Communications in Applied Analysis **14** (2010), no. 3, 273–300

GENERALIZED VARIATIONAL COMPARISON THEOREMS AND NONLINEAR ITERATIVE PROCESS UNDER RANDOM PARAMETRIC PERTURBATIONS

G. S. LADDE[1] AND M. SAMBANDHAM[2]

[1]Department of Mathematics and Statistics, University of South Florida
Tampa, FL 33620 USA
E-mail: gladde@cas.usf.edu

[2]Department of Mathematics, Morehouse College
Atlanta, GA 30314 USA
E-mail: msamband@morehouse.edu

ABSTRACT. In this article we develop basic mathematical tool to study the discrete time stochastic processes. Using these tools we develop comparison results, stability properties and error estimates.

1. INTRODUCTION

In this work, an attempt is made to investigate the qualitative properties of nonlinear and stationary iterative process with random parametric perturbation. The presented results extend and generalize the existing results [7, 8] in a systematic and unified way. For related results on random difference equations we refer [1, 2, 3, 4, 6, 9] and for random differential equations we refer [5, 10]. For deterministic theory of difference equations we refer [11].

In Section 2, we formulate a problem and present basic results. The presented results provide a fundamental mathematical tool to study the qualitative properties of iterative systems. Section 3 deals with the development of generalized comparison results. These results extend the existing results [7, 8]. In Section 4, we study the stability property and in Section 5, we outline the error estimate results [9]. Various examples are given to illustrate the significance of the results.

2. BASIC RESULTS

Let us present a mathematical description of discrete time dynamic processes in chemical, engineering, medical, physical and social sciences. It is described by

Received November 14, 2009

the following nonlinear and nonstationary iterative process under parametric random perturbations:

$$\Delta y(k) = f(k, y(k), \omega), \quad y(k_0, \omega) = y_0 \tag{2.1}$$

where for fixed $(k, y) \in \mathbb{I}(k_0) \times \mathbb{R}^n$, $y(k) \in \mathbb{R}^n$, $\Delta y(k) = y(k+1) - y(k)$; $f(k, y, \omega)$ and $y_0 \in \mathbb{R}^n$ are random vectors defined on a complete probability space $(\Omega, \mathcal{F}, P)$; for each (k, y), $f(k, y, \omega)$ describes random perturbations.

In the absence of random perturbations, mathematical description (2.1) reduces to:

$$\Delta m = F(k, m), \quad m(k_0) = m_0 = E[y_0]. \tag{2.2}$$

In our presentation, we also utilize the following initial value problem:

$$\Delta x = F(k, x), \quad x(k_0) = x_0. \tag{2.3}$$

For the sake of easy reference, we list the following assumptions with regard to (2.1)–(2.3).

Hypothesis-H$_{(2.1)}$: Assume that the initial state y_0 is $\mathcal{F}_{k_0}$-measurable. $f(k, y, \omega)$ is a sequence of random vectors. We designate by $y(k, k_0, y_0) = y(k)$ a solution process of (2.1) each $k \in \mathbb{I}(k_0)$ and $y(k_0) = y_0$.

Hypothesis-H$_{(2.2)}$: F is a sequence of continuous functions defined on $\mathbb{R}^n$ into $\mathbb{R}^n$, and it is twice continuously differentiable with respect to m. A solution process of (2.2) is denoted by $m(k, k_0, m_0) = m(k)$ for $k \geq k_0$. It is further assumed that its second derivative $\frac{\partial^2}{\partial m_0 \partial m_0} m(k, k_0, m_0)$ is locally Lipschitzian in z_0 for each (k, k_0).

Remark 2.1. For each $p \in \mathbb{I}(k_0)$ and $y \in \mathbb{R}^n$, we observe that $y(p+1)$ satisfies

$$y(p+1, p, y) = y(p+1) = y + f(p, y, \omega), \quad y(p) = y. \tag{2.4}$$

In the following, we present a few auxiliary results with respect to smooth system (2.2). These results will be used in this section as well as in the subsequent sections of this work.

Lemma 2.2. *Let the hypothesis $H_{(2.2)}$ be satisfied. Then the solution process $m(k) = m(k, k_0, m_0)$ of (2.2) is unique and it satisfies*

$$m(k, p, x) = m(k, p+1, x + F(p, x)), \tag{2.5}$$

where $m(k, p, x)$ is the solution process of (2.2) through the initial data (p, x) for $k_0 \leq p \leq k$, and $p, k \in \mathbb{I}(k_0)$.

Proof. We note that the uniqueness of the solution process $m(k, k_0, m_0)$ of (2.2) follows from the repeated composition of a map $G(x) = x + F(p, x)$ with itself.

To prove (2.5), we consider the left-hand side of (2.5)

$$\begin{aligned}
m(k,p,x) &= (G\circ G\circ\cdots\circ G\circ G)(x) \\
&\qquad\qquad \text{(by } (k-p) \text{ repeated composition of } G \text{ with itself)} \\
&= (G\circ G\circ\cdots\circ G)\circ G(x) \text{ (by regrouping)} \\
&= (G\circ G\circ\cdots\circ G)(m(p+1,p,x) \text{ (from the definition of } G \text{ and (2.4))} \\
&= m(k,p+1,(m(p+1,p,x)) \text{ (by the same reasoning as before)} \\
&= m(k,p+1,x+F(p,x)) \text{ (by the definition of } G \text{ and (2.4)).}
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.3. *Let the assumption of Lemma 2.2 be satisfied. Then, (a)* $\frac{\partial}{\partial m_0}m(k,k_0,m_0)$ *exists and satisfies the following linear homogeneous matrix iterative process:*

$$\Delta X = \frac{\partial}{\partial m}F(k,m(k))X,\quad X(k_0)=I, \tag{2.6}$$

along the solution process $m(k,k_0,m_0)=m(k)$ *of* (2.2) *and it is denoted by*

$$\frac{\partial}{\partial m_0}m(k,k_0,m_0)=\Phi(k,k_0,m_0); \tag{2.7}$$

(b) $\frac{\partial^2}{\partial m_0\partial m_0}m(k,k_0,m_0)$ *exists and satisfies the following linear nonhomogeneous matrix iterative process:*

$$\begin{aligned}
\Delta X &= \frac{\partial}{\partial m}F(k,m(k))X \\
&\quad + \frac{\partial^2}{\partial m^2}F(k,m(k))\cdot\Phi(k,k_0,m_0)\Phi(k,k_0,m_0),\quad X(k_0)=0,
\end{aligned} \tag{2.8}$$

along the solution process $m(k,k_0,m_0)=m(k)$ *of* (2.2)*, where* $\frac{\partial^2}{\partial m^2}F(k,mz(k))$ *and* $\frac{\partial^2}{\partial m_0^2}m(k,k_0,m_0)$ *are* $n\times n$ *Hessian matrices of* F *and* $m(k,k_0,m_0)$*, respectively, and their elements are* $1\times n$ *matrices, and*

$$\frac{\partial^2}{\partial m^2}F(k,,m(k))\cdot\Phi(k,k_0,mz_0)=\left(\frac{\partial^2}{\partial m\partial z_j}F_i(k,m(k))\Phi(k,k_0,m_0)\right)_{n\times n}.$$

Proof. The proof of the part (a) can be imitated by following the proof of Theorem A.1. The proof of the part (b) can be reformulated by following the steps in the proof of Theorem A.1. The details are left as an exercise to the reader. □

Lemma 2.4. *Let the hypotheses of Lemma 2.2 be satisfied. Then*

(a)

$$\Phi(k,p,m)=\Phi(k,p+1,m+F(p,m))\Phi(p+1,p,m); \tag{2.9}$$

(b)

$$\frac{\partial^2}{\partial m_0\partial m_0}m(k,p,m)=\frac{\partial}{\partial m_0}\Phi(k,p,m)$$

$$= \frac{\partial}{\partial m_0}\left(\Phi(k,p+2,m(p+2)) \cdot \Phi(p+2,p,m)\Phi(p+2,p,m)\right.$$
$$\left. + (\Phi(k,p+2,m(p+2)) \cdot \frac{\partial}{\partial m_0}\Phi(p+2,p,m)\Phi(p+1,p,m); \right. \tag{2.10}$$

(c)

$$\begin{aligned} m(k,p+1,u+v) - m(k,p,u) &= \Phi(k,p+1,u+F(p,u))\Delta(p,u,v) \\ &+ \frac{1}{2}\frac{\partial}{\partial m_0}\Phi(k,p+1,u+F(p,u)) \cdot \Delta(p,u,v)\Delta(p,u,v) \\ &+ O(k,p+1,u+F(p,u),\Delta(p,u,v)), \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &O(k,p+1,u+F(p,u),\Delta(p,u,v)) \\ &= \int_0^1\int_0^1 \theta\Big[\frac{\partial}{\partial m_0}\Phi(k,p+1,u+F(p,u)+\theta\psi\Delta(p,u,v)) \\ &\quad - \frac{\partial}{\partial m_0}\Phi(k,p+1,u+F(p,u))\Big] \cdot \Delta(p,u,v)d\psi\, d\theta\, \Delta(p,u,v), \end{aligned} \tag{2.12}$$

$$\Delta(p,u,v) = [v - F(p,u)]. \tag{2.13}$$

Proof. By differentiating partially both sides of (2.5) with respect to x, we have

$$\begin{aligned} \frac{\partial}{\partial m_0} m(k,p,m) &= \frac{\partial}{\partial m_0} m(k,p+1,m+F(p,m)) && \text{(by (2.4))} \\ &= \frac{\partial}{\partial m_0} m(k,p+1,m(p+1,p,m)) && \text{(by chain rule)} \\ &= \frac{\partial}{\partial m_0} m(k,p+1,m(p+1,p,m))\frac{\partial}{\partial m_0} m(p+1,p,m) && \text{(from (2.7))} \\ &= \Phi(k,p+1,m(p+1,p,m))\Phi(p+1,p,m) \end{aligned}$$

This completes the proof of the part (a).

