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GENERALIZED MONOTONE ITERATIVE TECHNIQUE FOR IMPULSIVE DIFFERENTIAL SYSTEMS

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ABSTRACT. Systems of differential equations with impulses can occur in the mathematical modeling of science and engineering. Using upper and lower solutions, we will develop the generalized monotone iterative method for impulsive differential systems where the forcing functions are sums of nondecreasing and nonincreasing functions.

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1. INTRODUCTION

It is well known that the method of upper and lower solutions coupled with the monotone iterative technique can be used to find the solutions to countless nonlinear ordinary differential equations [3], [4], [5], [6] as well as partial differential equations, see [2]. Most nonlinear differential equations cannot be solved analytically. The monotone iterative method offers theoretical as well as constructive existence results in a closed sector that is generated by upper and lower solutions of the nonlinear problem. In this paper, we will consider systems of nonlinear differential equations with impulse effects which we refer to as an impulsive differential system. Impulsive differential systems can occur in many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics and ecology, pharmacokinetics and frequency modulated systems, see [1]. These impulses can be characterized as sudden bursts, spikes, gains or losses, shocks, harvesting, natural disasters, etc. These impulses are sometimes referred to as perturbations or disturbances and are often instantaneous. We will develop a method to approximate the solution to nonlinear impulsive differential systems generated by upper and lower solutions. In our study, the forcing functions are the sums of nondecreasing and nonincreasing functions. The advantage of using the monotone method on systems of differential

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equations with impulse effects is that we can reduce the given nonlinear system into a much simpler one involving scalar equations, and we are guaranteed a solution.

2. PRELIMINARY RESULTS

The solution to an impulsive differential system of equations is a set of sufficiently differentiable functions which simultaneously satisfy all the equations in the system on some common interval, while concurrently satisfying the jumps generated by the impulses. To utilize this method to solve such a system, we would first construct a set of monotone sequences which approximate the set of solutions to the given nonlinear system. The elements of the monotone sequences are solutions which simultaneously satisfy all equations of the corresponding linear system or a simpler system. Secondly, we must show that the sequences converge uniformly and monotonically to the solutions of the original system.

From [1], a system with impulses can be described in the following process. Assume that

(i) a system of differential equations

$$x' = \mathbf{F}(t, x) \tag{2.1}$$

where $\mathbf{F} : \Omega \times R^+ \to R^n, \Omega \subset R^n$, R^n is the n-dimensional Euclidean space and R^+ is the nonnegative real line;

(ii) the sets $M(t), N(t) \subset \Omega$ for each $t \in R^+$;

(iii) the operator $A(t) : M(t) \to N(t)$ for each $t \in R^+$;

Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.1) starting at (t_0, x_0). The process begins with the point $P_t = (t, x(t))$ which begins its motion from the initial point $P_{1_0} = (t_0, x_0)$ and moves along the curve $\{(t, x) : t \geq t_0, x = x(t)\}$ until $t_1 > t_0$ at that time the point P_t meets the set $M(t)$. At $t = t_1$, the operator $A(t)$ transfers the point $P_{t_1} = (t_1, x(t_1))$ into $P_{t_1^+} = (t_1, x_1^+) \in N(t_1)$, where $x_1^+ = A(t_1)x(t_1)$. Then the point P_t continues to move along the curve with $x(t) = x(t, t_1, x_1^+)$ as the solution of (2.1) starting at $P_{t_1} = (t_1, x_1^+)$ until it hits the set $M(t)$ at the moment $t_2 > t_1$. Then, once again the point $P_{t_2} = (t_2, x(t_2))$ is transferred to the point $P_{t_2^+} = (t_2, x_2^+) \in N(t_2)$ where $x_2^+ = A(t_2)x(t_2)$. As before, the point P_t continues along the curve with $x(t) = (x, t_2, x_2^+)$ as the solution of (2.1) starting at $P_{t_2} = (t_2, x_2^+)$. The evolution process continues as long as the solution of (2.1) exists. The curves being described by P_t are integral curves, and the functions that define the integral curves are solutions of the system. An example of the integral curves described in this process is illustrated in Figure 1 below.

