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of h, is defined by

$$({}^cD^\alpha_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds.$$

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [13] and follow the terminology used in [18], but we will add some transformations. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$. The first two axioms on $\mathcal{B}$ are motivated by the fact that we want a solution of the problem (1.1)–(1.3) to be continuous on $(t_k, t_{k+1}]$ and the left hand limit exists for every t_k. We will assume that $\mathcal{B}$ satisfies the following axioms:

(A_1) If $y : (-\infty, b) \to \mathbb{R}, b > 0$, $y_0 \in \mathcal{B}$, and $y(t_k^-)$ and $y(t_k^+)$, $k = 1, \dots, m$ exist with $y(t_k^-) = y(t_k)$, $k = 1, \dots, m$ then for every $t \in [0, b) \setminus \{t_1, \dots, t_m\}$ the following conditions hold:

- (i) $y_t \in \mathcal{B}$; and y_t is continuous on $[0, b) \setminus \{t_1, \dots, t_m\}$;
- (ii) There exists a positive constant H such that $|y(t)| \le H\|y_t\|_{\mathcal{B}}$;
- (iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, independent of y, with K continuous and M locally bounded such that:

$$\|y_t\|_{\mathcal{B}} \le K(t)\sup\{\,|y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) The space $\mathcal{B}$ is complete.

Denote $K_b = \sup\{K(t) : t \in [0,b]\}$ and $M_b = \sup\{M(t) : t \in [0,b]\}$.

3. EXISTENCE OF SOLUTIONS

Consider the following space

$$PC(J,\mathbb{R}) = \{y : J \to \mathbb{R} : \mid y \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, \dots, m+1 \text{ and there exist } y(t_k^-) \text{ and } y(t_k^+), \quad k = 1, \dots, m \quad \text{with } y(t_k^-) = y(t_k)\}.$$

$PC(J,\mathbb{R})$ is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t\in J}|y(t)|.$$

Set

$$\mathcal{B}_b = \{y : (-\infty, b] \to \mathbb{R} \setminus y \in PC(J,\mathbb{R}) \cap \mathcal{B}\},$$

and let $\|\cdot\|_b$ the seminorm in $\mathcal{B}_b$ defined by

$$\|y\|_b = \|y_0\|_{\mathcal{B}} + \sup\{|y(t)| : 0 \le t \le b\}, \quad y \in \mathcal{B}_b.$$

Set

$$J' := J\backslash\{t_1, t_2, \dots, t_m\}.$$

Definition 3.1. A function $y \in \mathcal{B}_b$ with its α-derivative exists on J' is said to be a solution of (1.1)–(1.3) if y satisfies (1.1)–(1.3).

For the existence of solutions for the problem (1.1)–(1.3), we need the following auxiliary lemmas.

Lemma 3.2 ([31]). *If $\alpha > 0$, then the differential equation*

$$^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 3.3 ([31]). *If $\alpha > 0$, then*

$$I^{\alpha c}D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 3.4 ([15]). *Suppose $b \geq 0$, $\beta > 0$ and $a(t)$ is a nonnegative locally integrable function on $0 \leq t < T$ (some $T \leq \infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + c\int_0^t (t-s)^{\beta-1}u(s)ds, \quad 0 \leq t < T.$$

Then

$$u(t) \leq a(t) + \int_0^t \sum_{j=1}^{\infty} \frac{(c\Gamma(\beta))^j}{\Gamma(j\beta)}(t-s)^{j\beta-1}a(s)ds, \quad 0 \leq t < T. \tag{3.1}$$

If $a(t) \equiv a$, constant on $0 \leq t < T$, then the inequality (3.1) *is reduced to*

$$u(t) \leq aE_\beta\left(c\Gamma(\beta)t^\beta\right)$$

where E_β is the Mittag-Leffler function [19] *defined by*

$$E_\beta(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)}, \quad z \in \mathbb{C}, \quad \mathfrak{Re}(\beta) > 0.$$

For more generalized Gronwall inequalities see Ye *et al.* [30].