To prove (2.10), applying (2.4),

$$m(k,p,m) = m(k,p+2,m(p+1,p,m) + F(p+1,m(p+1,p,m))).$$

This together with the application of (2.9) yields

$$\frac{\partial}{\partial m_0} m(k,p,m) = \Phi(k,p,m) = \Phi(k,p+2,m(p+2,p,m))\Phi(p+2,p,m).$$

Again by differentiating this expression, partially, with respect m for each (k,p), we obtain

$$\begin{aligned} &\frac{\partial^2}{\partial m_0^2} m(k,p,m) = \frac{\partial}{\partial m_0}\Phi(k,p,m) \\ &= \frac{\partial}{\partial m_0}\Phi(k,p+2,m(p+2,p,m))\Phi(p+2,p,m) \end{aligned}$$

$$= \frac{\partial}{\partial m_0}\Phi(k,p+2,m(p+2,p,m))\cdot\frac{\partial}{\partial m_0}\,m(p+2)\Phi(p+2,p,m))$$
$$+\,\Phi(k,p+2,m(p+2,p,m))\frac{\partial}{\partial m_0}\Phi(p+2,p+1,m(p+1,p,m))$$
$$\text{(by product and chain rules)}$$
$$= \frac{\partial}{\partial m_0}\Phi(k,p+2,m(p+2))\cdot\Phi(p+2,p,m)\Phi(p+2,p,m) \qquad \text{(by notations)}$$
$$+\,\Phi(k,p+2,m(p+2))\frac{\partial}{\partial m_0}\Phi(p+2,p,m)\cdot\Phi(p+1,p,m).$$

This completes the proof (2.10).

To prove (2.11), by applying the generalized mean value Lemma A.1.1, we have

$$m(k,p+1,u+v)-m(k,p,u) \qquad \text{(from (2.5))}$$
$$= m(k,p+1,u+F(p,u)+v-F(p,u)))-m(k,p+1,u+F(p,u))$$
$$= \int_0^1 \Phi(k,p+1,u+F(p,u)+\theta\Delta(p,u,v))d\theta\Delta(p,u,v) \qquad \text{(from algebra and (2.5))}$$
$$= \Phi(k,p+1,u+F(p,u))\Delta(p,u,v)$$
$$+\int_0^1[\Phi(k,p+1,u+F(p,u)+\theta\Delta(p,u,v))-\Phi(k,p+1,u+F(p,u)]d\theta\Delta(p,u,v)$$
$$= \Phi(k,p+1,u+F(p,u))\Delta(p,u,v)$$
$$+\,\Psi(k,p+1,u+F(p,u),\Delta(p,u,v))\Delta(p,u,v), \tag{2.14}$$

where

$$\Psi(k,p+1,u+F(p,u),\Delta(p,u,v)) \qquad \text{(from (2.5))}$$
$$= \int_0^1[\Phi(k,p+1,u+F(p,u)+\theta\Delta(p,u,v))-\Phi(k,p+1,u+F(p,u))]d\theta. \tag{2.15}$$

Again by applying the generalized mean value Lemma A.1.1, we get

$$[\Phi(k,p+1,u+F(p,u)+\theta\Delta(p,u,v))-\Phi(k,p+1,u+F(p,u))]$$
$$= \int_0^1 \frac{\partial}{\partial m_0}\Phi(k,p+1,u+F(p,u)+\theta\psi\Delta(p,u,v))\cdot\Delta(p,u,v)\theta\,d\psi$$
$$= \frac{\partial}{\partial m_0}\Phi(k,p+1,u+F(p,u))\cdot\Delta(p,u,v)\theta$$
$$+\int_0^1\theta\Big[\frac{\partial}{\partial m_0}\Phi(k,p+1,u+F(p,u)+\theta\psi\Delta(p,u,v))$$
$$-\frac{\partial}{\partial m_0}\Phi(k,p+1,u+F(p,u))\Big]\cdot\Delta(p,u,v)d\psi \tag{2.16}$$

By integrating both sides with respect to θ from 0 to 1 and using notation (2.15), we have

$$\Psi(k,p+1,u+F(p,u),\Delta(p,u,v))$$

$$= \frac{1}{2}\frac{\partial}{\partial m_0}\Phi(k,p+1,u+F(p,u))\cdot\Delta(p,u,v)$$
$$+\int_0^1\int_0^1\theta\Big[\Big[\frac{\partial}{\partial m_0}\Phi(k,p+1,u+F(p,u)+\theta\psi\Delta(p,u,v))$$
$$-\frac{\partial}{\partial m_0}\Phi(k,p+1,u+F(p,u))\Big]\cdot\Delta(p,u,v)d\psi\Big]d\theta. \tag{2.17}$$

From (2.17), (2.14) reduces to

$$m(k,p+1,u+v)-m(k,p,u)$$
$$=\Phi(k,p+1,u+F(p,u))\Delta(p,u,v)$$
$$+\frac{1}{2}\frac{\partial}{\partial m_0}\Phi(k,p+1,u+F(p,u))\cdot\Delta(p,u,v)\Delta(p,u,v)$$
$$+\int_0^1\int_0^1\theta\Big[\Big[\frac{\partial}{\partial m_0}\Phi(k,p+1,u+F(p,u)+\theta\psi\Delta(p,u,v))$$
$$-\frac{\partial}{\partial m_0}\Phi(k,p+1,u+F(p,u))\Big]\cdot\Delta(p,u,v)d\psi\,d\theta\Big]\Delta(p,u,v). \tag{2.18}$$

This together with the notation (2.12), the proof of (2.11) follows, immediately. This completes the proof of the lemma. □

By utilizing a vector Lyapunov-like function $V\in C[\mathbb{R}_+\times\mathbb{R}^n\times\Omega,\mathbb{R}^N]$, we define an operator as follows:

$$\Delta V(p,m(k,p,u,v)) = V(p+1,m(k,p+1,u+v)) - V(p,m(k,p,u)) \tag{2.19}$$

where, $m(k,p,u)$ and $m(k,p+1,u+v)$ are the solution processes of (2.2) through (p,u) and $(p+1,u+v)$, v and $u\in\mathbb{R}^n$.

Remark 2.5. For $v=f(p,y,\omega)$ and $u=y$, $\Delta V(p,m(k,p,y))$ in (2.19) is denoted by $\Delta_{(2.1)}V(p,m(k,p,y))$. Furthermore, from the continuity of f guarantees the measurability of the difference operator $\Delta_{(2.1)}V(p,m(k,p,y))$.

In the following, we present a result that provides a basis for the definition of the generating operator L associated with a flow v. For this purpose, we need an additional condition on $V(t,y)$.

Theorem 2.6. *Let the hypotheses $H_{(2.1)}$ and $H_{(2.2)}$ be satisfied. Let $V\in C[\mathbb{R}_+\times\mathbb{R}^n,\mathbb{R}^N]$ and further assume that $\frac{\partial}{\partial t}V(t,m)$, $\frac{\partial}{\partial x}V(t,m)$ and $\frac{\partial^2}{\partial z\partial z}V(t,m)$ exist, and are continuous in all (t,m) in $\mathbb{R}_+\times\mathbb{R}^n$, and moreover $\frac{\partial^2}{\partial z\partial z}V(t,m)$ is locally Lipschitzian in m for each t. Then an operator L is defined by*

$$LV(p,m(k,p,u),v) = \mathcal{L}_aV(p,m(k,p,u),v) + \mathcal{L}_eV(p,m(k,p,u),v)$$
$$+\mathcal{L}_0V(p,m(k,p,u),v), \tag{2.20}$$

where

$$\mathcal{L}_a V(p, m(k,p,u), v) = V_t(p, m(k,p,u)) + V_m(p, m(k,p,u))A(k, p+1, \Delta(p,u,v)), \tag{2.21}$$

$$\mathcal{L}_e V(p, m(k,p,u), v) = \frac{1}{2}[V_m(p, m(k,p,u))E(k, p+1, \Delta(p,u,v)) + \frac{1}{2}\frac{\partial^2}{\partial m \partial m} V(p, m(k,p,u)) \cdot \Theta(k, p+1, \Delta(p,u,v))\Theta(k, p+1, \Delta(p,u,v)) \tag{2.22}$$

and

$$\begin{aligned}
&\mathcal{L}_0 V(p, m(k,p,u), v) \\
&= \int_0^1 [V_t(p+\theta, m(k,p,u) + \theta\Delta m) - V_t(p, m(k,p,u))]d\theta \\
&\quad + \int_0^1 \int_0^1 \theta \Big[\frac{\partial^2}{\partial z^2}[V(p+\theta, m(k,p,u) + \psi\theta\Delta m) - V(p, m(k,p,u))] \cdot \Delta m \Delta m\Big] d\psi\, d\theta \\
&\quad + \frac{1}{2}\Big[\frac{\partial^2}{\partial m \partial m} V(p, m(k,p,u)) \cdot \Theta(k, p+1, \Delta(p,u,v)) \\
&\quad \times O(k, p+1, u + F(p,u), \Delta(p,u,v)) \\
&\quad + \frac{\partial^2}{\partial m \partial m} V(p, m(k,p,u)) \cdot O(k, p+1, u + F(u,p), \Delta(p,y))\Theta(k, p+1, \Delta(p,y)), \\
&\quad + V_m(p, m(k,p,u))O(k, p+1, u + F(p,u), \Delta(p,u,v))\Big],
\end{aligned} \tag{2.23}$$

where $O(k, p+1, m(p+1), \Delta(p,u,v))$ *and* $\Delta(p,u,v)$ *are as defined in* (2.12) *and* (2.13), *respectively, and*

$$\Delta m = m(k, p+1, u+v) - m(k,p,u), \tag{2.24}$$

$$A(k, p+1, \Delta(p,u,v)) = \Phi(k, p+1, u + F(p,u))\Delta(p,u,v), \tag{2.25}$$

$$E(k, p+1, \Delta(p,u,v)) = \frac{1}{2}\frac{\partial}{\partial m_0}\Phi(k, p+1, u + F(p,u)) \cdot \Delta(p,u,v)\Delta(p,u,v), \tag{2.26}$$

and

$$\Theta(k, p+1, \Delta(p,u,v)) = A(k, p+1, \Delta(p,u,v)) + E(k, p+1, \Delta(p,u,v)). \tag{2.27}$$

