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EXISTENCE, NONEXISTENCE, AND UNIQUENESS FOR POSITIVE SOLUTIONS TO A THIRD ORDER BOUNDARY VALUE PROBLEM

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ABSTRACT. We consider a third order two point boundary value problem. Some new a priori estimates to positive solutions for the problem are obtained. Sufficient conditions for the existence, nonexistence, and uniqueness of positive solutions for the boundary value problem are established.

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1. INTRODUCTION

In 2007, Liu et al. [9] considered the third order two-point boundary value problem

$$u'''(t) + \lambda\alpha(t)f(t, u(t)) = 0, \quad a \le t \le b, \tag{1.1}$$

$$u(a) = u''(a) = u'(b) = 0. \tag{1.2}$$

Motivated by this work, we in this paper consider the third order boundary value problem

$$u'''(t) + g(t)f(u(t)) = 0, \quad 0 \le t \le 1, \tag{1.3}$$

$$u(0) = u''(0) = u'(1) = 0. \tag{1.4}$$

In this paper, we shall derive some new a priori estimates to positive solutions of the problem (1.3)–(1.4). These estimates improve the ones obtained in [9]. We shall also prove some existence, nonexistence, and uniqueness results for positive solutions of the problem (1.3)–(1.4). Here, by a positive solution, we mean a solution $u(t)$ such that $u(t) > 0$ for $t \in (0, 1)$.

The problem (1.3)–(1.4) is closely related to a boundary value problem for the fourth order beam equation, namely,

$$u''''(t) = g(t)f(u(t)), \quad 0 \le t \le 1, \tag{1.5}$$

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$$u(0) = u''(0) = u'(1) = u'''(1) = 0. \tag{1.6}$$

The boundary conditions (1.6) have definite physical meanings. The conditions $u(0) = u''(0) = 0$ mean that the beam is simply supported at $t = 0$, while the boundary conditions $u'(1) = u'''(1) = 0$ mean that the beam is supported by a sliding clamp at $t = 1$. Note that (1.4) is just a part of (1.6). A study of the problem (1.3)–(1.4) will give us more insight into the problem (1.5)–(1.6).

Throughout this paper, we assume that

(H1) $f : [0, \infty) \to [0, \infty)$ and $g : [0, 1] \to [0, \infty)$ are continuous functions, and $g(t) \not\equiv 0$ on $[0, 1]$.

This paper is organized as follows. In Section 2, we give the Green function for the problem (1.3)–(1.4), state the Krasnosel'skii fixed point theorem, and fix some notations. In Section 3, we present some a priori estimates to positive solutions to the problem (1.3)–(1.4). In Section 4, we establish some existence and nonexistence results for positive solutions to the problem (1.3)–(1.4). Then, in Section 5, we establish some uniqueness results for positive solutions to the problem (1.3)–(1.4).

2. PRELIMINARIES

The Green function $G : [0, 1] \times [0, 1] \to [0, \infty)$ for the problem (1.3)–(1.4) is given by

$$G(t, s) = \begin{cases} t(1 - s) - (t - s)^2/2, & \text{if } s \le t, \\ t(1 - s), & \text{if } t < s. \end{cases}$$

Then problem (1.3)–(1.4) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t, s)g(s)f(u(s))\, ds, \quad 0 \le t \le 1. \tag{2.1}$$

It is easy to verify that G is a continuous function, and $G(t, s) \ge 0$ if $(t, s) \in [0, 1]^2$. We will need the following fixed point theorem, which is due to Krasnosel'skii [7], to prove some of our results.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a Banach space over the reals, and let $P \subset X$ be a cone in X. Let H_1 and H_2 be real numbers such that $H_2 > H_1 > 0$, and let*

$$\Omega_i = \{v \in X \mid \|v\| < H_i\}, \quad i = 1, 2.$$

If $L : P \cap (\overline{\Omega_2} - \Omega_1) \to P$ is a completely continuous operator such that, either

(K1) $\|Lv\| \le \|v\|$ *if* $v \in P \cap \partial\Omega_1$, *and* $\|Lv\| \ge \|v\|$ *if* $v \in P \cap \partial\Omega_2$, *or*
(K2) $\|Lv\| \ge \|v\|$ *if* $v \in P \cap \partial\Omega_1$, *and* $\|Lv\| \le \|v\|$ *if* $v \in P \cap \partial\Omega_2$,

then L has a fixed point in $P \cap (\overline{\Omega_2} - \Omega_1)$.