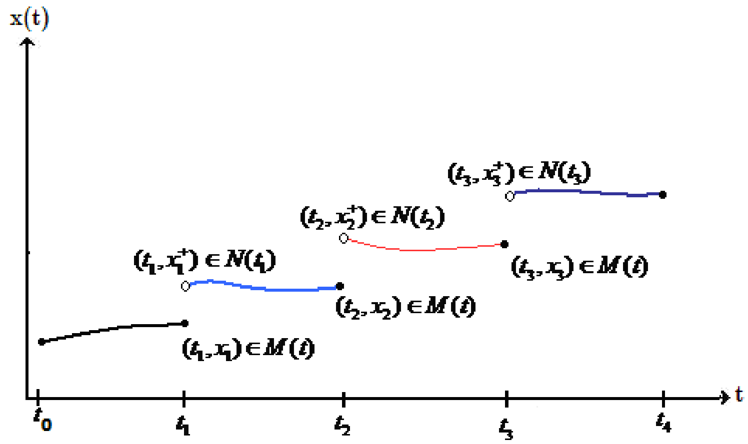

FIGURE 1. Integral Curves

Definition 2.1. A system of differential equations of the form

$$\begin{bmatrix} \mathbf{u}' = \mathbf{F}(t,u), t \neq t_k \\ \mathbf{u}(t_0) = \mathbf{u_0}, \mathbf{t_0} \geq 0 \\ \mathbf{u}(t_k^+) = \mathbf{u}(t_k) + \mathbf{I}_k(\mathbf{u}(t_k)) \end{bmatrix}$$

where $\mathbf{F} \in C[J \times R^N, R^N]$, $J = [t_0, T]$, $\mathbf{u}(t_k^+) = \lim_{h \to 0^+} \mathbf{u}(t_k + h)$ with $k = 1, 2, \ldots, m$, and where $\mathbf{I} \in R^N \to R^N$ are operators is called an impulsive differential system.

We can rewrite this system component-wise as follows:

$$\begin{bmatrix} u_i'(t) = F_i(t, u_1(t), u_2(t), \ldots, u_N(t)), t \neq t_k \\ u_i(t_0) = u_{0i}, i = 1, 2, \ldots, N \\ u_i(t_k^+) = u_i(t_k) + I_{ki}(u_1(t_k), u_2(t_k), \ldots, u_N(t_k)), k = 1, 2, \ldots, m \end{bmatrix}$$

In this paper, we will consider a system where the forcing functions are the sums of nonlinear, nondecreasing and nonincreasing functions in all components of u defined on a closed set $J = [t_0, T]$. For this purpose, we will consider the following system of nonlinear differential equations with impulse conditions:

$$\begin{bmatrix} \mathbf{u}' = \mathbf{f}(t,u) + \mathbf{g}(t,u), t \neq t_k \\ \mathbf{u}(t_0) = \mathbf{u_0}, \mathbf{t_0} \geq 0 \\ \mathbf{u}(t_k^+) = \mathbf{u}(t_k) + \mathbf{I}_k(\mathbf{u}(t_k)) + \mathbf{L}_k(\mathbf{u}(t_k)) \end{bmatrix}$$

where $\mathbf{f} : J \times R^N \to R^N$ are nonlinear, nondecreasing in all components of u and the $\mathbf{g} : J \times R^N \to R^N$ are nonlinear, nonincreasing in all components of u and $J = [t_0, T] \subset R$ such that $0 \leq t_0 < t_1 < t_2 < \cdots < t_m \leq T$. Also, $\mathbf{I} : R^N \to R^N$ are nonlinear, nondecreasing operators in all components of u and $\mathbf{L} : R^N \to R^N$ are nonlinear, nonincreasing operators in all components of u.

We can rewrite this system component-wise as follows:

$$\begin{cases} u_i'(t) = f_i(t, u_1(t), u_2(t), \ldots, u_N(t)) + g_i(t, u_1(t), u_2(t), \ldots, u_N(t)), \\ t \neq t_k, i = 1, 2, \ldots, N \\ u_i(t_0) = u_{0i} \\ u_i(t_k^+) = u_i(t_k) + I_{ki}(u_1(t_k), \ldots, u_N(t_k)) + L_{ki}(u_1(t_k), \ldots, u_N(t_k)), \\ k = 1, \ldots, m, t_k \in J \end{cases} \tag{2.2}$$

This leads to the possibility of having coupled lower and upper piecewise defined solutions described in the following definition.