As a consequence of Lemma 3.2 and Lemma 3.3, we have the following result which is useful in what follows.

Lemma 3.5 ([10]). *Let $0 < \alpha \leq 1$ and let $h : J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds & \text{if } t \in [0, t_1], \\ y_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}h(s)ds & \\ +\frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}h(s)ds & \\ +\sum_{i=1}^k I_i(y(t_i^-)), & \text{if } t \in (t_k, t_{k+1}],\ k = 1, \ldots, m, \end{cases} \tag{3.2}$$

if and only if y is a solution of the fractional IVP

$$^cD^\alpha y(t) = h(t), \quad \textit{for each } t \in J', \tag{3.3}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{3.4}$$

$$y(0) = y_0. \tag{3.5}$$

We will need to introduce the following hypotheses

(Hϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \rho(s,\varphi) \leq 0\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that $\|\phi_t\|_{\mathcal{B}} \leq L^\phi(t)\|\phi\|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.

(H1) The function $f : J \times \mathcal{B} \to \mathbb{R}$ is continuous.

(H2) There exist functions $p, q \in C(J, \mathbb{R}^+)$ such that

$$|f(t,u)| \leq p(t) + q(t)\|u\|_{\mathcal{B}} \text{ for each } t \in J \text{ and all } u \in \mathcal{B}.$$

The next result is a consequence of the phase space axioms.

Lemma 3.6 ([16, Lemma 2.1]). *If $y : (-\infty, b] \to \mathbb{R}$ is a function such that $y_0 = \phi$ and $y|_J \in PC(J : \mathbb{R})$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b \sup\{\|y(\theta)\|; \theta \in [0, \max\{0,s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where

$$L^\phi = \sup_{t \in \mathcal{R}(\rho^-)} L^\phi(t).$$

Remark 3.7. We remark that condition (H_ϕ) is satisfied by functions which are continuous and bounded. In fact, if the space $\mathcal{B}$ satisfies axiom C_2 in [18] then there exists a constant $L > 0$ such that $\|\phi\|_{\mathcal{B}} \leq L \sup\{\|\phi(\theta)\| : \theta \in [-\infty, 0]\}$ for every $\phi \in \mathcal{B}$ that is continuous and bounded (see [18, Proposition 7.1.1]) for details. Consequently,

$$\|\phi_t\|_{\mathcal{B}} \leq L \frac{\sup_{\theta \leq 0}\|\phi(\theta)\|}{\|\phi\|_{\mathcal{B}}}\|\phi\|_{\mathcal{B}}, \quad \text{for every } \phi \in \mathcal{B}\backslash\{0\}.$$

Theorem 3.8. *Assume that the hypotheses (H1)–(H2) and (H_φ) hold. Then the problem* (1.1)–(1.3) *has at least one solution on* $(-\infty, b]$.

Proof. The proof will be given in several steps.

Step 1: Consider the following problem

$$^cD^\alpha y(t) = f(t, y_{\rho(t,y_t)}), \quad \text{a.e, } t \in J = [0, t_1], \tag{3.6}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0]. \tag{3.7}$$

Define the operator $N : \mathcal{B}_{t_1} \to \mathcal{B}_{t_1}$ by:

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ \phi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_{\rho(s,y_s)})\, ds, & \text{if } t \in [0, t_1]. \end{cases} \tag{3.8}$$

Let $x(\cdot) : (-\infty, t_1] \to \mathbb{R}$ be the function defined by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ \phi(0), & \text{if } t \in [0, t_1]. \end{cases}$$

Then $x_0 = \phi$. For each $z \in \mathcal{B}_{t_1}$ with $z_0 = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0], \\ z(t), & \text{if } t \in [0, t_1]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation

$$y(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_{\rho(s,y_s)}) ds,$$

we can decompose $y(\cdot)$ into $y(t) = \overline{z}(t) + x(t)$, $0 \le t \le t_1$, which implies $y_t = \overline{z}_t + x_t$, for every $t \in [0, t_1]$, and the function $z(\cdot)$ satisfies

