

BIFURCATION IN A 3D HYBRID SYSTEM

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ABSTRACT. In this paper, we study a 3 dimensional Hybrid system which involves a switching mechanism such that at the moment of switching the differential equation that governs the model is changing. We first show that there is a center manifold and based on the results in [2] we show that the system under investigation has a Hopf bifurcation. An appropriate example is constructed to illustrate the theory.

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1. INTRODUCTION

In this study, we consider a 3 dimensional hybrid system which involves impulses and a switching mechanism, which implies that the moments are variable. The switching takes place while the impacts occur. Before and after each impact two different differential equations govern the model. While one of the system is active the other one is passive.

In [2] a planar discontinuous dynamical system has been studied and there the Hopf bifurcation theorem is proved. We have extended these results to 3 dimension and now we consider a different system in 3 dimensional space.

Methods of analysis of equations with variable moments of discontinuities can be found in [1]–[3], [6]–[25].

The paper is organized as follows. In the next section we give the non-perturbed system. Section 3 describes the center manifold. The bifurcation of periodic solutions is studied in section 4. Section 5 is devoted to an example in order to illustrate the theory.

2. THE NON-PERTURBED SYSTEM

We consider a system

$$\begin{aligned} x' &= A_{(i)}x, \quad x \notin \Gamma_{(i)}, \\ \Delta x|_{x \in \Gamma_{(i)}} &= B_{(i)}x, \\ \Delta i|_{x \in \Gamma_{(i)}} &= (-1)^{i+1}, \end{aligned} \tag{2.1}$$

where $i = 1, 2$ stands for the switching mechanism, $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, the matrices $A_{(i)}$ and $B_{(i)}$ are 3×3 real matrices of the form

$$\begin{aligned} A_{(1)} &= \begin{bmatrix} \alpha_1 & -\beta_1 & 0 \\ \beta_1 & \alpha_1 & 0 \\ 0 & 0 & \hat{b}_1 \end{bmatrix}, & A_{(2)} &= \begin{bmatrix} \alpha_2 & 0 & -\beta_2 \\ 0 & \hat{b}_2 & 0 \\ \beta_2 & 0 & \alpha_2 \end{bmatrix}, \\ B_{(1)} &= \begin{bmatrix} b_{11} & b_{12} & 0 \\ 0 & -1 & b_{23} \\ b_{31} & b_{32} & -1 \end{bmatrix}, & B_{(2)} &= \begin{bmatrix} c_{11} & 0 & c_{13} \\ c_{21} & -1 & c_{23} \\ 0 & c_{32} & -1 \end{bmatrix}, \end{aligned}$$

with $\beta_1, \beta_2 > 0$, $\Gamma_{(1)}(\Gamma_{(2)})$ is half plane $ax_1 + bx_2 = 0$ ($cx_1 + dx_3 = 0$), whose projection in $x_1x_2(x_1x_3)$ -plane starts from origin, for $a, b, c, d \in \mathbb{R}$, none of which is zero. Moreover, Δx denotes the jump operator. That is, if $x(t_{(i)})$ is the point on $\Gamma_{(i)}$ before jump and $x(\tilde{t}_{(i)})$ is the point on $\tilde{\Gamma}_{(i)}$ just after the jump, then $\Delta x = x(\tilde{t}_{(i)}) - x(t_{(i)})$.

The system (2.1) is described by the triplet (t, x, i) where t and x denote the time and the space variables, respectively, while the piecewise constant variable i denotes the switching mechanism. For each $i = 1, 2$, system (2.1) is a discontinuous dynamical system and satisfies the existence and uniqueness conditions given in [1], and hence there is a unique solution and this solution is continuable to $+\infty$ and $-\infty$.

A solution of (2.1) with the initial triplet (t_0, x_0, i_0) is defined as follows: Starting from the point (t_0, x_0) , the solution moves along the trajectory of $x' = A_{(i_0)}x$, with the initial condition $x(t_0) = x_0$, until the time when this trajectory meets the half plane $\Gamma_{(i_0)}$. The meeting is certain to exist because of the structure of the system all solutions must intersect the plane of discontinuity in a finite time. If $t = \theta_0$ is the time when the jump occurs, then $i(\theta_0) = i_0$. At this time the switching procedure takes place and the new value of the switching mechanism is evaluated as $i_1 = i(\theta_0^+) = i_0 + (-1)^{i_0+1}$. At the same time the space variable also performs a jump and it jumps to the point $x(\theta_0^+) = (I + B_{(i_0)})x(\theta_0)$. For each $i = 1, 2$ we define set $\tilde{\Gamma}_{(i)} = \{(I + B_{(i)})x : x \in \Gamma_{(i)}\}$. Thus, after the jump, the phase point is on the half plane $\tilde{\Gamma}_{(i_0)}$. Starting from that time, the solution moves along the trajectory of $x' = A_{(i_1)}x$, with the initial point at $x(\theta_0^+)$, until the time when this trajectory meets the half plane $\Gamma_{(i_1)}$. If $t = \theta_1$ is the time of meeting, then new values of the switching and the space variables are $i_2 = i_1 + (-1)^{i_1+1}$ and $x(\theta_1^+) = (I + B_{(i_1)})x(\theta_1)$,

respectively. After the impact, the system obeys the differential equation $x' = A_{(i_2)}x$, starting from the point $x(\theta_1^+)$, and so on.