Definition 2.2. The functions $\alpha_{0i}(t), \beta_{0i}(t) \in C[J \times R^N, R^N]$ with $I, L \in C[R^N, R^N]$ are said to be defined as:

Type 1

Coupled lower piecewise defined solutions of (2.2) if

$$\alpha_{0i}'(t) \leq f_i(t, \alpha_{01}(t), \alpha_{02}(t), \ldots, \alpha_{0N}(t)) + g_i(t, \beta_{01}(t), \beta_{02}(t), \ldots, \beta_{0N}(t)), t \neq t_k;$$
$$\alpha_{0i}(t_0) \leq u_{0i} \text{ on } J;$$
$$\alpha_{0i}(t_k^+) \leq \alpha_{0i}(t_k) + I_{ki}(t_k, \alpha_{01}(t_k), \ldots, \alpha_{0N}(t_k)) + L_{ki}(t_k, \beta_{01}(t_k), \ldots, \beta_{0N}(t_k)).$$

Coupled upper piecewise defined solutions of (2.2) if

$$\beta_{0i}'(t) \geq f_i(t, \beta_{01}(t), \beta_{02}(t), \ldots, \beta_{0N}(t)) + g_i(t, \alpha_{01}(t), \alpha_{02}(t), \ldots, \alpha_{0N}(t)), t \neq t_k;$$
$$\beta_{0i}(t_0) \geq u_{0i} \text{ on } J;$$
$$\beta_{0i}(t_k^+) \geq \beta_{0i}(t_k) + I_{ki}(t_k, \beta_{01}(t_k), \ldots, \beta_{0N}(t_k)) + L_{ki}(t_k, \alpha_{01}(t_k), \ldots, \alpha_{0N}(t_k)).$$

Type II

Coupled lower piecewise defined solutions of (2.2) if

$$\alpha_{0i}'(t) \leq f_i(t, \beta_{01}(t), \beta_{02}(t), \ldots, \beta_{0N}(t)) + g_i(t, \alpha_{01}(t), \alpha_{02}(t), \ldots, \alpha_{0N}(t)), t \neq t_k;$$
$$\alpha_{0i}(t_0) \leq u_{0i} \text{ on } J;$$
$$\alpha_{0i}(t_k^+) \leq \alpha_{0i}(t_k) + I_{ki}(t_k, \alpha_{01}(t_k), \ldots, \alpha_{0N}(t_k)) + L_{ki}(t_k, \beta_{01}(t_k), \ldots, \beta_{0N}(t_k)).$$

Coupled upper piecewise defined solutions of (2.2) if

$$\beta_{0i}'(t) \geq f_i(t, \alpha_{01}(t), \alpha_{02}(t), \ldots, \alpha_{0N}(t)) + g_i(t, \beta_{01}(t), \beta_{02}(t), \ldots, \beta_{0N}(t)), t \neq t_k;$$
$$\beta_{0i}(t_0) \geq u_{0i} \text{ on } J;$$
$$\beta_{0i}(t_k^+) \geq \beta_{0i}(t_k) + I_{ki}(t_k, \beta_{01}(t_k), \ldots, \beta_{0N}(t_k)) + L_{ki}(t_k, \alpha_{01}(t_k), \ldots, \alpha_{0N}(t_k)).$$

To correspond to the lower and upper solutions described in Definition 2.2, we can develop sequences using the following iterative schemes with $n = 0, 1, \ldots, N-1$:

Type (i)

$\alpha_{n+1,i}'(t) = f_i(t, \alpha_{n,1}(t), \ldots, \alpha_{n,N}(t)) + g_i(t, \beta_{n,1}(t), \ldots, \beta_{n,N}(t)), t \neq t_k;$
$\alpha_{n+1,i}(t_0) = u_{0i}$ on J;
$\alpha_{n+1,i}(t_k^+) = \alpha_{n,i}(t_k) + I_{ki}(t_k, \alpha_{n,1}(t_k), \ldots, \alpha_{n,N}(t_k)) + L_{ki}(t_k, \beta_{n,1}(t_k), \ldots, \beta_{n,N}(t_k)).$
and
$\beta_{n+1,i}'(t) = f_i(t, \beta_{n,1}(t), \ldots, \beta_{n,N}(t)) + g_i(t, \alpha_{n,1}(t), \ldots, \alpha_{n,N}(t)), t \neq t_k;$

$\beta_{n+1,i}(t_0) = u_{0i}$ on J;
$\beta_{n+1,i}(t_k^+) = \beta_{n,i}(t_k) + I_{ki}(t_k, \beta_{n,1}(t_k), \ldots, \beta_{n,N}(t_k)) + L_{ki}(t_k, \alpha_{n,1}(t_k), \ldots, \alpha_{n,N}(t_k))$.