Proof. Let $m(k, p+1, u+v)$ and $m(k,p,u)$ be solution processes of (2.2) through $(p+1, u+v)$ and (p,u), respectively. From (2.5) and (2.13) we recall that $m(k, p+1, u+F(p,u)+\Delta(p,u,v)) = m(k, p+1, u+v)$ and $m(k, p+1, u+F(p,u)) = m(k,p,u)$. Under the assumption of the lemma, applications of Lemmas 2.2, 2.3, and 2.4, the generalized mean value theorem with notations (2.5), imitating the argument used in the proof of Lemma 2.4(c), and algebraic simplifications, we have

$$\begin{aligned}
&\Delta_v V(p, m(k,p,u,v)) \\
&= V(p+1, m(k, p+1, u+v)) - V(p, m(k,p,u))
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 [V_t(p+\theta, m(k,p,u)+\theta\Delta m) \\
&\qquad + V_m(p+\theta, m(k,p,u)+\theta\Delta m)\Delta m] d\theta \\
&= V_t(p, m(k,p,u)) + V_m(p, m(k,p,u))\Delta m \\
&\qquad + \int_0^1 [V_t(p+\theta, m(k,p,u)+\theta\Delta m) - V_t(p, m(k,p,u)) \\
&\qquad + [V_m(p+\theta, m(k,p,u)+\theta\Delta m) - V_m(p, m(k,p,u))]\Delta m] d\theta \\
&= V_t(p, m(k,p,u)) + V_m(p, m(k,p,u))\Delta m \\
&\qquad + \frac{1}{2}\frac{\partial^2}{\partial m \partial m} V(p, m(k,p,u)) \cdot \Delta m(\Delta m)) \\
&\qquad + \int_0^1 [V_t(p+\theta, m(k,p,u)+\theta\Delta m) - V_t(p, m(k,p,u))] d\theta \\
&\qquad + \int_0^1 \int_0^1 \theta \Big[\Big[\frac{\partial^2}{\partial m^2} V(p+\theta, m(k,p,u)+\psi\theta\Delta m) \\
&\qquad - \frac{\partial^2}{\partial m^2} V(p, m(k,p,u))\Big] \cdot \Delta m \Delta m\Big] d\psi\, d\theta \\
&= V_t(p, m(k,p,u)) + V_m(p, m(k,p,u))[\Delta m] \\
&\qquad + \frac{1}{2}\Big[\frac{\partial^2}{\partial m^2} V(p, m(k,p,u)) \cdot \Delta m \Delta m\Big] \\
&\qquad + \Big[\int_0^1 [V_t(p+\theta, m(k,p,u)+\theta\Delta m) - V_t(p, m(k,p,u))] d\theta \\
&\qquad + \int_0^1 \int_0^1 \theta \Big[\frac{\partial^2}{\partial m^2} V(p+\theta, m(k,p,u)+\psi\theta\Delta m) \\
&\qquad - \frac{\partial^2}{\partial m^2} V(p, m(k,p,u))\Big] \cdot \Delta m \Delta m\Big] d\psi\, d\theta
\end{aligned} \tag{2.28}
$$

From (2.4), (2.5), (2.11), (2.12), and the definitions of operators $\mathcal{L}_a$, $\mathcal{L}_e$, $\mathcal{L}_o$, the notations and the definitions introduced in the theorem, and increments $\Delta(p, u, v)$ and Δm with algebraic simplifications and regrouping, (2.28) reduces to desired relation in (2.20). The details are left to the reader. □

Remark 2.7. We note that for $v = f(p, y, \omega)$ and $u = y$, the definition of $LV(p, m(k, p, y, v)) \equiv LV(p, m(k, p, y))$ in (2.20) does not depend on the knowledge of the solution process of (2.1). It just depends on $y \in \mathbb{R}^n$, the rate function f and $\Delta(p, u, v) = f(p, y, \omega) - F(p, y)$. Moreover, in order to characterize the effects of random perturbations (both internal as well as external), L is decomposed into three operators $\mathcal{L}_a$, $\mathcal{L}_e$ and $\mathcal{L}_o$ representing the absence of randomness, the presence of external disturbances, and round of error, respectively.

The following results illustrate the computational feasibility of L-operator.

Corollary 2.8. *Let* $V(p,m)=\alpha(p)m^Tm=\alpha(p)\|m\|^2$, $\alpha(p)>0$, *for any* $k\in\mathbb{I}(k_0)$, $k_0\le p\le k$. *Then*

$$V_t(p,m)=0,\quad V_m(p,m)=2\alpha(p)m^T,\quad \frac{\partial^2}{\partial m\partial m}V(p,m(k,p,y))=2I.$$

Then $LV(p,m(k,p,y))$ *reduces to*

$$LV(p,m(k,p,u))=\mathcal{L}_aV(p,m(k,p,u,v))+\mathcal{L}_eV(p,m(k,p,u,v))+\mathcal{L}_0V(p,m(k,p,u,v)),\tag{2.29}$$

where

$$\begin{aligned}\mathcal{L}_aV(p,m(k,p,u))&=2a(p)m(k,p,u)^T\Phi(k,p+1,m(p+1))\Delta(p,u,v)\\&=2\alpha(p)m(k,p,u)^TA(k,p+1,\Delta(p+1))\\\mathcal{L}_eV(p,m(k,p,u))&=[\Theta^T(k,p+1,m(p+1))\Theta(k,p+1,m(p+1))\\&\quad+\alpha(p)m(k,p,u)^TE(k,p+1,\Delta(p,u,v))\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_oV(p,z(k,p,u))&=2\alpha(p)m(k,p,u)^TO(k,p+1,m(p+1),\Delta(p,u,v))\\&\quad+O^T(k,p+1,m(p+1,p,u),\Delta(p,u,v)\Theta(k,p+1,m(p+1)\\&\quad+\Theta^T(k,p+1,m(p+1,p,u))O(k,p+1,m(p+1),\Delta(p,u,v))\\&\quad+O^T(k,p+1,m(p+1,p,u)O(k,p+1,m(p+1),\Delta(p,u,v)).\end{aligned}$$

Proof. From (2.11) and (2.12), we have

$$\begin{aligned}&V_m(p,m(k,p,u))\Delta m=V_m(p,m(k,p,y))\Big[\Phi(k,p+1,m(p+1))\Delta(p,y)\\&\quad+\frac{1}{2}\frac{\partial}{\partial m_0}\Phi(k,p+1,u+F(p,y))\Delta(p,u,v)\Delta(p,u,v)\\&\quad+O(k,p+1,u+F(p,y),\Delta(p,u,v)\Big]\end{aligned}$$

where

$$\begin{aligned}&\frac{1}{2}\frac{\partial^2}{\partial z^2}V(p,m(k,p,u))\Delta m^T\Delta m\\&\quad=\Theta^T(k,p+1,m(p+1))\Theta(k,p+1,m(p+1))\\&\qquad+\Theta^T(k,p+1,m(p+1))O(k,p+1,y+F(p,u)\Delta(p,u))\\&\qquad+O^T(k,p+1,u+F(p,u),\Delta(p,u,v))\Theta(k,p+1,m(p+1))\\&\qquad+O^T(k,p+1,u+F(p,u),\Delta(p,u,v))O(k,p+1,u+F(p,u),\Delta(p,u,v))\end{aligned}$$

After elementary computations and simplifications, (2.20) reduces to the desired relation in (2.29). We note that the scope of L-operator defined in (2.20) in the context of auxiliary system (2.2) is illustrated by following well-known special cases: □

Corollary 2.9. *If $F(p,y)$ in (2.2) is $F(p,y) \equiv 0$, then $m(k,p,u) = u$, $\Phi(k,p,u) = I$ and $\frac{\partial^2}{\partial m_0^2} m(k,p,u) \equiv 0$. Then the L-operator defined in (2.20) in Theorem 2.6 reduces to the L-operator: $LV(p,u,v) = \mathcal{L}_a V(p,u,v) + \mathcal{L}_e V(p,u,v) + \mathcal{L}_0 V(p,u,v)$, where*

$$
\begin{aligned}
\mathcal{L}_a V(p,u,v) &= E[V_t(p,u) + V_m(p,u)v,\\
\mathcal{L}_e V(p,u,v) &= \frac{1}{2}\frac{\partial^2}{\partial m^2} V(p,u)\cdot vv,\\
\mathcal{L}_0 V(p,u,v) &= \int_0^1 \Bigg\{ V_t(p+\theta, u+\theta v) - V_t(p,u)\\
&\quad + \int_0^1 \left[\theta \frac{\partial}{\partial m} V_m(p+\theta, u+\theta\psi v - \frac{\partial}{\partial m} V_m(k,y)\right]\cdot vv \Bigg\} d\psi\, d\theta
\end{aligned}
$$

Corollary 2.10. *If $F(p,m) = A(p)m$, then in this case: $m(k,p,u) = \prod_{i=p}^{k-1}(I + A(i))u$, $\frac{\partial}{\partial m}F(p,m) = A(p)$, $\frac{\partial^2}{\partial m^2}F(p,m) \equiv 0$, $\Phi(k,p,u) = \prod_{i=p}^{k-1}(I + A(i)) \equiv \Phi_a(k,p)$, $\frac{\partial^2}{\partial m_0^2} m(k,p,u) \equiv 0$. Then the L-operator defined in (2.20) in Theorem 2.6 reduces to:*

$$
\begin{aligned}
LV(p, m(k,p,u,v)) &= \mathcal{L}_a V(p, m(k,p,u), v) + \mathcal{L}_e V(p, m(k,p,u,v))\\
&\quad + \mathcal{L}_o V(p, m(k,p,u,v)),
\end{aligned}
\tag{2.30}
$$

where

$$
\mathcal{L}_a V(p, m(k,p,u,v)) = [V_t(p, m(k,p,u)) + V_m(p, m(k,p,u))][\Phi(k,p+1)\Delta(p,u,v))],
$$

$$
\mathcal{L}_e V(p, m(k,p,u,v)) = \frac{1}{2}\frac{\partial^2}{\partial m \partial m} V(p, m(k,p,u)) \cdot \Phi(k,p+1)v\Phi(k,p+1)v,
$$

and

$$
\begin{aligned}
\mathcal{L}_o V(p, m(k,p,u,v)) &= \int_0^1 [V_t(p+\theta, m(k,p,u) + \theta\Delta m)) - V_t(p, m(k,p,u))] d\theta\\
&\quad + \int_0^1\int_0^1 \theta\Big[\frac{\partial^2}{\partial m^2} V(p+\theta, m(k,p,u) + \psi\theta\Delta m)\\
&\quad - \frac{\partial^2}{\partial m^2} V(p, m(k,p,u))\Delta m(\Delta m)^T\Big]
\end{aligned}
$$

where,

$$
\begin{aligned}
\Delta(p,u,v) &= [v - A(p)u]\\
\Theta(k,p+1,m(p+1)) &= (\Delta(p,u,v))^T \Phi^T(k,p+1)\Phi(k,p+1)\Delta(p,u,v)
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta m = m(k,p+1,u+v) - m(k,p,u) &= m(k,p+1,v) - m(k,p+1,A(p)u)\\
&= \Phi(k,p+1)[v - A(p)u].
\end{aligned}
$$

3. COMPARISON THEOREMS

In this section, by developing results concerning random difference inequalities, we shall present several comparison theorems which are useful to study the error estimates and stability properties of stochastic difference systems. Again in the following we assume that all the inequalities and relations involving random quantities are true with probability one (w.p. 1).