For the rest of this paper, we let $X = C[0,1]$ with the norm

$$\|v\| = \max_{t\in[0,1]} |v(t)|, \quad \forall v \in X.$$

Clearly, X is a Banach space. We define $Y = \{v \in X \mid v(t) \geq 0 \text{ for } 0 \leq t \leq 1\}$, and define the operator $T : Y \to X$ by

$$(Tu)(t) = \int_0^1 G(t,s)g(s)f(u(s))\,ds, \quad 0 \leq t \leq 1. \tag{2.2}$$

It is clear that if (H1) holds, then $T(Y) \subset Y$ and $T : Y \to Y$ is a completely continuous operator. We also define the constants

$$F_0 = \limsup_{x\to 0^+} \frac{f(x)}{x}, \quad f_0 = \liminf_{x\to 0^+} \frac{f(x)}{x},$$
$$F_\infty = \limsup_{x\to +\infty} \frac{f(x)}{x}, \quad f_\infty = \liminf_{x\to +\infty} \frac{f(x)}{x}.$$

These constants, which are associated with the function f, will be used later in Section 4.

3. ESTIMATES FOR POSITIVE SOLUTIONS

In this section, we shall prove some new a priori estimates for positive solutions of the problem (1.3)–(1.4). To this purpose, we define the function $b : [0,1] \to [0,1]$ by

$$b(t) = 2t - t^2.$$

It is easily seen that $b(t) \leq 2t$ for $0 \leq t \leq 1$. It is also easy to see that $b(t) \geq t$ for $0 \leq t \leq 1$. In fact, we have

$$b(t) - t = t - t^2 = t(1-t) \geq 0, \quad 0 \leq t \leq 1.$$

Lemma 3.1. *If $u \in C^3[0,1]$ satisfies the boundary conditions* (1.4), *and*

$$u'''(t) \leq 0 \quad \text{for} \quad 0 \leq t \leq 1, \tag{3.1}$$

then

$$u''(t) \leq 0, \;\; u'(t) \geq 0, \;\; u(t) \geq 0 \quad \text{for} \quad 0 \leq t \leq 1. \tag{3.2}$$

Proof. Note that (3.1) implies that u'' is nonincreasing. Since $u''(0) = 0$, we have $u''(t) \leq 0$ on $[0,1]$. This means that u' is nonincreasing on $[0,1]$. Since $u'(1) = 0$, we have $u'(t) \geq 0$ on $[0,1]$. Since $u(0) = 0$, we have $u(t) \geq 0$ for $0 \leq t \leq 1$. The proof of the lemma is now complete. □

Lemma 3.2. *If $u \in C^3[0,1]$ satisfies* (1.4) *and* (3.1), *then*

$$u(t) \geq tu(1) \quad \text{for} \quad 0 \leq t \leq 1. \tag{3.3}$$

Proof. If we define

$$h(t) = u(t) - tu(1), \quad 0 \le t \le 1,$$

then $h'(t) = u'(t) - u(1)$, and

$$h''(t) = u''(t) \le 0, \quad 0 \le t \le 1.$$

To prove the lemma, it suffices to show that $h(t) \ge 0$ on $[0,1]$.