Type (ii)

$\alpha'_{n+1,i}(t) = f_i(t, \beta_{n,1}(t), \ldots, \beta_{n,N}(t)) + g_i(t, \alpha_{n,1}(t), \ldots, \alpha_{n,N}(t)), t \neq t_k$;
$\alpha_{n+1,i}(t_0) = u_{0i}$ on J;
$\alpha_{n+1,i}(t_k^+) = \alpha_{n,i}(t_k) + I_{ki}(t_k, \alpha_{n,1}(t_k), \ldots, \alpha_{n,N}(t_k)) + L_{ki}(t_k, \beta_{n,1}(t_k), \ldots, \beta_{n,N}(t_k))$.
and
$\beta'_{n+1,i}(t) = f_i(t, \alpha_{n,1}(t), \ldots, \alpha_{n,N}(t)) + g_i(t, \beta_{n,1}(t), \ldots, \beta_{n,N}(t)), t \neq t_k$;
$\beta_{n+1,i}(t_0) = u_{0i}$ on J;
$\beta_{n+1,i}(t_k^+) = \beta_{n,i}(t_k) + I_{ki}(t_k, \beta_{n,1}(t_k), \ldots, \beta_{n,N}(t_k)) + L_{ki}(t_k, \alpha_{n,1}(t_k), \ldots, \alpha_{n,N}(t_k))$.

Notice that the iterations are solutions of simple linear impulsive differential equations and component-wise each iterate is a scalar equation. This is the main advantage of the monotone method.

Before we give our main theorem, we will recall some known results from [1]. Let PC denote the class of piecewise left continuous functions from R^+ to R with discontinuities at $t = t_k$, $k = 1, 2, \ldots, m$. We will state the scalar comparison theorem which will be needed in our main results. We will not prove the comparison theorem here, but it can be proved by method of induction. For details see [1].

Theorem 2.3. (A_1) *the sequence* $\{t_k\}$ *satisfies* $0 \leq t_0 < t_1 < t_2 < \cdots \leq t_m = T$.
(A_2) $p \in PC[J, R]$ *and* $p(t)$ *is left-continuous at* t_k, $k = 1, 2, \ldots, m$;
(A_3) *for* $k = 1, 2, \ldots, m$ *and* $t \geq t_0$,

$$p'(t) \leq q(t)p(t) + v(t), t \neq t_k$$

and

$$p(t_k^+) \leq d_k p(t_k) + b_k$$

where $q, v \in C[R^+, R]$ *are continuous functions and* $a_k \geq 0$, b_k *are constants.*

Then

$$p(t) \leq p(t_0) \prod_{t_0 < t_k < t} a_k e^{\int_{t_0}^t q(s)ds} + \sum_{t_0 < t_k < t} \left(\prod_{t_k < t_j < t} a_j e^{\int_{t_k}^t q(s)ds} \right) b_k + \int_{t_0}^t \left(\prod_{s < t_k < t} a_k e^{\int_s^t q(\sigma)d\sigma} v(s) \right) ds, t \geq t_0$$

3. MAIN RESULTS

In this section, we will consider the system of nonlinear differential equations with impulse conditions (2.2). We will develop the generalized monotone method for the pair of upper and lower solutions of Type I given in Definition 2.2. In our theorem, we will start with the lower and upper solutions described in Definition 2.2, and use Type (i) iterative scheme to develop the sequence. This will result in natural monotone sequences. In the proof of our theorem, we will consider a special case of Theorem 2.3, that is, we will let $q(t) = 0$, $v(t) = 0$ and $a_k = 0$.

Theorem 3.1. (A_1) *$\alpha_{0i}(t), \beta_{0i}(t)$ are lower and upper PC solutions of (2.2) with $\alpha_{0i}(t) \le \beta_{0i}(t), t \in (t_k, t_{k+1}] \in J, t \neq t_k$;*

(A_2) *$f_i : J \times R^N \to R^N$ are nondecreasing in every component of u and $g_i : J \times R^N \to R^N$ are nonincreasing in every component of u.*

(A_3) *For $t_k \in J, I_{ki} : R^N \to R^N$ are nondecreasing in every component of u and $L_{ki} : R^N \to R^N$ are nonincreasing in every component of u*

Then there exists natural monotone sequences $\{\alpha_{ni}\}$ and $\{\beta_{ni}\}$ on $(t_k, t_{k+1}] \in J$ such that $\alpha_{ni}(t) \to \rho_i(t)$ and $\beta_{ni}(t) \to r_i(t)$ uniformly on each interval $(t_k, t_{k+1}] \in J$ where (ρ_i, r_i) are coupled minimal and maximal solutions respectively of (2.2). That is (ρ_i, r_i) satisfy the following sequences:

$$\left[\begin{array}{l} \rho_i'(t) = f_i(t, \rho_1(t), \ldots, \rho_N(t)) + g_i(t, r_1(t), \ldots, r_N(t)), \quad t \neq t_k; \\ \rho_i(t_0) = u_{0i}; \\ \rho_i(t_k^+) = \rho_i(t_k) + I_i(t_k, \rho_1(t_k), \ldots, \rho_N(t_k)) + L_i(t_k, r_1(t_k), \ldots, r_N(t_k)) \end{array}\right. \tag{3.1}$$

$$\left[\begin{array}{l} r_i'(t) = f_i(t, r_1(t), \ldots, r_N(t)) + g_i(t, \rho_1(t), \ldots, \rho_N(t)), \quad t \neq t_k; \\ r_i(t_0) = u_{0i}; \\ r_i(t_k^+) = r_i(t_k) + I_i(t_k, r_1(t_k), \ldots, r_N(t_k)) + L_i(t_k, \rho_1(t_k), \ldots, \rho_N(t_k)) \end{array}\right. \tag{3.2}$$

The iterations (3.1) *and* (3.2) *can be determined using the following Type (i) iterative schemes:*

$$\left[\begin{array}{l} \alpha_{n+1,i}'(t) = f_i(t, \alpha_{n,1}, \ldots, \alpha_{n,N}) + g_i(t, \beta_{n,1}, \ldots, \beta_{n,N}), \quad t \neq t_k \\ \alpha_{n+1,i}(t_0) = u_{0i} \text{ on } J \\ \alpha_{n+1,i}(t_k^+) = \alpha_{n,i}(t_k) + I_{ki}(t_k, \alpha_{n,1}(t_k), \ldots, \alpha_{n,N}(t_k)) \\ \qquad\qquad + L_{ki}(t_k, \beta_{n,1}(t_k), \ldots, \beta_{n,N}(t_k)) \end{array}\right. \tag{3.3}$$

$$\left[\begin{array}{l} \beta_{n+1,i}'(t) = f_i(t, \beta_{n,1}, \ldots, \beta_{n,N}) + g_i(t, \alpha_{n,1}, \ldots, \alpha_{n,N}), \quad t \neq t_k \\ \beta_{n+1,i}(t_0) = u_{0i} \text{ on } J; \\ \beta_{n+1,i}(t_k^+) = \beta_{n,i}(t_k) + I_{ki}(t_k, \beta_{n,1}(t_k), \ldots, \beta_{n,N}(t_k)) \\ \qquad\qquad + L_{ki}(t_k, \alpha_{n,1}(t_k), \ldots, \alpha_{n,N}(t_k)). \end{array}\right. \tag{3.4}$$

Proof. The solutions for (3.3) and (3.4) exist and are unique for $i = 1, 2, \ldots, N$, $n = 0, 1, \ldots N - 1$, and $k = 1, 2, \ldots, m$. We will prove that for $\alpha_{ki} \le \beta_{ki}$,

$$\alpha_{ki}(t), \beta_{ki}(t) \in [\alpha_{0i}(t), \beta_{0i}(t)],$$

$$t \in (t_k, t_{k+1}] \in J,$$

and

$$\alpha_{ki}(t_k^+), \beta_{ki}(t_k^+) \in [\alpha_{0i}(t_k^+), \beta_{0i}(t_k^+)]$$

where

$$[\alpha_{0i}(t), \beta_{0i}(t)] = \{u_i \in C[J \times R^N, R^N] : \alpha_{0i}(t, \alpha_1(t), \dots, \alpha_N(t)) \\ \leq u_i(t) \leq \beta_{0i}(t, \beta_1(t), \dots, \beta_N(t))\}.$$

Furthermore, the impulses are

$$[\alpha_{0i}(t_k^+), \beta_{0i}(t_k^+)] = \{u_i \in C[J \times R^N, R^N] : \alpha_{0i}(t_k^+, \alpha_1(t_k^+), \dots, \alpha_N(t_k^+)) \\ \leq u_i(t_k^+) \leq \beta_{0i}(t_k^+, \beta_1(t_k^+), \dots, \beta_N(t_k^+))\}.$$

Our aim is to show that

$$\alpha_{0i} \leq \alpha_{1i} \leq \alpha_{2i} \leq \cdots \leq \alpha_{ji} \leq u_i \leq \beta_{ji} \leq \cdots \leq \beta_{2i} \leq \beta_{1i} \leq \beta_{0i} \tag{3.5}$$

holds component-wise for all $t \in J$.