Despite many insertions on hybrid systems have been done so far, the problems that we shall consider in this study, like Hopf Bifurcation and the center manifold, are new.

To investigate the system we use the so-called generalized cylindrical coordinates. At any time, when $i = 1$, we let $x_1 = r \cos \phi_{(1)}$, $x_2 = r \sin \phi_{(1)}$, $x_3 = z$, and when $i = 2$ we let $x_1 = r \cos \phi_{(2)}$, $x_2 = z$, $x_3 = r \sin \phi_{(2)}$. In this change of variables, when $i = 1$, we consider the projection of the point $x = (x_1, x_2, x_3)$ on the x_1x_2 -plane and when $i = 2$, we consider the projection of the point $x = (x_1, x_2, x_3)$ on the x_1x_3 -plane and the line segment joining the projection point and the origin. Here r denotes the length of this line segment and $\phi_{(i)}$ denotes the angle between that line and positive x_1 -axis measured in positive direction.

Now, from the differential part of (2.1) we obtain

$$\begin{aligned} \frac{dr}{d\phi_{(i)}} &= \lambda_{(i)}r, \\ \frac{dz}{d\phi_{(i)}} &= b_{(i)}z, \end{aligned} \quad (2.2)$$

where $\lambda_{(i)} = \alpha_i/\beta_i$, $b_{(i)} = \hat{b}_i/\beta_i$, for $i = 1, 2$.

To find the jump part, we investigate the cases when $i = 1$ or 2 separately. Let $i = 1$. Assume that initially the system is at the point (x_{01}, x_{02}, x_{03}) , or in cylindrical coordinates at the point (r_0, z_0) when $\phi_{(1)} = 0$. The system will follow the differential equation until the time when it meets the half plane $\Gamma_{(1)}$. If the phase point is at (x_{11}, x_{12}, x_{13}) before the impact, using $\Delta x|_{x \in \Gamma_{(1)}} = B_{(1)}x$, we see that after the impact the phase point will be at $(\tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{13})$ where

$$\begin{aligned} \tilde{x}_{11} &= (b_{11} + 1)x_{11} + b_{12}x_{12}, \\ \tilde{x}_{12} &= b_{23}x_{13}, \\ \tilde{x}_{13} &= b_{31}x_{11} + b_{32}x_{12}. \end{aligned}$$

If we denote the angle of $\Gamma_{(1)}$ by $\gamma_{(1)}$ and use the cylindrical coordinates, $x_{11} = r_1 \cos \gamma_{(1)}$, $x_{12} = r_1 \sin \gamma_{(1)}$, $x_{13} = z_1$, we get

$$\begin{aligned} \tilde{x}_{11} &= r_1 \left((b_{11} + 1) \cos \gamma_{(1)} + b_{12} \sin \gamma_{(1)} \right), \\ \tilde{x}_{12} &= b_{23}z_1, \\ \tilde{x}_{13} &= r_1 \left(b_{31} \cos \gamma_{(1)} + b_{32} \sin \gamma_{(1)} \right). \end{aligned}$$

In cylindrical coordinates, after the jump, the phase point will be at $(\tilde{r}_1, \tilde{z}_1)$ with

$$\begin{aligned} \tilde{r}_1 &= k_{12}r_1, \\ \tilde{z}_1 &= b_{23}z_1, \end{aligned}$$

where

$$k_{12} = \sqrt{[(b_{11} + 1) \cos \gamma_{(1)} + b_{12} \sin \gamma_{(1)}]^2 + [b_{31} \cos \gamma_{(1)} + b_{32} \sin \gamma_{(1)}]^2}. \quad (2.3)$$

After this time, i is switched to 2, the first equation is paused, the second equation is activated, and the system will obey the differential equation $x' = A_{(2)}x$ until the time when it touches the half plane $\Gamma_{(2)}$. If the phase point is at (x_{21}, x_{22}, x_{23}) where the meeting occurs, and $\gamma_{(2)}$ is the angle of $\Gamma_{(2)}$, then after switching it will be on $\tilde{\Gamma}_{(2)}$ at the point $(\tilde{x}_{21}, \tilde{x}_{22}, \tilde{x}_{23})$ where

$$\tilde{x}_{21} = (c_{11} + 1)x_{21} + c_{13}x_{23},$$

$$\tilde{x}_{22} = c_{21}x_{21} + c_{23}x_{23},$$

$$\tilde{x}_{23} = c_{32}x_{22}.$$

In cylindrical coordinates, $x_{21} = r_2 \cos \gamma_{(2)}$, $x_{22} = z_2$, $x_{23} = r_2 \sin \gamma_{(2)}$, these lead to

$$\tilde{x}_{21} = r_2 ((c_{11} + 1) \cos \gamma_{(2)} + c_{13} \sin \gamma_{(2)}),$$

$$\tilde{x}_{22} = r_2 (c_{21} \cos \gamma_{(2)} + c_{23} \sin \gamma_{(2)}),$$

$$\tilde{x}_{23} = c_{32}z_2,$$

which means that after the jump if the phase point has cylindrical coordinates $(\tilde{r}_2, \tilde{z}_2)$, at the angle $\tilde{\gamma}_{(2)}$, then

$$\tilde{r}_2 = k_{21}r_2,$$

$$\tilde{z}_2 = c_{32}z_2,$$

where

$$k_{21} = \sqrt{[(c_{11} + 1) \cos \gamma_{(2)} + c_{13} \sin \gamma_{(2)}]^2 + [c_{21} \cos \gamma_{(2)} + c_{23} \sin \gamma_{(2)}]^2}. \quad (2.4)$$