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}) ds.$$

Set

$$C_0 = \{z \in \mathcal{B}_{t_1} : z_0 = 0\}.$$

Let $\|\cdot\|_0$ be the norm in C_0 defined by

$$\|z\|_0 = \|z_0\|_{\mathcal{B}} + \sup\{|z(s)| : 0 \le s \le t_1\} = \sup\{|z(s)| : 0 \le s \le t_1\}.$$

We define the operator $P : C_0 \to C_0$ by

$$P(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}) ds.$$

Obviously that the operator N has a fixed point is equivalent to P has one, so we need to prove that P has a fixed point. We shall use the Leray- Schauder alternative.

Claim 1: *P is continuous*

Let $\{z_n\}$ be a sequence such that $z_n \to z$ in C_0. Then

$$\begin{aligned} |P(z_n)(t) - P(z)(t)| &\le \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1} |f(s, \overline{z_n}_{\rho(s,\overline{z_n}_s+x_s)} + x_{\rho(s,\overline{z_n}_s+x_s)}) \\ &\quad - f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})| ds. \end{aligned}$$

Since f is a continuous function, we have

$$\|P(z_n) - P(z)\|_0 \to 0 \quad \text{as } n \to \infty.$$

Claim 2: *P maps bounded sets into bounded sets in C_0.*

Indeed, it is enough to show that for any $\eta > 0$, there exists a positive constant ℓ such that for each $z \in B_\eta = \{z \in C_0 : \|z\|_0 \le \eta\}$, by (H2) we have for each $t \in [0, t_1]$,

$$|P(z)(t)| \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [p(s) + q(s) \|\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}}] ds$$

$$
\begin{aligned}
&\leq \frac{b^\alpha}{\Gamma(\alpha+1)}\|p\|_\infty \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} q(s)(K_b\eta + K_b|\phi(0)| + M_b\|\phi\|_{\mathcal{B}})ds \\
&\leq \frac{b^\alpha}{\Gamma(\alpha+1)}\|p\|_\infty \\
&\quad + \frac{b^\alpha}{\Gamma(\alpha+1)}\|q\|_\infty (K_b\eta + K_b|\phi(0)| + M_b\|\phi\|_{\mathcal{B}}) := \ell.
\end{aligned}
$$

Claim 3: *N maps bounded sets into equicontinuous sets of C_0.*

Let $l_1, l_2 \in [0, t_1],\ l_1 < l_2$, let B_η a bounded set of C_0 as in Claim 2, and let $z \in B_\eta$. Then,

$$
\begin{aligned}
|P(z)(l_2) - P(z)(l_1)| &\leq \frac{1}{\Gamma(\alpha)}\int_{l_1}^{l_2} (l_2-s)^{\alpha-1}|f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})|ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{l_1} [(l_2-s)^{\alpha-1} - (l_1-s)^{\alpha-1}]|f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})|ds \\
&\leq \frac{\|p\|_\infty}{\Gamma(\alpha)}\left|\int_{l_1}^{l_2} (l_2-s)^{\alpha-1}ds + \int_0^{l_1} (l_2-s)^{\alpha-1}ds\right| \\
&+ \frac{\|q\|_\infty (K_b\eta + K_b|\phi(0)| + M_b\|\phi\|_{\mathcal{B}})}{\Gamma(\alpha)}\left|\int_{l_1}^{l_2} (l_2-s)^{\alpha-1}ds + \int_0^{l_1} [(l_2-s)^{\alpha-1}ds\right| \\
&\leq \frac{2\|p\|_\infty (l_2-l_1)^\alpha}{\Gamma(\alpha+1)} \\
&+ \frac{2\|q\|_\infty (l_2-l_1)^\alpha (K_b\eta + K_b|\phi(0)| + M_b\|\phi\|_{\mathcal{B}})}{\Gamma(\alpha+1)}.
\end{aligned}
$$

As $l_1 \to l_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claims 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that N is continuous and completely continuous.

Claim 4: *A priori bounds*.