We claim that $\alpha_{0i} \leq \alpha_{1i}$ and $\beta_{0i} \geq \beta_{1i}$. For this purpose, let $p_i(t) = \alpha_{0i}(t) - \alpha_{1i}(t)$. We know from Definition 2.2 that Type I lower solutions are

$$\alpha'_{0i}(t) \leq f_i(t, \alpha_{01}(t), \alpha_{02}(t), \dots, \alpha_{0N}(t)) + g_i(t, \beta_{01}(t), \beta_{02}(t), \dots, \beta_{0N}(t)), \quad t \neq t_k.$$

We also have for $n = 0$, Type (i) iterative scheme yields

$$\alpha'_{0+1,i}(t) = f_i(t, \alpha_{01}(t), \dots, \alpha_{0N}(t)) + g_i(t, \beta_{01}(t), \dots, \beta_{0N}(t)), \quad t \neq t_k;$$

Thus we have

$$\begin{aligned} p'_i(t) &= \alpha'_{0i}(t) - \alpha'_{1i}(t) \\ &\leq f_i(t, \alpha_{01}(t), \alpha_{02}(t), \dots, \alpha_{0N}(t)) + g_i(t, \beta_{01}(t), \beta_{02}(t), \dots, \beta_{0N}(t)) \\ &\quad - f_i(t, \alpha_{01}(t), \alpha_{02}(t), \dots, \alpha_{0N}(t)) - g_i(t, \beta_{01}(t), \beta_{02}(t), \dots, \beta_{0N}(t)) = 0 \end{aligned}$$

This implies that $p'_i(t) \leq 0$ for $t \in (t_k, t_{k+1})$. Also, we have

$$p_i(t_0) = \alpha_{0i}(t_0) - \alpha_{1i}(t_0) \leq u_{0i} - u_{0i} = 0$$

and

$$\begin{aligned} p_i(t_k^+) = \alpha_{0i}(t_k^+) - \alpha_{1i}(t_k^+) &\leq \alpha_{0i}(t_k) + I_{ki}(t, \alpha_{01}(t_k), \alpha_{02}(t_k), \dots, \alpha_{0N}(t)) \\ &\quad + L_{ki}(t, \beta_{01}(t_k), \beta_{02}(t_k), \dots, \beta_{0N}(t_k)) - \alpha_{0i}(t_k) \\ &\quad - I_{ki}(t, \alpha_{01}(t_k), \alpha_{02}(t_k), \dots, \alpha_{0N}(t)) \\ &\quad - L_{ki}(t, \beta_{01}(t_k), \beta_{02}(t_k), \dots, \beta_{0N}(t_k)) = 0 \end{aligned}$$

It follows that $p_i(t) = \alpha_{0i}(t) - \alpha_{1i}(t) \leq 0$ on J. This proves that $\alpha_{0i} \leq \alpha_{1i}$ holds component-wise for all $t \in (t_k, t_{k+1}] \in J$. Similarly, we can show $\beta_{0i} \geq \beta_{1i}$.

Our next step is to show $\alpha_{1i} \leq \beta_{1i}$. For this purpose, we set $p_i(t) = \alpha_{1i}(t) - \beta_{1i}(t)$. We must use the assumptions (A_1) and (A_2). That is, we will use the assumptions

$\alpha_{0i} \le \beta_{0i}$ and the monotone nature of f_i and g_i. We get

$$\begin{aligned} p_i'(t) = \alpha_{1i}'(t) - \beta_{1i}'(t) = {} & f_i(t, \alpha_{01}(t), \alpha_{02}(t), \ldots, \alpha_{0N}(t)) \\ & + g_i(t, \beta_{01}(t), \beta_{02}(t), \ldots, \beta_{0N}(t)) - f_i(t, \beta_{01}(t), \beta_{02}(t), \ldots, \beta_{0N}(t)) \\ & - g_i(t, \alpha_{01}(t), \alpha_{02}(t), \ldots, \alpha_{0N}(t)) \le 0 \end{aligned}$$

Also, we have

$$p_i(t_0) = \alpha_{1i}(t_0) - \beta_{1i}(t_0) \le u_{0i} - u_{0i} = 0$$

and by the assumptions (A_1) and (A_3); that is, by $\alpha_{0i} \le \beta_{0i}$ and the monotone nature of I_{ki} and J_{ki}, we can conclude that

$$\begin{aligned} p_i(t_k^+) = \alpha_{1i}(t_k^+) - \beta_{1i}(t_k^+) \le {} & \alpha_{0i}(t_k) + I_{ki}(t, \alpha_{01}(t_k), \alpha_{02}(t_k), \ldots, \alpha_{0N}(t)) \\ & + L_{ki}(t, \beta_{01}(t_k), \beta_{02}(t_k), \ldots, \beta_{0N}(t_k)) - \beta_{0i}(t_k) \\ & - I_{ki}(t, \beta_{01}(t_k), \beta_{02}(t_k), \ldots, \beta_{0N}(t)) \\ & - L_{ki}(t, \alpha_{01}(t_k), \alpha_{02}(t_k), \ldots, \alpha_{0N}(t_k)) \le 0 \end{aligned}$$

We have shown that $\alpha_{1i} \le \beta_{1i}$ holds component-wise for all $t \in (t_k, t_{k+1}] \in J$. Thus we have shown that $\alpha_{0i} \le \alpha_{1i} \le \beta_{1i} \le \beta_{0i}$. Hence (3.5) is holds for $k = 1$.