Therefore, the part of the system (2.1) when $i = 1$ can be written as

$$\begin{aligned} \frac{dr}{d\phi_{(1)}} &= \lambda_{(1)}r, \\ \frac{dz}{d\phi_{(1)}} &= b_{(1)}z, \quad \phi_{(1)} \neq \gamma_{(1)} \pmod{2\pi}, \\ \Delta r|_{\phi_{(1)}=\gamma_{(1)} \pmod{2\pi}} &= k_{(1)}r, \\ \Delta \phi_{(1)}|_{\phi_{(1)}=\gamma_{(1)} \pmod{2\pi}} &= \theta_{(1)}, \\ \Delta z|_{\phi_{(1)}=\gamma_{(1)} \pmod{2\pi}} &= c_{(1)}z, \end{aligned} \quad (2.5)$$

where $k_{(1)} = k_{12}k_{21}e^{\lambda_{(2)}(2\pi-\theta_{(2)})} - 1$, $c_{(1)} = b_{23}c_{32}e^{b_{(2)}(2\pi-\theta_{(2)})} - 1$, $\theta_{(1)} = \tilde{\gamma}_{(2)} - \gamma_{(1)}$ and $\theta_{(2)} = 2\pi + \tilde{\gamma}_{(1)} - \gamma_{(2)}$. Similarly, consecutive two parts of the system corresponding

to $i = 2$ can be written as

$$\begin{aligned}
\frac{dr}{d\phi_{(2)}} &= \lambda_{(2)}r, \\
\frac{dz}{d\phi_{(2)}} &= b_{(2)}z, \quad \phi_{(2)} \neq \gamma_{(2)} \pmod{2\pi}, \\
\Delta r|_{\phi_{(2)}=\gamma_{(2)} \pmod{2\pi}} &= k_{(2)}r, \\
\Delta \phi_{(2)}|_{\phi_{(2)}=\gamma_{(2)} \pmod{2\pi}} &= 2\pi - \theta_{(2)}, \\
\Delta z|_{\phi_{(2)}=\gamma_{(2)} \pmod{2\pi}} &= c_{(2)}z,
\end{aligned} \tag{2.6}$$

where $k_{(2)} = k_{12}k_{21}e^{\lambda_{(1)}\theta_{(1)}} - 1$, $c_{(2)} = b_{23}c_{32}e^{b_{(1)}\theta_{(1)}} - 1$. The domains of (2.5) and (2.6) are

$$\mathbb{R}_{\phi_{(1)}} = \bigcup_{n \in \mathbb{Z}} (2n\pi + \tilde{\gamma}_{(2)}, 2(n+1)\pi + \gamma_{(1)})]$$

and

$$\mathbb{R}_{\phi_{(2)}} = \bigcup_{n \in \mathbb{Z}} (2n\pi + \tilde{\gamma}_{(1)}, 2n\pi + \gamma_{(2)}],$$

respectively. We, now, construct the Poincaré section on x_1x_3 -plane, starting with $i = 1$, and consider the solution $(r(\phi_{(1)}), z(\phi_{(1)}))$ with the initial position at (r_0, z_0) when $\phi_{(1)} = 0$. The solution will move along the trajectory of (2.5) with the initial position at (r_0, z_0) until the time when the angle is $\phi_{(1)} = \gamma_{(1)}$. At this time the solution is at (r_1, z_1) where $r_1 = \exp(\lambda_{(1)}\gamma_{(1)})r_0$, $z_1 = \exp(b_{(1)}\gamma_{(1)})z_0$, and switching occurs. After that time the phase point is at $(\tilde{r}_1, \tilde{z}_1)$ where $\tilde{r}_1 = k_{12}r_1 = k_{12}\exp(\lambda_{(1)}\gamma_{(1)})r_0$, $\tilde{z}_1 = b_{23}z_1 = b_{23}\exp(b_{(1)}\gamma_{(1)})z_0$, and the solution moves along the trajectory of (2.6) until the time when it meets the half plane $\Gamma_{(2)}$. The solution of (2.6) with the initial condition $r(\tilde{\gamma}_{(1)}) = \tilde{r}_1$, $z(\tilde{\gamma}_{(1)}) = \tilde{z}_1$, for $\tilde{\gamma}_{(1)} < \phi_{(2)} \leq \gamma_{(2)}$ is given by $r(\phi_{(2)}) = \exp(\lambda_{(2)}(\phi_{(2)} - \tilde{\gamma}_{(1)}))\tilde{r}_1$, $z(\phi_{(2)}) = \exp(b_{(2)}(\phi_{(2)} - \tilde{\gamma}_{(1)}))\tilde{z}_1$. At the angle $\gamma_{(2)}$, just before the next impact, we have

$$\begin{aligned}
r_2 &:= r(\gamma_{(2)}) = k_{12}\exp(\lambda_{(1)}\gamma_{(1)} + \lambda_{(2)}(2\pi - \theta_{(2)}))r_0, \\
z_2 &:= z(\gamma_{(2)}) = c_{23}\exp(b_{(1)}\gamma_{(1)} + b_{(2)}(2\pi - \theta_{(2)}))z_0.
\end{aligned}$$