Now we prove a fundamental result concerning difference inequalities. This result plays an important role for further development of comparison theorems for random difference systems.

Theorem 3.1 (VARIATIONAL COMPARISON THEOREM). *Let the hypotheses of Theorem* 2.6 *be satisfied. Further assume that*

i) $G \in C[\mathbb{R}\times\mathbb{R}^N\times\Omega, \mathbb{R}^N]$ *and* $G(k,a,\omega)$ *satisfies the one-sided Lipschitz condition* $G(k,b,\omega) - G(k,a,\omega) \geq -\Gamma(k,\omega)(b-a)$ *for all* $a, b \in \mathbb{R}^N$ *and* $b \geq a$ *where* $\Gamma(k,\omega) = diag\{\gamma_1,\ldots,\gamma_N\}$ *with* $0 \leq \gamma_i \leq 1$ *for* $1 \leq i \leq N$*;*

ii) $LV(p, m(k,p,y)) \leq G(p, V(p, m(k,p,y)), \omega)$ *for all* $(p,y) \in \mathbb{I}(k_0)\times\mathbb{R}^n$ *where the generating operator* L *is defined as*

$$LV = [LV_1, LV_2, \ldots, LV_i, \ldots, LV_N]^T$$

where $LV(k, m(t,p,x))$ *is defined analogous to* L *in* (2.20)*;*

iii) $r(k)$ *is the solution of the system of comparison difference equations*

$$u(k+1) = u(k) + G(k,u(k),\omega), \qquad u(k_0,\omega) = u_0(\omega); \tag{3.1}$$

iv) $E[V(p, m(k,p,y(p)))]$ *exists for any solution* $y(p) = y(p,k_0,y_0)$ *of the iterative stochastic process* (2.1) *for all* $p \in \mathbb{I}(k_0)$ *and* $k_0 \leq p \leq k$*, and*

$$V(k_0, m(k,k_0,y_0)) \leq u_0. \tag{3.2}$$

Then

$$V(p, m(k,p,y(p))) \leq r(p,\omega), \qquad \text{for all } p \in \mathbb{I}(k_0). \tag{3.3}$$

Moreover, (3.3) *reduces to*

$$V(k, y(k)) \leq r(k,\omega) \quad \text{for all } k \in \mathbb{I}(k_0). \tag{3.4}$$

Proof. For $k_0 \leq p+1 \leq k$, let $m(k,p,y) = m(k,p+1,y+F(p,y))$ and $y(p,\omega) = y(k,k_0,y_0)$ be the solution processes of (2.2) and (2.1) through $(p, y+F(p,y))$ and (k_0,y_0), respectively. Define

$$V(p+1, m(k,p+1,y(p+1))) = V(p+1, m(k,p+1,y(p)+f(p,y(p),\omega))) \tag{3.5}$$

We also note that

$$V(p, m(k,p,y(p))) = V(p, m(k,p+1,y(p)+F(p,y(p)))). \tag{3.6}$$

From (3.5), (3.6), (2.1), we have

$$\begin{aligned} &V(p+1, m(k,p+1,y(p+1))) - V(p, m(k,p,y(p))) \\ &\quad = [\Delta_{(2.1)} V_1(p, m(k,p,y(p))), \ldots, \\ &\qquad \Delta_{(2.1)} V_i(p, m(k,p,y(p))), \ldots, \Delta_{(2.1)} V_N(p, m(k,p,y(p)))]^T. \end{aligned}$$

From the hypotheses of Theorem 2.6 and assumption ii) of the theorem, we have

$$\begin{aligned} &\Delta V(p, m(k,p,y(p))) = V(p+1, m(k,p+1,y(p+1))) - V(p, m(k,p,y(p))) \\ &= [LV_1(p, m(k,p,y(p))), \ldots, LV_i(p, m(k,p,y(p))), \ldots, LV_N(p, m(k,p,y(p)))]^T \\ &\le G(p, V(p, m(k,p,y(p))), \omega). \end{aligned} \tag{3.7}$$

By choosing $u_0 \ge V(k_0, m(k, k_0, y_0(\omega)))$ and applying Theorem 2.2 [7], one concludes

$$V(p, m(k,p,y(p))) \le r(p, k_0, u_0, \omega), \quad \text{for all } k_0 \le p \le k, \quad k \in \mathbb{I}(k_0). \tag{3.8}$$

In particular, for $p = k$, (3.8) reduces to

$$V(k, y(k,\omega)) \le r(k, k_0, u_0, \omega)), \quad \text{for all } k \in \mathbb{I}(k_0).$$

This completes the proof of the theorem. □

The following corollary demonstrates the scope of the comparison Theorem 3.1. This corollary is based on Corollaries 2.8, 2.9 and 2.10.

Corollary 3.2. *By considering Corollaries* 2.8 *and* 2.10 *and noting the fact that* $\mathcal{L}_o V(p, m(k,p,x)) = 0$ *and* $\Delta(p,y) = f(p,y,\omega) - A(p)y$*, we assume that one can find* $\mu(p) \in \mathbb{R}$*, and* $\nu(p)$*,* $\beta(p) \in \mathbb{R}^+$ *such that*

$$m(k,p,y)^T \Phi(k,p)[f(p,y,\omega) - F(p,y)] \le \mu(p,\omega) m(k,p,y)^T m(k,p,y), \tag{3.9}$$

$$\begin{aligned} &[f(p,y,\omega) - F(p,y)]^T \Phi^T(k,p) \Phi(k,p) [f(p,y,\omega) - F(p,y)] \\ &\qquad \le \nu(p,\omega) m(k,p,y)^T m(k,p,y) + \beta(p,\omega), \end{aligned} \tag{3.10}$$

and

$$2\mu(p,\omega) + \nu(p,\omega) \ge -1, \qquad k_0 \le p \le k$$

for all $k_0 \le p \le k$ *and any* $k \in \mathbb{I}(k_0)$*. Then*

$$V(k, y(k, k_0, y_0)) \le r(k, k_0, u_0, \omega),$$

provided $V(k_0, y_0) \le u_0$*.*

Proof. From (3.2), algebraic calculations and simplifications with notation $V(p,z) = \alpha(p) z^T z$, one can obtain

$$LV(p, m(k,p,y)) \le g(p,\omega) V(p, m(k,p,y)) + \beta(p,\omega),$$

where $g(p,\omega)$ is defined by $g(p,\omega) = -(2\mu(p,\omega)+\nu(p,\omega)+1)+1$. In this setup, the system of comparison equations then becomes

$$\Delta u(k) = G(k,u(k),\omega), \qquad u_{k_0} = u_0 \tag{3.11}$$

where

$$G(k,u,\omega) = g(p,\omega)u(k) + \beta(k,\omega).$$

It is obvious that the above comparison function $G(k,u,\omega)$ satisfies all the hypotheses of Theorem 2.2 [7]. One only needs to assume condition iv) holds and then pick $u_0 \geq V(k_0,x_0))$. Thus by the application of Theorem 2.2 [7], one may conclude

$$V(k,y(k)) \leq r(k,k_0,u_0,\omega), \qquad \forall\, k \in \mathbb{I}(k_0) \tag{3.12}$$

where $r(k,k_0,u_0)$ is the solution process of the comparison difference equation (3.11). □

Corollary 3.3. *Let us assume that hypotheses $H_{(2.1)}$ and $H_{(2.2)}$ be satisfied.*

(a) *Assume that the conditions in Corollary* 2.10 *are fulfilled. In addition the operator L defined in* (2.30) *satisfies the following inequality*

$$LV(p,m(k,p,y)) \leq G(p,m(k,p,y),\omega) \tag{3.13}$$

where G satisfies the conditions of Theorem. Under the conditions (iii) and (iv) of Theorem 3.1*. The conclusion of Theorem* 3.1 *remains true.*

(b) *Assume that conditions of Corollaries* 2.8 *and* 2.9 *are valid. Further assume*

$$\begin{cases} 2\alpha y^T[f(p,y,\omega)-F(p,y)] \leq \mu(p,\omega)y^Ty \\ \alpha[f(p,y,\omega)-F(p,y)]^T[f(p,y,\omega)-F(p,y)] \leq \nu(p,\omega)y^Ty \\ 2\mu(p,\omega)+\gamma(p,\omega) \geq -1. \end{cases} \tag{3.14}$$

Then the conclusion of Corollary 3.2 *remains valid, that is,*

$$V(k,y(k,k_0,y_0)) \leq r(k,k_0,u_0,\omega) \tag{3.15}$$

provided

$$V(k_0,y_0) \leq u_0$$

where $r(k,k_0,u_0)$ is the solution process of (3.11).

Proof. The proofs of these results can be constructed by following the arguments used in Theorem 3.1 and Corollary 3.2. □

To investigate qualitative properties of (2.1), we use the corresponding properties of comparison system (3.1). If the dimension of system (3.1) (dimension "N") is very high, it is difficult to obtain such information about the system. Therefore, it forces to reduce the dimension to either lower or to obtain an information about such system by comparing with lower order system. Here, we choose to provide an estimate on the solution process of (2.1) in terms of a scalar iterative process.

Theorem 3.4. *Assume that conditions i), ii), iii) and iv) of Theorem* 3.1 *are satisfied. Further assume that satisfies*

$$\sum_{i=1}^{N} d_i G_i(p, m(k,p,y), \omega) \leq \gamma(p,\omega) \sum_{i=1}^{m} d_i V_i(p, m(k,p,y)) + \mu(p,\omega), \tag{3.16}$$

and define

$$\bar{v}(p, m(k,p,y)) = \sum_{i=1}^{m} d_i V_i(p, m(k,p,y)), \tag{3.17}$$

for $d_i > 0$ for $1 \leq i \leq N$. Then

$$\bar{v}(p, y(p,\omega)) \leq r(p,\omega) \quad \text{for all } k_0 \leq p \leq k, \quad k \in \mathbb{I}(k_0), \tag{3.18}$$

whenever

$$\bar{v}(k_0, m(k, k_0, x_0)) \leq u_0(\omega) \tag{3.19}$$

where $r(p,\omega)$ is the solution of the scalar comparison difference equation

$$\Delta u(p) = \gamma(p,\omega) u(p) + \mu(p,\omega), \qquad u_{k_0} = u_0 \tag{3.20}$$

with $u(p) \in \mathbb{R}_+$.

Proof. The proof of the result follows by using $\bar{v}(p, m(k,p,y))$ defined in (3.17) and the application of Corollary 3.2. The details are left to the reader. □

We demonstrate the scope of Theorem 3.1 in the following remark.