Now assume that (3.5) holds component-wise for some $j > 1$, such that

$$\alpha_{j-1,i} \le \alpha_{j,i} \le \beta_{j,i} \le \beta_{j-1,i}. \tag{3.6}$$

Using induction, we need to show that (3.5) holds for $j + 1$. So we must prove that

$$\alpha_{j,i} \le \alpha_{j+1,i} \le \beta_{j+1,i} \le \beta_{j,i}$$

holds component-wise on J. For this purpose, let $p_i(t) = \alpha_{j,i}(t) - \alpha_{j+1,i}(t)$ and note that $p_i(t_0) = \alpha_{j,i}(t_0) - \alpha_{j+1,i}(t_0) = u_{0i} - u_{0i} = 0$. Also, by (3.6), we have $\alpha_{j-1,i} \le \alpha_{j,i}$ and $\beta_{j,i} \le \beta_{j-1,i}$. Thus by the monotone nature of f_i and g_i, we get

$$\begin{aligned} p_i'(t) = \alpha_{j,i}'(t) - \alpha_{j+1,i}'(t) = {} & f_i(t, \alpha_{j-1,1}(t), \alpha_{j-1,2}(t), \ldots, \alpha_{j-1,N}(t)) \\ & + g_i(t, \beta_{j-1,1}(t), \beta_{j-1,2}(t), \ldots, \beta_{j-1,N}(t)) \\ & - f_i(t, \alpha_{j,1}(t), \alpha_{j,2}(t), \ldots, \alpha_{j,N}(t)) \\ & - g_i(t, \beta_{j,1}(t), \beta_{j,2}(t), \ldots, \beta_{j,N}(t)) \le 0 \end{aligned}$$

Furthermore by (3.6) and the monotone nature of I_{ki} and J_{ki} we have

$$\begin{aligned} p_i(t_k^+) = \alpha_{j,i}(t_k^+) - \alpha_{j+1,i}(t_k^+) \le {} & \alpha_{j-1,i}(t_k) + I_{ki}(t, \alpha_{j-1,1}(t_k), \ldots, \alpha_{j-1,N}(t)) \\ & + L_{ki}(t, \beta_{j-1,1}(t_k), \ldots, \beta_{j-1,N}(t_k)) - \alpha_{j,i}(t_k) \\ & - I_{ki}(t, \alpha_{j,1}(t_k), \ldots, \alpha_{j,N}(t)) - L_{ki}(t, \beta_{j,1}(t_k), \ldots, \beta_{j,N}(t_k)) \le 0. \end{aligned}$$

This proves that $\alpha_{j,i} \le \alpha_{j+1,i}$ holds component-wise on J.

Now we will show that $\beta_{j,i} \ge \beta_{j+1,i}$ holds component-wise on J. For this purpose, let $p_i(t) = \beta_{j,i} - \beta_{j+1,i}$ and note that $p_i(t_0) = \beta_{j,i}(t_0) - \beta_{j+1,i}(t_0) = u_{0i} - u_{0i} = 0$.

Again, by (3.6), $\alpha_{j-1,i} \leq \alpha_{j,i}$ and $\beta_{j,i} \leq \beta_{j-1,i}$. Thus by the monotone nature of f_i, g_i, I_{ki} and J_{ki}, we get

$$\begin{aligned} p_i'(t) = \beta_{j,i}'(t) - \beta_{j+1,i}'(t) &= f_i(t, \beta_{j-1,1}(t), \ldots, \beta_{j-1,N}(t)) + g_i(t, \alpha_{j-1,1}(t), \ldots, \alpha_{j-1,N}(t)) \\ &\quad - f_i(t, \beta_{j,1}(t), \ldots, \beta_{j,N}(t)) - g_i(t, \alpha_{j,1}(t), \ldots, \alpha_{j,N}(t)) \geq 0 \end{aligned}$$

and

$$\begin{aligned} p_i(t_k^+) = \beta_{j,i}(t_k^+) - \beta_{j+1,i}(t_k^+) &= \beta_{j-1,i}(t_k) + I_{ki}(t, \beta_{j-1,1}(t_k), \ldots, \beta_{j-1,N}(t)) \\ &\quad + L_{k,i}(t, \alpha_{j-1,1}(t_k), \ldots, \alpha_{j-1,N}(t_k)) - \beta_{j,i}(t_k) - I_{ki}(t, \beta_{j,1}(t_k), \ldots, \beta_{j,N}(t)) \\ &\quad - L_{ki}(t, \alpha_{j,1}(t_k), \ldots, \alpha_{j,N}(t_k)) \geq 0 \end{aligned}$$