Let z be a possible solution of the equation $z = \lambda P(z)$ for some $\lambda \in (0, 1)$. Then for each $t \in [0, t_1]$, we have

$$
|z(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} q(s)\|\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}}]ds + \frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)}.
$$

But

$$
\begin{aligned}
\|\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}} &\leq \|\overline{z}_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}} + \|x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}} \\
&\leq K(t)\sup\{|z(s)| : 0 \leq s \leq t\} + M(t)\|z_0\|_{\mathcal{B}} \\
&\quad + K(t)\sup\{|x(s)| : 0 \leq s \leq t\} + M(t)\|x_0\|_{\mathcal{B}} \\
&\leq K_b\sup\{|z(s)| : 0 \leq s \leq t\} + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|.
\end{aligned}
$$

Let us take the right-hand side of the above inequality as $w(t)$. Then, we have

$$|z(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s) w(s) ds + \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)}.$$

By the definition of w we obtain

$$w(t) \leq M_b \|\phi\|_{\mathcal{B}} + K_b |\phi(0)| + \frac{K_b b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + \frac{K_b \|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(s) ds.$$

Set

$$a = M_b \|\phi\|_{\mathcal{B}} + K_b |\phi(0)| + \frac{K_b b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)}$$

and

$$c = \frac{K_b \|q\|_\infty}{\Gamma(\alpha)}.$$

Then Lemma 3.4 implies that for each $t \in [0, t_1]$

$$|w(t)| \leq a E_\alpha \left(c \Gamma(\alpha) t^\alpha \right) := M.$$

Then,

$$\begin{aligned} |z(t)| &\leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s) ds + \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} \\ &\leq M \|I^\alpha q\|_\infty + \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} := M^*. \end{aligned}$$

Set

$$U_0 = \{z \in C_0 : \|z\|_0 < M^* + 1\}.$$

From the choice of U_0, there is no $z \in \partial U_0$ such that $z = \lambda P(z)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that P has a fixed point $z \in U_0$. Hence N, has a fixed point that is solution to the problem (1)-(3). Denote this solution by y_0.

Step 2: Consider the following problem

$$^cD^\alpha y(t) = f(t, y_{\rho(t,y_t)}), \quad \text{a.e, } t \in J = (t_1, t_2], \tag{3.9}$$

$$y(t_1^+) - y(t_1^-) = I_1(y_0(t_1^-)), \tag{3.10}$$

$$y(t) = y_0(t), \quad t \in (-\infty, t_1]. \tag{3.11}$$

Let

$$C_1 = \{y \in \mathcal{B}_{t_2} : y(t_1^+) \text{ exists}\},$$

and define the operator $N_1 : C_1 \to C_1$ by

$$N_1(y)(t) = \begin{cases} y_0(t), & \text{if } t \in (-\infty, t_1] \\ y_0(t_1^-) + I_1(y_0(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f(s, y_{\rho(s,y_s)})\, ds, & \text{if } t \in [t_1, t_2]. \end{cases} \tag{3.12}$$

Let $x(\cdot) : (-\infty, t_2] \to \mathbb{R}$ defined by

$$x(t) = \begin{cases} y_0(t), & \text{if } t \in (-\infty, t_1], \\ y_0(t_1^-) + I_1(y_0(t_1^-)), & \text{if } t \in [t_1, t_2]. \end{cases}$$

Then $x_{t_1} = y_0$. For each $z \in C_1$ with $z(t_1) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \in (-\infty, t_1]; \\ z(t), & \text{if } t \in [t_1, t_2]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation

$$y(t) = y_0(t_1^-) + I_1(y_0(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} f(s, y_{\rho(s,y_s)})\, ds,$$

we can decompose $y(\cdot)$ into $y(t) = \overline{z}(t) + x(t)$, $t_1 \le t \le t_2$ which implies $y_t = \overline{z}_t + x_t$, for every $t \in [t_1, t_2]$, and the function $z(\cdot)$ satisfies

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{z}_{\rho(s,\overline{z}_s + x_s)} + x_{\rho(s,\overline{z}_s + x_s)}) ds.$$