Once the impact takes place, the solution will be on the half plane $\tilde{\Gamma}_{(2)}$ at the point $(\tilde{r}_2, \tilde{z}_2) = (k_{21}r_2, c_{32}z_2)$ and system (2.5) will be active. Till the solution meets the half plane $\Gamma_{(1)}$, that is, for $\tilde{\gamma}_{(2)} \leq \phi_{(1)} \leq 2\pi + \gamma_{(1)}$, we have $r(\phi_{(1)}) = \exp(\lambda_{(1)}(\phi_{(1)} - \tilde{\gamma}_{(2)}))\tilde{r}_2$, $z(\phi_{(1)}) = \exp(b_{(1)}(\phi_{(1)} - \tilde{\gamma}_{(2)}))\tilde{z}_2$, and on the Poincare section, we have

$$\begin{aligned}
r(2\pi) &= k_{12}k_{21}\exp(\lambda_{(1)}(2\pi - \theta_{(1)}) + \lambda_{(2)}(2\pi - \theta_{(2)}))r_0, \\
z(2\pi) &= b_{23}c_{32}\exp(b_{(1)}(2\pi - \theta_{(1)}) + b_{(2)}(2\pi - \theta_{(2)}))z_0.
\end{aligned}$$

We, now, define

$$q_1 = k_{12}k_{21}\exp(\lambda_{(1)}(2\pi - \theta_{(1)}) + \lambda_{(2)}(2\pi - \theta_{(2)})), \tag{2.7}$$

and

$$q_2 = b_{23}c_{32} \exp(b_{(1)}(2\pi - \theta_{(1)}) + b_{(2)}(2\pi - \theta_{(2)})). \quad (2.8)$$

Note that $q_1 > 0, q_2 > 0$. Depending on q_1 and q_2 we have the following lemmas.

Lemma 2.1. *Assume that $q_1 = 1$. If*

- (i) $q_2 = 1$, then all solutions are periodic with period $T = \frac{2\pi - \theta_{(1)}}{\beta_1} + \frac{2\pi - \theta_{(2)}}{\beta_2}$;
- (ii) $q_2 < 1$, then a solution with initial position on x_1x_2 -plane is periodic with period T , and all other solutions lie on the surface of cylinders and they move toward the x_1x_2 -plane, i.e., x_1x_2 -plane is center manifold and x_3 -axis is stable manifold;
- (iii) $q_2 > 1$, then a solution with initial position on x_1x_2 -plane is periodic with period T , and all other solutions lie on the surface of cylinders and they move away from x_1x_2 -plane, i.e., x_1x_2 -plane is center manifold and x_3 -axis is unstable manifold.

Lemma 2.2. *Assume that $q_1 < 1$. If*

- (i) $q_2 = 1$, then a solution with initial position on x_3 -axis is T -periodic and all other solutions will approach to x_3 -axis, i.e., x_3 -axis is center manifold and x_1x_2 -plane is stable manifold;
- (ii) $q_2 < 1$, then all solutions will spiral toward the origin, i.e., origin is asymptotically stable;
- (iii) $q_2 > 1$, then a solution with initial position on x_1x_2 -plane spirals to origin, and a solution with initial position on x_3 -axis will move away from the origin, i.e., origin is half stable (or conditionally stable).

Lemma 2.3. *Assume that $q_1 > 1$. If*

- (i) $q_2 = 1$, then a solution with initial position on x_3 -axis is T -periodic and all other solutions will move away from x_3 -axis, i.e., x_3 -axis is center manifold and x_1x_2 -plane is unstable manifold;
- (ii) $q_2 < 1$, then a solution with initial position on x_1x_2 -plane moves away from origin, and a solution with initial position on x_3 -axis will move toward the origin, i.e., origin is half stable (or conditionally stable);
- (iii) $q_2 > 1$, then all solutions will move away from origin i.e., origin is unstable focus.

From now on, we assume that $q_1 = 1, q_2 < 1$, i.e., x_1x_2 -plane is center manifold and x_3 -axis is a stable manifold for (2.1).