It is easy to see that $h(0) = h(1) = 0$. Since h is concave downward on $[0,1]$, we have $h(t) \ge 0$ for $0 \le t \le 1$. The proof is complete. □

Lemma 3.3. *If* $u \in C^3[0,1]$ *satisfies* (1.4) *and* (3.1)*, then*

$$u(t) \le u(1)b(t) \quad \textit{for} \quad t \in [0,1]. \tag{3.4}$$

Proof. If we define

$$h(t) = b(t)u(1) - u(t) = (2t - t^2)u(1) - u(t), \quad 0 \le t \le 1,$$

then

$$h'(t) = (2 - 2t)u(1) - u'(t), \quad h''(t) = -2u(1) - u''(t),$$
$$h'''(t) = -u'''(t) \ge 0, \quad 0 \le t \le 1. \tag{3.5}$$

The last inequality implies that h' is concave upward on $[0,1]$. It is easy to see that $h(0) = h(1) = h'(1) = 0$. By the Mean Value Theorem, because $h(0) = h(1) = 0$, there exists $p \in (0,1)$ such that $h'(p) = 0$. Now we have $h'(p) = h'(1) = 0$. Since $h'(t)$ is concave upward, we have

$$h'(t) \ge 0 \text{ on } (0,p), \quad h'(t) \le 0 \text{ on } (p,1).$$

Since $h(0) = h(1) = 0$, we have $h(t) \ge 0$ on $(0,1)$. The proof is complete. □

Theorem 3.4. *Suppose that (H1) holds. If* $u(t)$ *is a nonnegative solution to the problem* (1.3)–(1.4)*, then* $u(t)$ *satisfies* (3.2), (3.3)*, and* (3.4)*.*

Proof. If $u(t)$ is a nonnegative solution to the problem (1.3)–(1.4), then $u(t)$ satisfies the boundary conditions (1.4), and

$$u'''(t) = -g(t)f(u(t)) \le 0, \quad 0 \le t \le 1.$$

Now Theorem 3.4 follows directly from Lemmas 3.1, 3.2, and 3.3. The proof is complete. □

Now we define

$$P = \{v \in X : v(1) \ge 0,\ tv(1) \le v(t) \le b(t)v(1) \text{ on } [0,1]\}.$$

Clearly P is a positive cone in X. It is obvious that if $u \in P$, then $u(1) = \|u\|$. We see from Theorem 3.4 that if $u(t)$ is a nonnegative solution to the problem (1.3)–(1.4), then $u \in P$. In a similar fashion to Theorem 3.4, we can show that $T(P) \subset P$. To find a positive solution to the problem (1.3)–(1.4), we need only to find a fixed point u of T such that $u \in P$ and $u(1) = \|u\| > 0$.

Remark 3.5. In [9], Liu et al. considered the following cone for the problem (1.3)–(1.4), namely,

$$P' = \{v \in X : v(t) \geq (t/2)\|v\| \text{ on } [0,1]\}.$$

It is easy to see that P is a subset of P'. In other words, P is a finer cone than P'. If we apply the Krasnosell'skii fixed point theorem on this finer cone, we will obtain sharper existence and nonexistence results for positive solutions to the problem (1.3)–(1.4). Our cone P is finer because our upper and lower estimates for positive solutions for the problem (1.3)–(1.4), which are given in Lemmas 3.2 and 3.3, are sharper than those in [9].

4. EXISTENCE AND NONEXISTENCE RESULTS

Now we define some important constants. Let

$$A = \int_0^1 G(1,s)g(s)s\,ds, \quad B = \int_0^1 G(1,s)g(s)b(s)\,ds.$$

The next two theorems provide sufficient conditions for the existence of at least one positive solution for the problem (1.3)–(1.4).