Remark 3.5. If $u_0(\omega) = V(k_0, x(k, k_0, y_0(\omega)), \omega)$, then (3.4) becomes

$$V(k, y(k,\omega), \omega) \leq r(k, k_0, V(k_0, x(k,\omega), \omega)), \qquad k \geq k_0. \tag{3.21}$$

We remark that the comparison Theorems 3.1 and 3.4 are not exactly like the comparison results [7]. From (3.21) it is obvious that Theorems 3.1 and 3.4 relate the solution processes of three kinds of difference equations, namely (2.1), (2.3), and (3.1)/(3.11). On the other hand, the usual comparison results relate the solutions of two kinds of initial value problems namely (2.1) and (3.1)/(3.11). Another important factor regarding Theorems 3.1 and 3.4 is that the initial state u_0 of the sample solution process $r(p, k_0, u_0, \omega)$ of (3.1)/(3.11) depends upon k.

In the following, we shall give a few examples to illustrate the scope of Theorems 3.1 and 3.4.

Example 3.6. Consider the linear stochastic difference equation

$$\Delta y(k,\omega) = \frac{y(k,\omega)}{k+1} + H(k, y(k,\omega),\omega), \qquad y(k_0,\omega) = y_0(\omega), \tag{3.22}$$

where $H \in M[I(k_0+1) \times R, R[\Omega, R]]$. We assume

$$\begin{aligned} yH(p,y,\omega) &\le -\lambda_1(p,\omega)y^2, \qquad \lambda_1(p,\omega) > 0 \\ H^2(p,y,\omega) &\le \lambda_2(p,\omega)y^2, \qquad \lambda_2(p,\omega) \ge 0. \end{aligned} \tag{3.23}$$

Further we consider the difference equation

$$\Delta x(k) = \frac{1}{k+1}\, x(k), \qquad x(k_0,\omega) = x_0(\omega) = y_0(\omega). \tag{3.24}$$

We notice that

$$m(k) = \begin{cases} y_0, & k = p \ge k_0 \\ \frac{y_0(k+2)}{(p+1)}, & k \ge p+1 \ge k_0 \end{cases}$$

solution of (3.24). We assume $V(k,x) = |x|^2$. From Theorem 3.1 and Corollary 2.8, we have

$$LV(p, m(k,p,y)) = \mathcal{L}_a V(p, m(k,p,y) + \mathcal{L}_e V(p, m(k,p,y)) + \mathcal{L}_0 V(p, m(k,p,y)),$$

where $m(k,p,y)$ is the solution process of (3.24) through (p,y), $\mathcal{L}_a V(p, m(k,p,y)) = m(k,p,y)\left(\frac{k+2}{p+1}\right) H(p,y,\omega)$, $\mathcal{L}_e V(p, x(k,p,y)) = \left(\frac{k+2}{p+1}\right)^2 H^2(p,y,\omega)$, and $\mathcal{L}_0 V(p, x(k,p,y)) = 0$. In the context of the (3.22) and (3.23), we obtain

$$\begin{aligned} LV(p, x(k,p,y,\omega)) &= \frac{(k+2)^2 yH(p,y,\omega)}{(p+1)} + \frac{(k+2)^2 H^2(\ell,y,\omega)}{(p+1)^2} \\ &\le (-2\lambda_1(p,\omega) + \lambda_2(p,\omega)) \left(\frac{k+2}{p+1}\right)^2 y^2 \\ &\le \lambda V(p, m(k,p,y)), \end{aligned} \tag{3.25}$$

where

$$\lambda(p,\omega) = -2\lambda_1(p,\omega) + \lambda_2(p,\omega).$$

In this case the comparison equation is

$$\Delta u(p,\omega) = \lambda(p,\omega) u(p,\omega), \qquad u(k_0,\omega) = u_0(\omega) \tag{3.26}$$

and the solution is given by

$$r(k, k_0, u_0(\omega), \omega) = \begin{cases} u_0(\omega), & k = k_0 \\ u_0(\omega) \prod_{p=k_0}^{k-1} \lambda(p,\omega), & k \ge k_0 \end{cases} \tag{3.27}$$

where $u_0(\omega) = |m(k, k_0, y_0(\omega))|^2$. Therefore from (3.25), (3.26) and (3.27) and applying Theorem 2.2 [7], relation (3.4) becomes

$$|y(k,\omega)|^2 \le |m(k,\omega)|^2 \prod_{p=k_0}^{k-1} (1 + \lambda(p)) \le |m(k,\omega)|^2 \exp\left[\sum_{p=k_0}^{k-1} \lambda(p)\right] \tag{3.28}$$

This justifies Remark 3.5.

Example 3.7. Consider the linear stochastic difference equation

$$\Delta y(k,\omega) = y(k,\omega) + H(k, y(k,\omega),\omega), \qquad y(k_0,\omega) = y_0(\omega) \tag{3.29}$$

where H is as defined in (3.22) and it satisfies

$$\begin{aligned} yH(p,y,\omega)y &\le -\lambda_3(p,\omega)y^2, \qquad \lambda_3(p,\omega) > 0 \\ H^2(p,y,\omega) &\le \lambda_4(p,\omega)y^2, \qquad \lambda_4(p,\omega) > 0. \end{aligned} \tag{3.30}$$

Further we consider the difference equation

$$\Delta x(k) = x(k), \qquad x(k_0,\omega) = y_0(\omega). \tag{3.31}$$

We notice $x(k) = 2^{k-k_0}y_0(\omega)$ is the solution of (3.31). By taking $V(k,x,\omega) = |x|^2$, we have

$$V(p, x(k,p,y)) = |m(k,p,y)|^2 = 2^{2(k-p)}y^2,$$

where $m(k,p,y)$ is the solution process of (3.31) through (p,y). We compute $LV(p, m(k,p,y))$ with respect to (3.29). By using (3.29) and following the argument used in Example 3.6, we have

$$\begin{aligned} LV(p,x(k,p,y)) &= \mathcal{L}_a V(p,m(k,p,y)) + \mathcal{L}_e V(p,m(k,p,y)) + \mathcal{L}_0 V(p,m(k,p,y)) \\ &= 2^{2(k-p)}\left[2yH(p,y,\omega) + H^2(p,y,\omega)\right] \\ &\le [-2\lambda_3(p,\omega) + \lambda_4(p,\omega)]2^{2(k-p)}y^2 \\ &\le \lambda(p,w)V(p,m(k,p,y)), \end{aligned} \tag{3.32}$$

where

$$\lambda(p,\omega) = [-2\lambda_3(p,\omega) + \lambda_4(p,\omega)].$$

In this case the comparison equation is

$$\Delta u(p,\omega) = \lambda(p,\omega)u(p,\omega), \qquad u(k_0,\omega) = u_0(\omega) \tag{3.33}$$

and its solution is given by

$$r(k,k_0,u_0(\omega),\omega) = \begin{cases} u_0(\omega), & k = k_0 \\ u_0(\omega)\prod_{p=k_0+1}^{k-1}(1+\lambda(p,\omega)), & k \ge k_0 \end{cases} \tag{3.34}$$

Therefore, from (3.32), (3.33), (3.34), and invoking Theorem 3.1 relation (3.4) reduces to

$$|y(k,\omega)|^2 \le |x(k,\omega)|^2 \prod_{s=k_0+1}^{k}(1+\lambda(s)) \le |x(k,\omega)|^2 \exp\left[\sum_{s=k_0+1}^{k}\lambda(s)\right] \tag{3.35}$$

Remark 3.8. In particular, if $\lambda_3(p,\omega) = \sqrt{\lambda_4(p,\omega)} = \alpha(p,\omega)$ and $2|1-\alpha(p,\omega)| < 1$ w.p. 1, then (3.35)

$$|y(k,\omega)|^2 \le \left(\frac{1}{4}\right)^{k-k_0} |x(k,\omega)|^2. \tag{3.36}$$

Example 3.9. We consider the following system of difference equation

$$\Delta y(k,\omega) = \hat{A}(k,e)y(k,\omega) + R(k,y,\omega)y(k,\omega), \qquad y(k_0,\omega) = y_0(\omega) \tag{3.37}$$

where $R(k,y,\omega) = A(k,y,\omega) - \hat{A}(k,e)$ and e is an n-dimensional parameter. Further we consider the difference system

$$\Delta x(k) = \hat{A}(k,e)x(k), \qquad x(k_0,\omega) = x_0(\omega). \tag{3.38}$$

We notice that the solution of (3.38) is

$$x(k,k_0,x_0(\omega)) = \begin{cases} y_0(\omega), & k = k_0 \\ \prod_{p=k_0}^{k-1}(I + \hat{A}(p,e))y_0(\omega) = \Phi(k,k_0,e)y_0(\omega), & k \ge k_0 + 1, \end{cases} \tag{3.39}$$

where

$$\Phi(k,k_0,e) = \begin{cases} I, & k = k_0 \\ \prod_{p=k_0}^{k-1}(I + \hat{A}(p,e)), & k \ge k_0. \end{cases}$$

Let us assume that

$$\begin{aligned} &m(k,p)\|y\|^2 \le y^T\Phi^T(k,p,e)\Phi(k,p,e)y \le M(k,p)\|y\|^2, \\ &y^T\Phi^T(k,p,e)\Phi(k,p,e)R(p,y,\omega)y \le -\lambda(p,\omega)\|y\|^2 \\ &y^TR^T\Phi^T(k,p,e)\Phi(k,p,e)Ry \le M_1(k,p)\|y\|^2, \qquad M_1 > 0 \end{aligned} \tag{3.40}$$

where m, M are minimal and maximal eigenvalues of $\Phi^T\Phi$ and M_1 is the maximal eigenvalue of $R^T\Phi^T(k,p,e)\Phi(k,p,e)R$.