3. CENTER MANIFOLD

Consider the system

$$\begin{aligned} x' &= A_{(i)}x + f_{(i)}(x), \quad x \notin \Gamma_{(i)}, \\ \Delta x|_{x \in \Gamma_{(i)}} &= B_{(i)}x, \\ \Delta i|_{x \in \Gamma_{(i)}} &= (-1)^{i+1}, \end{aligned} \tag{3.1}$$

where $A_{(i)}$, $B_{(i)}$ and $\Gamma_{(i)}$ are the same as in (2.1), $f_{(i)}(x) = \mathcal{O}(\|x\|^2)$, and $f_{(i)} = (f_{i1}, f_{i2}, f_{i3})$ for $i = 1, 2$. We assume that $f_{11}(0, 0, x_3) = f_{12}(0, 0, x_3) = 0$, $f_{21}(0, x_2, 0) = f_{31}(0, x_2, 0) = 0$ for all $x_2, x_3 \in \mathbb{R}$. Using the cylindrical coordinates, we obtain

$$\begin{aligned} \frac{dr}{d\phi_{(1)}} &= \lambda_{(1)}r + P_{(1)}(r, \phi_{(1)}, z), \\ \frac{dz}{d\phi_{(1)}} &= b_{(1)}z + Q_{(1)}(r, \phi_{(1)}, z), \quad \phi_{(1)} \neq \gamma_{(1)} \pmod{2\pi}, \\ \Delta r|_{\phi_{(1)} = \gamma_{(1)} \pmod{2\pi}} &= k_{(1)}r, \\ \Delta \phi_{(1)}|_{\phi_{(1)} = \gamma_{(1)} \pmod{2\pi}} &= \theta_{(1)}, \\ \Delta z|_{\phi_{(1)} = \gamma_{(1)} \pmod{2\pi}} &= c_{(1)}z, \end{aligned} \tag{3.2}$$

when $i = 1$, and

$$\begin{aligned} \frac{dr}{d\phi_{(2)}} &= \lambda_{(2)}r + P_{(2)}(r, \phi_{(2)}, z), \\ \frac{dz}{d\phi_{(2)}} &= b_{(2)}z + Q_{(2)}(r, \phi_{(2)}, z), \quad \phi_{(2)} \neq \gamma_{(2)} \pmod{2\pi}, \\ \Delta r|_{\phi_{(2)} = \gamma_{(2)} \pmod{2\pi}} &= k_{(2)}r, \\ \Delta \phi_{(2)}|_{\phi_{(2)} = \gamma_{(2)} \pmod{2\pi}} &= 2\pi - \theta_{(2)}, \\ \Delta z|_{\phi_{(2)} = \gamma_{(2)} \pmod{2\pi}} &= c_{(2)}z, \end{aligned} \tag{3.3}$$

when $i = 2$, where functions $P_{(i)}$ and $Q_{(i)}$ are 2π periodic in $\phi_{(i)}$, continuously differentiable and $P_{(i)} = \mathcal{O}(r^2, z)$, $Q_{(i)} = \mathcal{O}(r^2, z)$, for $i = 1, 2$. The domain of (3.2) is $\mathbb{R}_{\phi_{(1)}}$ and the domain of (3.3) is $\mathbb{R}_{\phi_{(2)}}$. We now use the ψ -substitution which was introduced in [2]. When $i = 1$ we let $\phi = \psi(\phi_{(1)})$, and when $i = 2$ we let $\phi = \psi(\phi_{(2)})$, so that we obtain

$$\begin{aligned} \frac{dr}{d\phi} &= \lambda_{(i)}r + F_{(i)}(r, \phi, z), \\ \frac{dz}{d\phi} &= b_{(i)}z + G_{(i)}(r, \phi, z), \quad \phi \neq \varphi_{(i)}, \\ \Delta r|_{\phi = \varphi_{(i)}} &= k_{(i)}r, \\ \Delta z|_{\phi = \varphi_{(i)}} &= c_{(i)}z, \end{aligned} \tag{3.4}$$

where $F_{(i)}(r, \phi, z) = P_{(i)}(r, \psi^{-1}(\phi), z)$, $G_{(i)}(r, \phi, z) = Q_{(i)}(r, \psi^{-1}(\phi), z)$ and $\varphi_{(i)} = \psi(\gamma_{(i)})$, for $i = 1, 2$. Following the methods given in [4] we see that system (3.4) has

two integral equations whose equations are given by

$$\begin{aligned}\Phi_0(\phi, r) &= \int_{-\infty}^{\phi} X_{(1)+}(\phi, s)G_{(1)}(r(s, \phi, r), s, z(s, \phi, r))ds \\ &+ \int_{-\infty}^{\psi(\gamma_{(2)}+2n\pi)} X_{(2)+}(\phi, s)G_{(2)}(r(s, \phi, r), s, z(s, \phi, r))ds,\end{aligned}\quad (3.5)$$

and

$$\begin{aligned}\Phi_{-}(\phi, z) &= - \int_{\phi}^{\infty} X_{(1)-}(\phi, s)F_{(1)}(r(s, \phi, z), s, z(s, \phi, z))ds \\ &- \int_{\psi(\tilde{\gamma}_{(1)}+2(n+1)\pi)}^{\infty} X_{(2)-}(\phi, s)F_{(2)}(r(s, \phi, z), s, z(s, \phi, z))ds,\end{aligned}\quad (3.6)$$

when $i = 1$, $2n\pi + \tilde{\gamma}_{(2)} < \phi_{(1)} \leq 2(n+1)\pi + \gamma_{(1)}$ for some integer n , and