Theorem 4.1. *Suppose that (H1) holds. If $BF_0 < 1 < Af_\infty$, then the problem* (1.3)–(1.4) *has at least one positive solution.*

Proof. First, we choose $\varepsilon > 0$ such that $(F_0+\varepsilon)B \leq 1$. By the definition of F_0, there exists $H_1 > 0$ such that $f(x) \leq (F_0+\varepsilon)x$ for $0 < x \leq H_1$. Now for each $u \in P$ with $\|u\| = H_1$, we have $Tu \in P$ and

$$\begin{aligned}(Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &\leq \int_0^1 G(1,s)g(s)(F_0+\varepsilon)u(s)\,ds \\ &\leq (F_0+\varepsilon)\|u\| \int_0^1 G(1,s)g(s)b(s)\,ds \\ &= (F_0+\varepsilon)\|u\|B \leq \|u\|,\end{aligned}$$

which means $\|Tu\| = (Tu)(1) \leq \|u\|$. Thus, if we let $\Omega_1 = \{u \in X \mid \|u\| < H_1\}$, then

$$\|Tu\| \leq \|u\| \;\text{ for }\; u \in P \cap \partial\Omega_1.$$

To construct Ω_2, we choose $\delta > 0$ and $\tau \in (0, 1/4)$ such that

$$\int_\tau^1 G(1,s)g(s)s\,ds \cdot (f_\infty - \delta) \geq 1.$$

There exists $H_3 > 0$ such that $f(x) \geq (f_\infty - \delta)x$ for $x \geq H_3$. Let $H_2 = H_3/\tau + H_1$. If $u \in P$ such that $\|u\| = H_2$, then for each $t \in [\tau, 1]$, we have

$$u(t) \geq H_2 t \geq H_2 \tau \geq H_3.$$

Therefore, for each $u \in P$ with $\|u\| = H_2$, we have

$$\begin{aligned}(Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &\geq \int_\tau^1 G(1,s)g(s)f(u(s))\,ds \\ &\geq \int_\tau^1 G(1,s)g(s)(f_\infty - \delta)u(s)\,ds \\ &\geq \int_\tau^1 G(1,s)g(s)s\,ds \cdot (f_\infty - \delta)\|u\| \geq \|u\|,\end{aligned}$$

which means $\|Tu\| \geq \|u\|$. Thus, if we let $\Omega_2 = \{u \in X \mid \|u\| < H_2\}$, then $\overline{\Omega_1} \subset \Omega_2$, and

$$\|Tu\| \geq \|u\| \quad \text{for} \quad u \in P \cap \partial\Omega_2.$$

Now that the condition (K1) of Theorem 2.1 is satisfied, there exists a fixed point of T in $P \cap (\overline{\Omega_2} - \Omega_1)$. The proof is now complete. □

Theorem 4.2. *Suppose that (H1) holds. If $BF_\infty < 1 < Af_0$, then the problem* (1.3)–(1.4) *has at least one positive solution.*

The proof of Theorem 4.2 is very similar to that of Theorem 4.1 and is therefore omitted. The next two theorems provide sufficient conditions for the nonexistence of positive solutions to the problem (1.3)–(1.4).

Theorem 4.3. *Suppose that (H1) holds. If $Bf(x) < x$ for all $x > 0$, then the problem* (1.3)–(1.4) *has no positive solutions.*

Proof. Assume the contrary that $u(t)$ is a positive solution of the problem (1.3)–(1.4). Then $u \in P$, $u(t) > 0$ for $0 < t \leq 1$, and

$$\begin{aligned}u(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &< B^{-1}\int_0^1 G(1,s)g(s)u(s)\,ds \\ &\leq B^{-1}\int_0^1 G(1,s)g(s)b(s)\,ds \cdot u(1) \\ &= B^{-1}Bu(1) = u(1),\end{aligned}$$

which is a contradiction. The proof is complete. □

In a very similar fashion, we can prove the next nonexistence theorem.