By taking $V(k,x) = \|x\|^2$, we get

$$V(p,x(k,p,y)) = \|x(k,p,y)\|^2 = \|\Phi(k,p,e)y\|^2,$$

where $x(k,p,y)$ is the solution process of (3.38) through (p,y). Now we compute $LV(p,x(k,p,y(p)))$ with regard to (3.37). By using (3.40), (3.37), Theorem 2.6, Corollary 2.8, we have

$$\begin{aligned} LV(p,x(k,p,y)) &= \mathcal{L}_aV(p,x(k,p,y)) + \mathcal{L}_eV(p,x(k,p,y)) + \mathcal{L}_0V(p,x(k,p,y)) \\ &= 2x^T(k,p,y)\Phi(k,p,e)R(p,y,\omega)y \\ &\qquad + y^TR^T(p,y,\omega)\Phi^T(k,p,e)\Phi(k,p,e)R(p,y,\omega)y \\ &= 2y^T\Phi^T(k,p,e)\Phi(k,p,e)R(p,y,\omega)y \\ &\qquad + y^TR^T(p,y,\omega)\Phi^T(k,p,e)\Phi(k,p,e)R(p,y,\omega)y. \end{aligned} \tag{3.41}$$

By using (3.40), we have

$$\|y\|^2 \le \frac{1}{m(k,p)} y^T \Phi^T(k,p,e)\Phi(k,p,e) y \le \frac{1}{m(k,p)} V(p, x(k,p,y))$$

$$\begin{aligned} y^T \Phi^T(k,p,e)\Phi(k,p,e) R(p,y,\omega) y &\le -\lambda(p,\omega)\|y\|^2 \\ &\le -\frac{\lambda(p,\omega)}{M(k,p)} V(p, x(k,p,y),\omega), \qquad \lambda(p,\omega) > 0 \end{aligned}$$

$$\begin{aligned} y^T R^T(p,y,\omega)\Phi^T(k,p,e)\Phi(k,p,e) R(p,y,\omega) y &\le M_1(p)\|y\|^2 \\ &\le \frac{M_1(p)}{m(p)} V(p, x(k,p,y)). \end{aligned} \tag{3.42}$$

Using estimates (3.42), (3.41) reduces to

$$LV(p, x(k,p,y)) \le \nu(p) V(p, x(k,p,y)) \tag{3.43}$$

where $\nu(p,\omega) = -\frac{2\lambda(p,\omega)}{M(k,p)} + \frac{M_1(p)}{m(k,p)}$. In this case the comparison equation is

$$\Delta u(p,\omega) = \nu(p,\omega) u(p,\omega), \qquad u(k_0,\omega) = u_0(\omega) \tag{3.44}$$

and its solution is given by

$$r(p, k_0, u_0(\omega), \omega) = \begin{cases} u_0(\omega), & p = k_0 \\ u_0(\omega)\prod_{s=k_0+1}^{p} (\nu(s,\omega)+1), & p \ge s \ge k_0 \end{cases} \tag{3.45}$$

where $u_0(\omega) = V(k_0, x(k, k_0, y_0(\omega))) = \|x(k,\omega)\|^2$. From (3.43), (3.44), (3.45) and applying Theorem 3.1, relation (3.4) becomes

$$\|y(k,\omega)\|^2 \le \|x(k,\omega)\|^2 \prod_{s=k_0+1}^{k} (1+\nu(s,\omega)) \le \|x(k,\omega\|^2 \exp\left[\sum_{s=k_0+1}^{k} \nu(s,\omega)\right].$$

We now state and prove a comparison theorem which has wide range of applications in the theory of error estimates and stability analysis of stochastic difference systems. For this purpose, we consider the following system

$$\Delta w(k) = h(k, w(k), \omega), \quad w(k_0) = w_0. \tag{3.46}$$

Theorem 3.10. *Let the hypotheses of Theorem* 3.1 *be satisfied except* $LV(p, m(k,p,y))$, $V(p, m(k,p,y(p)))$ *and* $L(k_0, m(k,k_0,y_0))$ *are replaced by* $LV(p, m(k,p,y(p)-w(p)))$, $V(t, m(k,p,y(p)-w(p)))$ *and* $V(k_0, m(k,k_0,y_0-w_0))$, *respectively where* $y(p) = y(p,k_0,y_0)$ *and* $w(p) = w(p,k_0,w_0)$ *are solution process of* (2.1) *and* (3.46) *for* $k_0 \le p \le k$. *Then*

$$V(p, m(k,p,y(p)-w(p))) \le r(p,\omega) \quad \textit{for } k_0 \le p \le k, \tag{3.47}$$

provided

$$V(k_0, m(k,k_0,y_0-w_0)) \le u_0. \tag{3.48}$$

Moreover, for $p = k$, (3.47) *reduces to*

$$V(k, y(k) - w(k)) \le r(k, \omega), \quad \text{for } k \ge k_0 \tag{3.49}$$

whenever (3.48) *remains valid.*

Proof. Let $y(p)$ and $w(p)$ be solution processes of (2.1) and (3.46) through (k_0, y_0) and (k_0, w_0), respectively. By repeating the argument used in the proof of Theorem 3.1, the proof of the theorem can be constructed, analogously. The details are left to the reader. □

To illustrate the significance of Theorem 3.10, we outline a few particular cases.

Corollary 3.11. *Let us suppose that the hypotheses* $H_{(2.1)}$ *and* $H_{(2.2)}$ *remain true.*

(i) *Assume that all the conditions of Corollary* 2.9 *and Theorem* 3.10 *remain valid. Then*

$$V(p, y(p) - w(p)) \le r(p, \omega) \quad \text{for } k_0 \le p \le k$$

and

$$V(k, y(k) - w(k)) \le r(k, \omega) \quad \text{for } k \ge k_0 \tag{3.50}$$

provided

$$V(k_0, y_0 - w_0) \le u_0,$$

LV *in Theorem* 3.10 *is* $LV(p, y(p) - w(p))$ *and* $r(k, \omega)$ *is as defined by* (3.1).

(ii) *If the conditions of Corollary* 2.10 *and Theorem* 3.10, *then*

$$V(p, \Phi(k, p)(y(p) - w(p))) \le r(p, \omega) \quad \text{for } k_0 \le p \le k$$

and hence

$$V(k, y(k) - w(k)) \le r(k, \omega) \quad \text{for } k \ge k_0 \tag{3.51}$$

provided

$$V(k_0, \Phi(k, k_0)(y_0 - w_0)) \le u_0,$$

and $r(k, \omega)$ *is solution process of* (3.1).

(iii) *If the assumptions of Corollaries* 2.8 *and* 2.9, 3.3(b), *then*

$$\alpha \|y(k, k_0, y_0) - w(k, k_0, w_0)\|^2 \le r(k, k_0, \omega) \quad \text{for } k \ge k_0 \tag{3.52}$$

whenever

$$\alpha \|y_0 - w_0\|^2 \le u_0.$$

We note that

$$\begin{cases} 2\alpha(y - w)^T[f(p, y, \omega) - h(p, w, \omega) - F(p, y - w)] \le \mu(p, \omega)\|y - w\|^2 \\ \alpha\|f(p, y, \omega) - h(p, w, \omega) - F(p, y - w)\|^2 \le \nu(p, \omega)\|y - w\|^2 \\ 2\mu(p, \omega) + \nu(p, \omega) \ge -1 \end{cases} \tag{3.53}$$

and $r(k, \omega)$ *is the solution process of* (3.11).

(iv) *If the assumptions of Corollaries* 2.8 *and* 2.9, *then*

$$\alpha(k)\|y(k,k_0,y_0)-w(k,k_0,w_0)\|^2 \le r(k,\omega) \quad \text{for } k \ge k_0 \tag{3.54}$$

when

$$\alpha(k_0)\|\Phi(k,k_0)(y_0-w_0)\|^2 \le u.$$

We remark that

$$\begin{cases} m^T(k,p,y-w)\Phi(k,p)(f(p,y,\omega)-h(p,w,\omega)-F(p,y-w)) \\ \qquad \le \mu(p,\omega)\|m(k,p,y-w)\|^2 \\ \|\Phi(k,p)(f(p,y,\omega)-h(p,w,\omega)-F(p,y-w))\|^2 \\ \qquad \le \nu(p,\omega)\|m(k,p,y-w)\|^2+\beta(p,\omega) \\ 2\mu(p,\omega)+\nu(p,\omega)\ge -1 \end{cases}$$

and $r(p,\omega)$ *is the solution process of comparison equation* (3.1).

(v) *If the assumptions of (iv) and with* $F(k,y)\equiv 0$ *are fulfilled, then*

$$V(k,y(k)-w(k)) \le r(k,\omega) \quad \text{for } k\ge k_0 \tag{3.55}$$

provided that $V(k_0,y_0-w_0)\le u_0(\omega)$.

Theorem 3.12. *Assume that all the hypotheses of Corollary* 3.11 *are satisfied with* $h=F$. *Further assume that* L *in Theorem* 3.1 *satisfies*

$$V(p,m(k,p,y-w),\omega)+LV(p,m(k,p,y-w),\omega)\le G(p,V(p,m(k,p-w),\omega) \tag{3.56}$$

and

$$V(k_0,m(k,k_0,y_0-m_0,\omega),\omega)\le u_0(\omega), \tag{3.57}$$

where $x(k,k_0,z_0)=\bar{x}(k)$ *is the solution process of either* (2.2) *or* (2.3) *depending on the choice of* z_0; $LV(p,m(k,p,y-w)) = V(p+1,m(k,p+1,y-w+f(p,y,\omega)-F(p,y-w))-V(p,m(k,p,y-w))$. *Then*

$$V(k,y(k)-w(k),\omega)\le r(k,k_0,u_0(\omega),\omega), \qquad k\ge k_0 \tag{3.58}$$

Proof. Let $y(k,\omega)$ and $w(k,\omega)$ be the solution processes of (2.1) and (2.3) respectively. Let $m(k)=x(k,k_0,x_0)$ be a solution process of either (2.2) or (2.3) depending upon the choice of z_0. Set

$$\begin{aligned} v(p+1,\omega) &= V(p+1,x(k,p+1,y(p+1,\omega)-m(p+1)),\omega), \\ w(k_0,\omega) &= V(k_0,m(k,k_0,y_0-m_0),\omega). \end{aligned}$$

By following the proof of Corollary 3.11, the proof of theorem can be completed. □

To demonstrate the scope of Theorem 3.10 we present a few examples.