Set

$$\tilde{C}_{t_1} = \{z \in C_1 : z_{t_1} = 0\},$$

and consider the operator $P_1 : \tilde{C}_{t_1} \to \tilde{C}_{t_1}$ defined by

$$P(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{z}_{\rho(s,\overline{z}_s + x_s)} + x_{\rho(s,\overline{z}_s + x_s)}) ds.$$

As in Step 1, we can show that P_1 is continuous and completely continuous, and if z is a possible solution of the equation $z = \lambda P_1(z)$ for some $\lambda \in (0,1)$, then there exists $M_{*1} > 0$ such that

$$\|z\|_{t_1} \le M_{*1}.$$

Set

$$U_1 = \{z \in \tilde{C}_{t_1} : \|z\|_0 < M_{*1} + 1\}.$$

From the choice of U_0, there is no $z \in \partial U_1$ such that $z = \lambda P(z)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that P_1 has a fixed point $z \in U_1$. Hence N_1 has a fixed point that is solution to the problem (1)–(3). Denote this solution by y_1.

Step 3: We continue this process and taking into account that $y_m := y|_{[t_m,b]}$ is a solution to the problem

$$^cD^\alpha y(t) = f(t, y_{\rho(t,y_t)}), \quad \text{a.e, } t \in J = (t_m, b], \tag{3.13}$$

$$y(t_m^+) - y_{m-1}(t_{m-1}^-) = I_m(y_{m-1}(t_m^-)) \tag{3.14}$$

$$y(t) = y_{m-1}(t), \quad t \in (-\infty, t_{m-1}]. \tag{3.15}$$

The solution of problem (1.1)–(1.3) is then defined by

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in (-\infty, t_1] \\ y_1(t), & \text{if} t \in (t_1, t_2], \\ \dots & \\ y_m(t), & \text{if} t \in (t_m, b]. \end{cases}$$

□

4. AN EXAMPLE

To apply our results, we consider the functional differential equation with state dependent delay of the form

$$y'(t) = \frac{e^t y(t - \sigma(y(t)) + 2}{1 + y^2(t - \sigma(y(t))}, \quad t \in [0, b], \tag{4.1}$$

$$y(\theta) = \phi(\theta), \quad \theta \in (-\infty, 0], \tag{4.2}$$

$$\Delta y(t_i) = \int_{-\infty}^{t_i} \gamma_i(t_i - s) y(s) ds, \tag{4.3}$$

where $\gamma_i \in C([0, \infty), \mathbb{R})$, $\sigma \in C(\mathbb{R}, [0, \infty))$, $0 < t_1 < t_2 < \cdots < t_n < b$.

Set $\gamma > 0$. For the phase space, we choose $\mathcal{B}$ to be defined by

$$\mathcal{B} = PC^\gamma = \{\phi \in PC((-\infty, 0], \mathbb{R}) : \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exists}\}$$

with the norm

$$\|\phi\|_\gamma = \sup_{\theta \in (-\infty, 0]} e^{\gamma\theta}|\phi(\theta)|, \quad \phi \in PC^\gamma.$$

Set

$$\rho(t, \varphi) = t - \sigma(\varphi(0)), \quad (t, \varphi) \in J \times \mathcal{B},$$

$$f(t, \varphi) = \frac{e^t \varphi + 2}{1 + \varphi^2}, \quad (t, \varphi) \in J \times \mathcal{B},$$

$$I_k(y(t_k)) = \int_{-\infty}^{t_i} \gamma_i(t_i - s) y(s) ds.$$

It is clear that (H1) and (H2) are satisfied with

$$|f(t, \varphi)| \leq e^t \|\varphi\|_{\mathcal{B}} + 2 \quad \texttt{for all} \quad (t, \varphi) \in J \times \mathcal{B}.$$

Theorem 4.1. *Let $\varphi \in \mathcal{B}$ be such that H_φ is valid and $t \to \varphi_t$ is continuous on $\mathcal{R}(\rho^-)$. Then there exists a solution of* (4.1)-(4.3).

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