This proves that $\beta_{j,i} \geq \beta_{j+1,i}$ holds component-wise on J.

To complete the induction process, we must show that $\alpha_{j+1,i} \leq \beta_{j+1,i}$. So we will consider $p_i(t) = \alpha_{j+1,i} - \beta_{j+1,i}$ and note that $p_i(t_0) = \alpha_{j+1,i}(t_0) - \beta_{j+1,i}(t_0) = 0$. Also, by (3.6) we know that $\alpha_{j,i} \leq \beta_{j,i}$ and by the monotone nature of f_i, g_i, I_{ki} and J_{ki}, we have

$$\begin{aligned} p_i'(t) = \alpha_{j+1,i}' - \beta_{j+1,i}' &= f_i(t, \alpha_{j,1}(t), \ldots, \alpha_{j,N}(t)) + g_i(t, \beta_{j,1}(t), \ldots, \beta_{j,N}(t)) \\ &\quad - f_i(t, \beta_{j,1}(t), \ldots, \beta_{j,N}(t)) - g_i(t, \alpha_{j,1}(t), \ldots, \alpha_{j,N}(t)) \leq 0 \end{aligned}$$

and

$$\begin{aligned} p_i(t_k^+) = \alpha_{j+1,i}(t_k^+) - \beta_{j+1,i}(t_k^+) &= \alpha_{j,i}(t_k) + I_{ki}(t, \alpha_{j,1}(t_k), \ldots, \alpha_{j,N}(t)) \\ &\quad + L_{k,i}(t, \beta_{j,1}(t_k), \ldots, \beta_{j,N}(t_k)) \\ &\quad - \beta_{j,i}(t_k) - I_{ki}(t, \beta_{j,1}(t_k), \ldots, \beta_{j,N}(t)) \\ &\quad - L_{ki}(t, \alpha_{j,1}(t_k), \ldots, \alpha_{j,N}(t_k)) \leq 0 \end{aligned}$$

Therefore, $\alpha_{j+1,i} \leq \beta_{j+1,i}$. Hence, we have proven by induction that (3.5) holds for $j+1$. Thus the inequalities given in (3.5) hold for all $j = 0, 1, 2, \ldots$

Furthermore, the sequences $\{\alpha_{n,i}\}$ and $\{\beta_{n,i}\}$ can be shown to be equicontinuous and uniformly bounded. Thus by Ascoli-Arzela's Theorem, the subsequences $\{\alpha_{n_j,i}\}$ and $\{\beta_{n_j,i}\}$ converge to ρ_i and r_i respectively on J. Since the sequences $\{\alpha_{n,i}\}$ and $\{\beta_{n,i}\}$ are monotone, the sequences converge uniformly and monotonically to ρ_i and r_i respectively on J. Therefore ρ_i and r_i satisfy equations (3.1) and (3.2).

Finally, we claim that ρ_i and r_i are coupled minimal and maximal solutions of equation (2.2). Suppose that u_i are any solutions of (2.2), such that $\alpha_{0i} \leq u_i \leq \beta_{0i}$ on J, then we can prove that

$$\alpha_{j,i} \leq u_i \leq \beta_{j,i}$$

utilizing a similar process used to prove (3.5). Once we have shown $\alpha_{ki} \leq u_i \leq \beta_{ki}$, taking the limit as $k \to \infty$ gives

$$\lim_{ki \to \infty} \alpha_{ki} = \rho_i \quad \text{and} \quad \lim_{ki \to \infty} \beta_{ki} = r_i.$$

Hence $\rho_i \leq u_i \leq r_i$ on J. □

4. CONCLUSION

Our results prove that we can approximate a nonlinear impulsive differential system using the generalized monotone iterative method. In particular, we have shown that we can approximate the solution when the forcing functions and impulses are the sums of nondecreasing and nonincreasing functions. We showed that we obtain natural sequences when starting with natural lower and upper solutions of Type I when utilizing the iterative schemes Type (i).

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