$$\begin{aligned}\Phi_0(\phi, r) &= \int_{-\infty}^{\phi} X_{(2)+}(\phi, s)G_{(2)}(r(s, \phi, r), s, z(s, \phi, r))ds \\ &+ \int_{-\infty}^{\psi(\gamma_{(1)}+2n\pi)} X_{(1)+}(\phi, s)G_{(1)}(r(s, \phi, r), s, z(s, \phi, r))ds,\end{aligned}\quad (3.7)$$

and

$$\begin{aligned}\Phi_{-}(\phi, r) &= - \int_{\phi}^{\infty} X_{(2)-}(\phi, s)F_{(2)}(r(s, \phi, z), s, z(s, \phi, r))ds \\ &- \int_{\psi(\tilde{\gamma}_{(2)}+2n\pi)}^{\infty} X_{(1)-}(\phi, s)F_{(1)}(r(s, \phi, z), s, z(s, \phi, z))ds,\end{aligned}\quad (3.8)$$

when $i = 2$, $\tilde{\gamma}_{(1)} + 2n\pi < \phi_{(2)} \leq \gamma_{(2)} + 2n\pi$ for some integer n . In the above integrals

$$X_{(i)+}(\phi, s) = e^{b_{(i)}(\phi-s)} \prod_{s \leq \varphi_{(j)} < \phi} (1 + c_{(i)})$$

and

$$X_{(i)-}(\phi, s) = e^{\lambda_{(i)}(\phi-s)} \prod_{s \leq \varphi_{(j)} < \phi} (1 + k_{(i)}).$$

Moreover, in (3.5) and (3.7), the pair $(r(s, \phi, r), z(s, \phi, r))$ denotes a solution of (3.4) with $r(\phi, \phi, r) = r$, and in (3.6) and (3.8), the pair $(r(s, \phi, z), z(s, \phi, z))$ denotes a solution of (3.4) with $z(\phi, \phi, z) = z$.

As it has been done in [4], one can show that there are positive constants K_0 , M_0 , σ_0 such that Φ_0 satisfies:

$$\Phi_0(\phi, 0) = 0, \quad (3.9)$$

$$\|\Phi_0(\phi, r_1) - \Phi_0(\phi, r_2)\| \leq K_0 \ell \|r_1 - r_2\| \quad (3.10)$$

for all $r_1, r_2 \geq 0$ such that a solution $w(\phi) = (r(\phi), z(\phi))$ of (3.4) with the initial condition $w(\phi_0) = (r_0, \Phi_0(\phi_0, r_0))$, $r_0 \geq 0$, is defined on \mathbb{R} and satisfies

$$\|w(\phi)\| \leq M_0 r_0 e^{-\sigma_0(\phi-\phi_0)}, \quad \phi \geq \phi_0. \quad (3.11)$$

Similarly, there exist positive constants K_- , M_- , σ_- such that Φ_- satisfies:

$$\Phi_-(\phi, 0) = 0, \tag{3.12}$$

$$\|\Phi_-(\phi, z_1) - \Phi_-(\phi, z_2)\| \leq K_- \ell \|z_1 - z_2\| \tag{3.13}$$

for all z_1, z_2 such that a solution $w(\phi) = (r(\phi), z(\phi))$ of (3.4) with the initial condition $w(\phi_0) = (\Phi_-(\phi_0, z_0), z_0)$, $z_0 \in \mathbb{R}$ is defined on \mathbb{R} and satisfies

$$\|w(\phi)\| \leq M_- \|z_0\| e^{-\sigma_-(\phi-\phi_0)}, \quad \phi \leq \phi_0.$$

Set $S_0 = \{(r, \phi, z) : z = \Phi_0(\phi, r)\}$ and $S_- = \{(r, \phi, z) : r = \Phi_-(\phi, z)\}$. In this setting S_0 is called the *center manifold* and S_- is called the *stable manifold*.

The analogues of the following Lemma together with its proof can be found in [4].

Lemma 3.1. *If the Lipschitz constant ℓ is sufficiently small, then for every solution $(r(\phi), z(\phi))$ of (3.4) there exists a solution $(u(\phi), v(\phi))$ of (3.4) on the center manifold S_0 such that*

$$\begin{aligned} \|r(\phi) - u(\phi)\| &\leq 2M_0 \|r(\phi_0) - u(\phi_0)\| e^{-\sigma_0(\phi-\phi_0)}, \\ \|z(\phi) - v(\phi)\| &\leq M_0 \|z(\phi_0) - v(\phi_0)\| e^{-\sigma_0(\phi-\phi_0)}, \quad \phi \geq \phi_0, \end{aligned} \tag{3.14}$$

where M_0 and σ_0 are the constants used in (3.11).