Theorem 4.4. *Suppose that (H1) holds. If* $Af(x) > x$ *for all* $x > 0$*, then the problem* (1.3)–(1.4) *has no positive solutions.*

Example 4.5. Consider the third order boundary value problem

$$u'''(t) = -\lambda(1 + 2t + t^2)u(t)(1 + 3u(t))/(1 + u(t)), \quad 0 \le t \le 1, \tag{4.1}$$

$$u(0) = u''(0) = u'(1) = 0, \tag{4.2}$$

where $\lambda > 0$ is a parameter. In this example, $g(t) = 1 + 2t + t^2$ and $f(u) = \lambda u(1 + 3u)/(1 + u)$. It is easy to see that $f_0 = F_0 = \lambda$, $f_\infty = F_\infty = 3\lambda$, and

$$\lambda x < f(x) < 3\lambda x \quad \text{for} \quad x > 0.$$

Calculations indicate that

$$A = 3/10, \quad B = 59/140.$$

By Theorem 4.1, if

$$1.111 \approx 1/(3A) < \lambda < 1/B \approx 2.373,$$

then the problem (4.1)-(4.2) has at least one positive solution. From Theorems 4.3 and 4.4 we see that if

$$\lambda \le 1/(3B) \approx 0.791 \quad \text{or} \quad \lambda \ge 1/A \approx 3.333,$$

then the problem (4.1)–(4.2) has no positive solutions.

This example shows that our existence and nonexistence results work quite well.

5. UNIQUENESS RESULTS AND CONVERGENCE OF ITERATION

The next theorem is a uniqueness result for the problem (1.3)–(1.4).

Theorem 5.1. *In addition to (H1), assume that*

(A1) $f(x)$ *is nondecreasing in* x*, and there exists a real number* $r > 0$ *such that* $f(r) > 0$*;*
(A2) *there exists* $\beta \in (0, 1)$ *such that*

$$f(\theta x) \ge \theta^\beta f(x) \ \text{for all} \ \theta \in (0, 1) \ and \ x \ge 0.$$

Then the boundary value problem (1.3)–(1.4) *has exactly one positive solution.*

Proof. First we show that the boundary value problem (1.3)–(1.4) has at least one positive solution. If $x > r$, then by (A2) we have $f(x) \le (x/r)^\beta f(r)$, which implies that $F_\infty = 0$. If $x < 1$, then by (A2) we have $f(x) \ge (x/r)^\beta f(r)$, which implies that $f_0 = +\infty$. Now Theorem 4.2 implies that the problem (1.3)–(1.4) has at least one positive solution.

Next, we shall show that the problem (1.3)–(1.4) has at most one positive solution. If the boundary value problem (1.3)–(1.4) has two positive solutions $u(t)$ and $v(t)$, then $u = Tu$ and $v = Tv$. Note that

$$\begin{aligned} u(t) &\geq tu(1) \geq b(t)u(1)/2 \\ &\geq \frac{u(1)}{2v(1)} b(t)v(1) \geq \frac{u(1)}{2v(1)} v(t), \quad 0 \leq t \leq 1. \end{aligned}$$

If we let M be the largest positive number such that

$$u(t) \geq Mv(t) \quad \text{for} \quad 0 \leq t \leq 1,$$

then $M \geq u(1)/(2v(1))$.

Now we show that $M \geq 1$. Assume that contrary that $M < 1$, then

$$\begin{aligned} u(t) = Tu(t) &= \int_0^1 G(t,s)g(s)f(u(s))ds \\ &\geq \int_0^1 G(t,s)g(s)f(Mv(s))ds \\ &\geq M^\beta \int_0^1 G(t,s)g(s)f(v(s))ds \\ &= M^\beta Tv(t) \\ &= M^\beta v(t), \quad 0 \leq t \leq 1, \end{aligned}$$

which contradicts the maximality of M since $M^\beta > M$. This contradiction shows that $M \geq 1$.