Example 3.13. We consider the stochastic difference systems (3.37), (3.38) and

$$\Delta m(k) = \hat{A}(k,e)m(k-1), \qquad m(k_0) = m_0. \tag{3.59}$$

Let $V(k,x,\omega) = \|x\|^2$. By following the discussion in Corollary 3.11(iv), we compute $LV(p, m(k,p,y(p) - m(p))$ as follows. Since $m(k) = \Phi(k,k_0,e)m_0(\omega)$. We assume that

$$2m^T(k,p,y-w)[f(p,y,\omega) - h(p,w,\omega) - F(p,y-w)] \leq \mu(p,\omega)V(p,m(k,p,y-w)),$$

$$\|\Phi(k,p)[f(p,y,\omega) - h(p,w,\omega) - F(p,y-w)]\|^2 \leq \nu(p,\omega)V(p,m(k,p,y-w)),$$

and

$$2\mu(p,\omega) + \nu(p,\omega) \geq -\gamma.$$

Hence,

$$LV(p,m(k,p,y-w),\omega) \leq [2\mu(p,\omega) + \nu(p,\omega)]V(p,m(k,p,y-w)) \tag{3.60}$$

Therefore, the comparison equation is

$$\Delta u(k) = 2\mu(k,\omega) + \nu(k,\omega)u, \quad u(k_0,\omega) = u_0(\omega) \tag{3.61}$$

where $u_0(\omega) = \|\Phi(k,k_0)(y_0 - w_0)\|$. By an application of Corollary 3.11(iv), we obtain

$$\|y(k,\omega) - m(k)\|^2 \leq \|\Phi(k,k_0)(y_0 - v_0)\|^2 \exp\left[\sum_{p=k_0}^{k-1} 2\mu(p,\omega) + \nu(p,\omega)\right] \tag{3.62}$$

4. STABILITY ANALYSIS

Let $y(k,\omega) = y(k,k_0,y_0(\omega),\omega)$ be any solution process of (2.1) and let $x(k,\omega) = x(k,k_0,y_0(\omega))$ be the solution process of (2.3) through $(k_0, y_0(\omega))$. Furthermore, let $\bar{x}(k) = x(k,k_0,z_0)$ be the solution process of either (2.2) or (2.3) depending upon the choice of z_0. Without loss of generality, we assume that $f(k,0,\omega) \equiv 0$ w.p. 1 and $F(k,0) \equiv 0$ for all $k \geq k_0$. $y(k,\omega) \equiv 0$, $x(k,\omega) \equiv 0$, $\bar{x}(k) \equiv 0$ are the unique solutions of the respective initial value problems.

In the following for the sake of easy reference, we present the qualitative properties solutions of (2.1). For more details, we refer to Ladde and Sambandham [10].

Definition 4.1. The trivial solution process of (2.1) is said to be

(DSM$_1$): *stable in the q^{th} moment*, if for each $\epsilon > 0$, $k_0 \in I(k_0)$ and $q \geq 1$, there exists a positive function $\delta(k_0,\epsilon)$ such that the inequality $\|y_0(\omega)\|_q \leq \delta$ implies

$$\|y(k,\omega)\|_q < \epsilon, \qquad k \geq k_0$$

where $\|y(k,\omega)\|_q = (E\|y(k,\omega)\|^p)^{1/p}$;

(DSM_2): *asymptotically stable in the q^{th} moment*, if it is stable in the q^{th} moment and if for any $\epsilon > 0$, $k_0 \in I(k_0)$, there exist $\delta_0(k_0)$ and $T = T(k_0, \epsilon)$ such that the inequality $\|y_0(\omega)\|_q \le \delta_0$ implies

$$\|y(k,\omega)\|_q < \epsilon, \qquad k \ge k_0 + T.$$

Remark 4.2. We note that depending on the mode of convergence in the probabilistic analysis, one can formulate other definitions of stability and boundedness. For the differential equations we refer to Ladde and Lakshmikantham [5].

For our further use we formulate a concept of relative stability.

Definition 4.3. The two systems (2.1) and (3.46) are said to be

(DRM_1): *relatively stable in q^{th} moment*, if for each $\epsilon > 0$, $k_0 \in I(k_0)$ and $q \ge 1$, there exists a positive function $\delta = \delta(k_0, \epsilon)$ such that the inequality $\|y_0(\omega) - w_0\|_q \le \delta$ implies

$$\|y(k,\omega) - w(k)\|_q < \epsilon, \qquad k \ge k_0.$$

(DRM_2): *relatively asymptotically stable in the q^{th} moment* if it is stable in the q^{th} moment and if for any $\epsilon > 0$, $k_0 \in I(k_0)$, there exist $\delta_0 = \delta_0(k_0)$ and $T = T(k_0, \epsilon)$ such that the inequality $\|y_0 - w_0\| \le \delta_0$ implies

$$\|y(k,\omega) - w(k)\|_q < \epsilon, \qquad k \ge k_0 + T.$$

Remark 4.4. Based on Definition 4.3, a definition relative to (2.1) and (2.2) or (2.3) can be formulated, analogously.

To study the stability analysis of (2.1) by an application of comparison method we require the stability of comparison difference system (3.1). The stability concepts in Definition 4.1 relative to (3.1) will be denoted by (DSM_1^*) and (DSM_2^*). In the present framework, we need a joint stability property of (3.1) and (2.2), or (3.1) and (2.3). We remark that this does not imply that each pair of the system (3.1) and (2.2) (or (3.1) and (2.3)) possesses the same kind of stability property. This property offers more flexibility in applications than the existing approaches [11].

Let $m(k,\omega) = m(k, k_0, y_0(\omega))$ and $u(k, k_0, u_0)$ be solutions of (2.3) and (3.1) through $(k_0, y_0(\omega))$ and (k_0, u_0), respectively. Then we define

$$\nu(k, k_0, y_0(\omega)) = u(k, k_0, m(k, k_0, y_0(\omega)), \omega) \tag{4.1}$$

and note that $\nu(k_0, k_0, y_0(\omega)) = V(k_0, y_0(\omega))$ and V is as defined in Theorem 3.1. We now formulate stability concepts relative to (2.2) and (3.1).

Definition 4.5. The trivial solution processes $m = 0$ and $u = 0$ of (2.2) and (3.1) are said to be

(DJM$_1$): *jointly stable in the mean,* if for $\epsilon > 0$, $k_0 \in I(k_0)$ there exists a $\delta = \delta(k_0, \epsilon) > 0$ such that

$$E\|y_0(\omega)\|_q \leq \delta \text{ implies } \sum_{i=1}^{N} E(\nu_i(k, k_0, y_0(\omega)) < \epsilon, \qquad k \geq k_0.$$

(DJM$_2$): *jointly asymptotically stable in the mean* if it is jointly stable in the mean and if for any $\epsilon > 0$, $k_0 \in I(k_0)$, there exist $\delta_0 = \delta_0(k_0) > 0$ and $T = T(k_0, \epsilon) > 0$ such that the inequality $E\|y_0(\omega)\|_q \leq \delta_0$ implies

$$\sum_{i=1}^{N} E[\nu_i(k, k_0, y_0(\omega))] < \epsilon, \qquad t \geq t_0 + T.$$

The joint relative stability of (3.1) and (2.2) ((3.1) and (2.3)) is defined as follows.

Definition 4.6. The systems (2.1), (3.46), (2.2) or (2.3) are said to be

(DJR$_1$): *jointly relatively sable in the mean* if for each $\epsilon > 0$, $k_0 \in I(k_0)$, there exists $\delta = \delta(k_0, \epsilon) > 0$ such that the inequality $E\|y_0(\omega) - w_0\| \leq \delta$ implies

$$\sum_{i=1}^{N} E[\nu_i(k, k_0, y_0(\omega)) - w_0)] < \epsilon, \qquad k \geq k_0.$$

We shall present some stability criteria that assures the stability in the q^{th} moment of the trivial solution processes of (2.1). Furthermore, some illustrations are given to show that the stability conditions are connected with the statistical properties of random rate functions of systems of difference equations. Examples are worked out to exhibit the advantage of the joint stability concepts.

Theorem 4.7. *Let the hypotheses of Theorem* 3.1 *be satisfied. Further assume that* $F(k, 0) \equiv 0$, $f(k, 0, \omega) \equiv 0$ *and* $G(k, 0, \omega) \equiv 0$ *w.p.* 1, *and for* $(k, x) \in I(k_0 + 1) \times R^n$

$$b(\|y\|)^q \leq \sum_{i=1}^{N} V_i(k, y, \omega) \leq a(k, \|y\|^q) \tag{4.2}$$

whenever $b \in \mathcal{VK}$, $a \in \mathcal{CK}$, *and* $q \geq 1$. *Then*

(DJM$_1$): *of* (3.1) *and* (2.3) *implies (DSM$_1$) of* (2.1)
and
(DJM$_2$): *of* (3.1) *and* (2.3) *implies (DSM$_2$) of* (2.1).

Proof. Let $\epsilon > 0$, $k_0 \in I(k_0)$ be given. Assume that (DJM$_1$) holds. Then for $b(\epsilon) > 0$ and $k_0 \in I(k_0)$, there exists a $\delta = \delta(\epsilon, k_0)$ such that $\|y_0(\omega)\|_p \leq \delta$ implies

$$\sum_{i=1}^{N} E(\nu_i(k, k_0, y_0(\omega), \omega) < b(\epsilon^q), \qquad k \geq k_0 \tag{4.3}$$

where

$$\nu(k, k_0, y_0(\omega), \omega) = r(k, k_0, V(k, k_0, y_0(\omega), \omega), \omega), \tag{4.4}$$

$r(k, k_0, u_0, \omega)$ is the solution process of (3.1) and $m(k, k_0, y_0(\omega))$ is the solution process of (2.2) through $(k_0, y_0(\omega))$. Now we claim that if $\|y_0(\omega)\|_q \leq \delta$ then $\|y(k,\omega)\|_q < \epsilon$, $k \geq k_0$. Suppose that this is false. Then there would exist a solution process $y(k, k_0, y_0(\omega), \omega)$ with $\|y_0(\omega)\|_q \leq \delta$ and a $k_1 > k_0$ such that

$$\|y(k_1,\omega)\|_q = \epsilon \quad \text{and} \quad \|y(k,\omega)\|_q \leq \epsilon, \quad k_0 \leq k \leq k_1. \tag{4.5}$$

On the other hand by Theorem 3.1 in the context of Remark 3.5, we have

$$V(k, y(k,\omega), \omega) \leq r(k, k_0, V(k_0, m(k,\omega), \omega), \omega), \qquad k \geq k_0. \tag{4.6}$$

From (4.2), (4.4), (4.6) and using convexity of b, we obtain

$$\begin{aligned} b(E\|y(k,\omega)\|^q) &\leq \sum_{i=1}^{N} EV_i(k, y(k,\omega), \omega) \\ &\leq \sum_{i=1}^{N} \nu_i(k, k_0, y_0(\omega), \omega), \end{aligned} \tag{4.7}$$

for $k \geq k_0$. Equations (4.3), (4.5) and (4.7) lead to the contradiction

$$\begin{aligned} b(\epsilon^q) &\leq \sum_{i=1}^{N} E(V_i(k_1, y(k_1,\omega), \omega)) \\ &\leq \sum_{i=1}^{N} \nu_i(k_1, k_0, y_0(\omega), \omega) < b(\epsilon^q). \end{aligned}$$

This proves (DSM_1). The proof of (DSM_2) can be proved analogously. This completes the proof of the theorem. □

The following example illustrates the scope and the usefulness of joint stability concept and Theorem 4.7.