On the local center manifold S_0 , the first coordinate of the solutions of (3.4) satisfies the following system:

$$\begin{aligned} \frac{dr}{d\phi} &= \lambda_{(i)} r + F_{(i)}(r, \phi, \Phi_0(\phi, r)), \quad \phi \neq \varphi_{(i)}, \\ \Delta r|_{\phi=\varphi_{(i)}} &= k_{(i)} r. \end{aligned} \tag{3.15}$$

Theorem 3.2. *The trivial solution of (3.4) is stable, asymptotically stable or unstable if the trivial solution of (3.15) is stable, asymptotically stable or unstable, respectively.*

4. BIFURCATION OF PERIODIC SOLUTIONS

In this section we investigate the bifurcation of periodic solutions. Let us consider the system

$$\begin{aligned} x' &= A_{(i)}x + f_{(i)}(x) + \mu \tilde{f}_{(i)}(x, \mu), \quad x \notin \Gamma_{(i)}, \\ \Delta x|_{x \in \Gamma_{(i)}} &= B_{(i)}x, \\ \Delta i|_{x \in \Gamma_{(i)}} &= (-1)^{i+1}, \end{aligned} \tag{4.1}$$

where $A_{(i)}$, $B_{(i)}$, $f_{(i)}$ and $\Gamma_{(i)}$ are the same as in (3.1). For each $i = 1, 2$, the function $\tilde{f}_{(i)}$ is assumed to be analytic in $x \in \mathbb{R}^3$, $\mu \in (-\mu_0, \mu_0)$ for sufficiently small μ_0 , and $\tilde{f}_{(i)}(0, \mu) = 0$ for all $\mu \in (-\mu_0, \mu_0)$. Using the generalized cylindrical coordinates, as it

is done for (3.1), and then using ψ -substitution as it is done for (3.2) and (3.3), we obtain

$$\begin{aligned}\frac{dr}{d\phi} &= \lambda_{(i)}(\mu)r + F_{(i)}(r, \phi, z, \mu), \\ \frac{dz}{d\phi} &= b_{(i)}(\mu)z + G_{(i)}(r, \phi, z, \mu), \quad \phi \neq \varphi_{(i)}, \\ \Delta r|_{\phi=\varphi_{(i)}} &= k_{(i)}(\mu)r, \\ \Delta z|_{\phi=\varphi_{(i)}} &= c_{(i)}(\mu)z,\end{aligned}\tag{4.2}$$

where all of the new elements depending on μ can be evaluated as it was done for (3.4). Following the methods, as we did to obtain (3.5)–(3.8) one can see that (4.2) has two integral manifolds whose equations are given by

$$\begin{aligned}\Phi_0(\phi, r, \mu) &= \int_{-\infty}^{\phi} X_{(1)+}(\phi, s, \mu)G_{(1)}(r(s, \phi, r, \mu), s, z(s, \phi, r, \mu), \mu)ds \\ &+ \int_{-\infty}^{\psi_{(2)}} X_{(2)+}(\phi, s, \mu)G_{(2)}(r(s, \phi, r, \mu), s, z(s, \phi, r, \mu), \mu)ds,\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}\Phi_{-}(\phi, z, \mu) &= - \int_{\phi}^{\infty} X_{(1)-}(\phi, s, \mu)F_{(1)}(r(s, \phi, z, \mu), s, z(s, \phi, z, \mu), \mu)ds \\ &- \int_{\tilde{\psi}_{(1)}}^{\infty} X_{(2)-}(\phi, s, \mu)F_{(2)}(r(s, \phi, z, \mu), s, z(s, \phi, z, \mu), \mu)ds,\end{aligned}\tag{4.4}$$

when $i = 1$, $2n\pi + \tilde{\gamma}_{(2)} < \phi_{(1)} \leq 2(n+1)\pi + \gamma_{(1)}$ for some integer n , $\psi_{(2)} = \psi(\gamma_{(2)} + 2n\pi)$, $\tilde{\psi}_{(1)} = \psi(\tilde{\gamma}_{(1)} + 2(n+1)\pi)$, and

$$\begin{aligned}\Phi_0(\phi, r, \mu) &= \int_{-\infty}^{\phi} X_{(2)+}(\phi, s, \mu)G_{(2)}(r(s, \phi, r, \mu), s, z(s, \phi, r, \mu), \mu)ds \\ &+ \int_{-\infty}^{\psi_{(1)}} X_{(1)+}(\phi, s, \mu)G_{(1)}(r(s, \phi, r, \mu), s, z(s, \phi, r, \mu), \mu)ds,\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}\Phi_{-}(\phi, r, \mu) &= - \int_{\phi}^{\infty} X_{(2)-}(\phi, s, \mu)F_{(2)}(r(s, \phi, z, \mu), s, z(s, \phi, r, \mu), \mu)ds \\ &- \int_{\tilde{\psi}_{(2)}}^{\infty} X_{(1)-}(\phi, s, \mu)F_{(1)}(r(s, \phi, z, \mu), s, z(s, \phi, z, \mu), \mu)ds,\end{aligned}\tag{4.6}$$

when $i = 2$, $\tilde{\gamma}_{(1)} + 2n\pi < \phi_{(2)} \leq \gamma_{(2)} + 2n\pi$ for some integer n , $\psi_{(1)} = \psi(\gamma_{(1)} + 2n\pi)$, $\tilde{\psi}_{(2)} = \psi(\tilde{\gamma}_{(2)} + 2n\pi)$. In the above integrals

$$X_{(i)+}(\phi, s, \mu) = e^{b_{(i)}(\mu)(\phi-s)} \prod_{s \leq \varphi_{(j)} < \phi} (1 + c_{(i)}(\mu))$$

and

$$X_{(i)-}(\phi, s, \mu) = e^{\lambda_{(i)}(\mu)(\phi-s)} \prod_{s \leq \varphi_{(j)} < \phi} (1 + k_{(i)}(\mu)).$$

Moreover, in (4.3) and (4.5), the pair $(r(s, \phi, r, \mu), z(s, \phi, r, \mu))$ denotes a solution of (4.2) with $r(\phi, \phi, r, \mu) = r$, and the pair $(r(s, \phi, z, \mu), z(s, \phi, z, \mu))$, in (4.4) and (4.6), denotes a solution of (4.2) with $z(\phi, \phi, z, \mu) = z$.