Since $M \geq 1$, we have $u(t) \geq v(t)$. In a similarly way we can show that $v(t) \geq u(t)$. This implies that $u \equiv v$. The proof is complete. □

We define two positive constants

$$K_1 := \int_0^1 G(1,s)g(s)f(1)ds \text{ and } K_2 := \int_0^1 G(1,s)g(s)f(1)s^\beta ds.$$

Lemma 5.2. *Assume that (H1), (A1), and (A2) hold. Let M be a positive constant such that $M > 1$ and*

$$M \geq K_1^{1/(1-\beta)}.$$

Then $(Tw_0)(t) \leq w_0(t)$ for $0 \leq t \leq 1$ where $w_0(t) = M$ for all $0 \leq t \leq 1$.

Proof. In fact, we have

$$\begin{aligned} (Tw_0)(1) &= \int_0^1 G(1,s)g(s)f(M)ds \\ &\leq M^\beta \int_0^1 G(1,s)g(s)f(1)ds \\ &\leq M^\beta M^{1-\beta} \\ &= M. \end{aligned}$$

Since $Tw_0 \in P$, we have

$$(Tw_0)(t) \le (Tw_0)(1) \le M = w_0(t), \quad 0 \le t \le 1.$$

The proof is complete. □

Lemma 5.3. *Assume that (H1), (A1), and (A2) hold. Let m be a positive constant such that $m < 1$ and*

$$m \le K_2^{1/(1-\beta)}.$$

Then $(Tu_0)(t) \ge u_0(t)$ for $0 \le t \le 1$ where $u_0(t) = mt$ for all $0 \le t \le 1$.

Proof. First we have

$$\begin{aligned}(Tu_0)(1) &= \int_0^1 G(1,s)g(s)f(ms)ds \\ &\ge m^\beta \int_0^1 G(1,s)g(s)f(1)s^\beta ds \\ &= m^\beta K_2.\end{aligned}$$

Then, since $Tu_0 \in P$, we have

$$(Tu_0)(t) \ge t \cdot (Tu_0)(1) \ge m^\beta K_2 t \ge mt, \quad 0 \le t \le 1.$$

The proof is complete. □

The next theorem shows that, under certain conditions, if we start with any function $u \in P$ with $u(1) > 0$ and apply the operator T to this u again and again, then the iterative process will always converge to the unique solution to the problem (1.3)–(1.4).

Theorem 5.4. *Assume that (H1), (A1), and (A2) hold. Let $v^*(t)$ be the unique positive solution for the problem* (1.3)–(1.4)*. If $v_0 \in P$ is such that $v_0(1) > 0$, then*

$$\lim_{n\to\infty} T^n v_0 = v^*.$$

Proof. Choose $M > 1$ such that $M \ge K_1^{1/(1-\beta)}$ and $M \ge \|v_0\|$. Choose $m \in (0,1)$ such that $m \le K_2^{1/(1-\beta)}$ and $m \le \|v_0\|$. Let

$$w_0(t) \equiv M, \quad 0 \le t \le 1$$

and

$$u_0(t) = mt, \quad 0 \le t \le 1.$$

Then we have $Tu_0 \ge u_0$, $Tw_0 \le w_0$, and $u_0 \le v_0 \le w_0$.

Now we let $u_n = T^n u_0$, $v_n = T^n v_0$, and $w_n = T^n w_0$ for $n \ge 1$. Since T is an increasing operator, we have

$$u_0 \le u_1 \le u_2 \le \cdots\cdots \le w_2 \le w_1 \le w_0.$$

Therefore both $\{u_n\}$ and $\{w_n\}$ are convergent sequences. Let $u^* = \lim u_n$ and $w^* = \lim w_n$, then both u^* and w^* are positive solutions to the boundary value problem (1.3)–(1.4). Since the positive solution to the boundary value problem (1.3)–(1.4) is unique, we have $u^* = v^* = w^*$. It is also easy to see that for every positive integer n, we have

$$u_n \leq v_n \leq w_n.$$

The squeeze theorem implies that $\lim v_n = v^*$. The proof of the theorem is complete. □

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