Example 4.8. Consider Example 3.6. We further assume that $H(k, 0, \omega) \equiv 0$ w.p. 1. Furthermore, let λ in (3.24) satisfy

$$E\left[\exp\left[\sum_{p=k_0}^{k-1}(\lambda(p,\omega))\right]\right] \leq M \quad \text{and} \quad \left(\frac{\sum_{p=k_0}^{k-1}\lambda(p,\omega)}{k-k_0}\right) \to \alpha$$

as $(k - k_0) \to \infty$ for some positive numbers α, M and independent of $y_0(\omega)$ with $\|y_0(\omega)\|_2 < \infty$. It is clear that

$$\nu(k, k_0, y_0(\omega), \omega) = |m(k,\omega)|^2 \exp\left[\sum_{s=k_0}^{k} \lambda(s,\omega)\right].$$

This together with the assumptions about H, $y_0(\omega)$ and $\lambda(k,\omega)$, it follows that the trivial solution processes $m \equiv 0$ and $u \equiv 0$ of (3.24) and (3.26) are jointly stable in the mean.

Moreover, from the conditions on λ and $y_0(\omega)$, one can conclude that $m \equiv 0$ and $u \equiv 0$ of (3.24) and (3.26) are jointly asymptotically stable in the mean. From this and an application of Theorem 3.1, one can conclude that the trivial solution process of (3.22) is asymptotically stable in the second moment. We remark that for the trivial solution of (3.24) is unbounded.

Example 4.9. Consider Example 3.7, where we replace (3.32) by

$$LV(p, x(k, p, y)) \leq (\bar{\alpha}(p) + \eta(p, \omega))V(p, x(k, p, y)) \tag{4.8}$$

where $\lambda(p, \omega) = \bar{\alpha}(p) + \eta(p, \omega)$ and $\eta(p, \omega)$ is a stationary Gaussian process with mean $E(\eta(p, \omega)) = 0$ and the covariance function $C(k - p) = E(\eta(k, \omega)\eta(p, \omega))$. Let $H(k, 0, \omega) \equiv 0$ w.p. 1. By following the discussion in Example 3.7, we have

$$\Delta u(p, \omega) = (\bar{\alpha}(p) + \eta(p, \omega))u(p, \omega), u(k_0, \omega) = u_0(\omega), \tag{4.9}$$

and

$$V(k, y(k, \omega)) \leq V(k_0, m(k, \omega)) \exp\left[\sum_{s=k_0}^{k} (\bar{\alpha}(s) + \eta(s, \omega))\right] \tag{4.10}$$

Let $y_0(\omega)$ and $\eta(s, \omega)$ be independent processes. By taking expectation on both sides of (4.10), we obtain

$$E|y(k, \omega)|^2 \leq E|m(k, \omega)|^2 E\left(\exp\sum_{s=k_0}^{k} (\eta(s, \omega) + \bar{\alpha}(s))\right). \tag{4.11}$$

From the properties of Gaussian process, we have

$$E\left[\exp\sum_{s=k_0}^{k} (\eta(s, \omega) + \bar{\alpha}(s))\right] = \exp\sum_{s=k_0}^{k}\left[(\bar{\alpha}(s)) + \frac{1}{2}\sum_{u=k_0}^{k}\sum_{s=k_0}^{k} C(u - s)\right].$$

The trivial solution of (3.31) and (4.9) are jointly stable in mean if $\|y_0\|_2 < \infty$, and

$$\frac{1}{(k - k_0)}\left[\sum_{s=k_0}^{k} \bar{\alpha}(s) + \frac{1}{2}\sum_{u=k_0}^{k}\sum_{s=k_0}^{k} C(u - s)\right] < -1 \qquad \text{for } k \geq k_0. \tag{4.12}$$

From this and (4.10) we conclude that the trivial solution of (3.29) is stable in mean square. Moreover, it is asymptotically stable in the 2^{nd} moment.

Remark 4.10. Example 4.9 shows that the trivial solution process $m \equiv 0$ of (3.31) is unstable (exponentially with base 2) in the mean square sense. However joint stability is guaranteed by Theorem 4.7 provides a greater advantage to study the stability problems.

5. ERROR ESTIMATES AND RELATIVE STABILITY

By using the comparison results, we present results concerning error estimates and relative stability.

Theorem 5.1. *Let the hypotheses of Theorem* 3.1 *be satisfied. Further assume that*

$$b(\|y\|^q) \leq \sum_{i=1}^{m} v_i(k, y, w) \tag{5.1}$$

where $b \in k$, $q \geq 1$. *Then*

$$b(\|m(k, p, y(p) - w(p))\|^q) \leq \sum_{i=1}^{N} E[r_i(p, \omega)], \quad k_0 \leq p \leq k \tag{5.2}$$

and

$$b(E[\|y(k, \omega) - w(k)\|^q) \leq \sum_{i=1}^{N} E[r_i(k, \omega)], \quad k \leq p \leq k_0. \tag{5.3}$$

Proof. By the choice of $u_0 = v(k_0, m(k, k_0, y_0 - w_0))$, (3.47) reduces to

$$\sum_{i=1}^{N} v_i(p, m(k, p, y(p) - w(p))) \leq \sum_{i=1}^{N} r_i(p, w).$$

This together with (5.1) and the convexity of b yields

$$b(E[\|m(k, p, y(p) - w(p))\|^q \leq \sum_{i=1}^{N} E[r_i(p, w)]. \tag{5.4}$$

Moreover, for $p = k$, (5.4) reduces to (5.3). This completes the proof of the theorem. □

Remark 5.2. By Using Theorem 5.1, one can easily find error estimate results with regard to solution process of (3.59). The details are left to the reader. Moreover, from Corollary 3.11 and Theorem 5.1 one can obtain the corresponding error estimate results.

In the following, we present relative stability results by use of comparison method.

Theorem 5.3. *Later the hypotheses of Theorem* 5.1 *be satisfied except* (5.1) *is replaced by*

$$b(\|y\|^q) \leq \sum_{i=1}^{N} v_i(k, y, w) \leq a(k, \|y\|^q) \tag{5.5}$$

where b *and* q *are as defined in* (5.1) *and ???.*

(DJM$_1$): *of* (3.1) *and* (2.3) *implies* (DRM$_1$) *of* (2.1)
(DJM$_2$): *of* (3.1) *and* (2.3) *implies* (DRM$_2$) *of* (2.1)

Proof. The proof of the theorem can be formulated by following the argument used in the proofs of Theorems 4.7 and 5.1. The details are left to the reader. □

REFERENCES

[1] J. F. Chamayou and J. L. Dunau, Random difference equations: an asymptotical result, *J. Comput. Appl. Math.* **154** (2003), no. 1, 183–193.

[2] J. F. Chamayou and J. L. Dunau, Random difference equations with logarithmic distribution and the triggered shot noise, *Adv. in Appl. Math.* **29** (2002) no. 3, 454–470.

[3] Francesco S. De Blasi and Józef Myjak, Some remarks on random difference equations, *Opuscula Math.* **6** (1990) 29–39.

[4] Tsukasa Fujiwara, Limit theorems for random difference equations driven by mixing processes, *J. Math. Kyoto Univ.* **32** (1992) no. 4, 763–795.

[5] G. S. Ladde and V. Lakshmikantham, Random differential inequalities, *Mathematics in Science and Engineering* **150** Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.

[6] G. S. Ladde and M. Sambandham, Stochastic versus deterministic, *Math. Comput. Simulation* **24** (1982) no. 6, 507–514.

[7] G. S. Ladde and M. Sambandham, Random difference inequalities, *Trends in the Theory and Practice of Nonlinear Analysis*, 231–240, North-Holland, Amsterdam, 1985.

[8] G. S. Ladde and M. Sambandham, Numerical solutions to stochastic difference equations, *Nonlinear Analysis and Applications* 279–287, Lecture Notes in Pure and Appl. Math., 109, Dekker, New York, 1987.

[9] G. S. Ladde and M. Sambandham, Variation of constants formula and error estimate to stochastic difference equations, *J. Math. Phys. Sci.* **22** (1988) no. 5, 557–584.

[10] G. S. Ladde and M. Sambandham, *Stochastic Versus Deterministic Systems of Differential Equations*, Marcel Dekker, Inc., New York, 2004.

[11] V. Lakshmikantham and Donato Trigiante, *Theory of Difference Equations: Numerical Methods and Applications*, second edition, Marcel Dekker, Inc.m, New York, 2002.

APPENDIX I

A.1. Fundamental Results in System of Iterative Process

Let us consider the nonlinear system of difference equations with random parameters

$$x_k = h(k-1, x_{k-1}, A(\omega)), \quad x_{k_0} = x_0, \quad k \geq k_0 + 1 \tag{A.1.1}$$

where $x_0 \in \mathbb{R}^n$ and $A \in \mathbb{R}^m$ are random vectors defined on a complete probability space, $(\Omega, \mathcal{F}, P)$. It is also assumed that $h(k, x, a)$ is continuously differentiable with respect to $(x, a)^T$. We denote by $x_k = x(k, k_0, x_0, A)$ a solution process of (A.1.1). The system can be rewritten as follows:

$$z_k = H(k-1, z_{k-1}), \quad z_{k_0} = z_0, k \geq k_0 + 1 \tag{A.1.2}$$

where $z_k \in \mathbb{R}^{n+m}$, $z_k = [x_k^T, A^T]^T$, $H = [h^T, A^T]^T$, and $z_0 = [x_0^T, A^T]^T$. We establish the differentiability of the solution process, z_k of (A.1.2) with respect to the initial data z_0. This theorem stochasticizes (Ladde/Lak book 1980) and also provides a rigorous and systematic proof.

Theorem A.1 *Let h be a continuously differentiable function with respect to $z = (x^T, A^T)$ satisfying* (A.1.2). *Then for $1 \leq i \leq m + n$, $\frac{\partial z}{\partial z_{0_i}}$ exists and, moreover $\frac{\partial z}{\partial z_{0_i}}$ is a solution of the following linear difference equation*

$$y_k = H_z(k-1, z_{k-1})y_{k-1}, \quad y_{k_0} = y_0 = e_i, \tag{A.1.3}$$

where $z_k = z(k, k_0, z_0)$

$$H_z = \frac{\partial H}{\partial z}, \qquad e_i \in \mathbb{R}^{n+m},$$

$z_k = z(k, k_0, z_0)$ is the solution process of (A.1.2), *and e_i is the vector with i-th component 1 and 0 elsewhere.*