Set $S_0(\mu) = \{(r, \phi, z, \mu) : z = \Phi_0(\phi, r, \mu)\}$ and $S_-(\mu) = \{(r, \phi, z, \mu) : r = \Phi_-(\phi, z, \mu)\}$. On the local integral manifold $S_0(\mu)$, the first coordinate of the solutions of (4.2) satisfies the following system:

$$\begin{aligned} \frac{dr}{d\phi} &= \lambda_{(i)}(\mu)r + F_{(i)}(r, \phi, \Phi_0(\phi, r, \mu), \mu), \quad \phi \neq \varphi_{(i)}, \\ \Delta r|_{\phi=\varphi_{(i)}} &= k_{(i)}(\mu)r. \end{aligned} \quad (4.7)$$

Similar to (2.7) and (2.8) one can define

$$q_1(\mu) = k_{12}k_{21} \exp(\lambda_{(1)}(\mu)(2\pi - \theta_{(1)}) + \lambda_{(2)}(\mu)(2\pi - \theta_{(2)})), \quad (4.8)$$

and

$$q_2(\mu) = b_{23}c_{32} \exp(b_{(1)}(\mu)(2\pi - \theta_{(1)}) + b_{(2)}(\mu)(2\pi - \theta_{(2)})). \quad (4.9)$$

System (4.7) is a system studied in [2] and there it was shown that this system, for sufficiently small μ , has a periodic solution with period $T' = T(\mu) \approx T$. Here we will show that if the first coordinate of a solution of (4.2) is T' -periodic, then so is the second one.

We first note that

$$\begin{aligned} X_{(i)+}(\phi + T', s + T', \mu) &= X_{(i)+}(\phi, s, \mu), \\ r(s + T', \phi + T', r, \mu) &= r(s, \phi, r, \mu), \\ z(s + T', \phi + T', z, \mu) &= z(s, \phi, z, \mu), \end{aligned}$$

and each $G_{(i)}$ is T' periodic in ϕ . Assume without loss of any generality that $i = 1$. Now,

$$\begin{aligned} &\Phi_0(\phi + T', r, \mu) \\ &= \int_{-\infty}^{\phi+T'} X_{(1)+}(\phi + T', s, \mu) G_{(1)}(r(s, \phi + T', r, \mu), s, z(s, \phi + T', r, \mu), \mu) ds \\ &\quad + \int_{-\infty}^{\psi_{(2)}+T'} X_{(2)+}(\phi + T', s, \mu) G_{(2)}(r(s, \phi + T', r, \mu), s, z(s, \phi + T', r, \mu), \mu) ds \\ &= \int_{-\infty}^{\phi} X_{(1)+}(\phi, t, \mu) G_{(1)}(r(t, \phi + T', r, \mu), t, z(t, \phi, r, \mu), \mu) dt \\ &\quad + \int_{-\infty}^{\psi_{(2)}} X_{(2)+}(\phi, t, \mu) G_{(2)}(r(t, \phi, r, \mu), t, z(t, \phi, r, \mu), \mu) dt \\ &= \Phi_0(\phi, r, \mu), \end{aligned}$$

where we used the substitution $s = t + T'$ after the first equality. Then we have the following theorem which, in case of two dimension, can be found in [2].

Theorem 4.1. *Assume that $q_1(0) = 1$, $q_1'(0) \neq 0$, $|q_2(0)| < 1$, and the origin is a focus for (3.1). Then, for sufficiently small r_0 and z_0 , there exists a function $\mu = \delta(r_0, z_0)$ such that the solution $(r(\phi, \delta(r_0, z_0)), z(\phi, \delta(r_0, z_0)))$ of (4.2), with the initial condition $r(0, \delta(r_0, z_0)) = r_0, z(0, \delta(r_0, z_0)) = z_0$, is periodic with a period, $T' = (2\pi - \theta_{(1)})/\beta_1 + (2\pi - \theta_{(2)})/\beta_2 + o(|\mu|)$.*

5. AN EXAMPLE

As an example consider the system

$$\begin{aligned} x' &= A_{(i)}x + f_{(i)}(x) + \mu \tilde{f}_{(i)}(x, \mu), \quad x \notin \Gamma_{(i)}, \\ \Delta x|_{x \in \Gamma_{(i)}} &= B_{(i)}x, \\ \Delta i|_{x \in \Gamma_{(i)}} &= (-1)^{i+1}, \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} A_{(1)} &= \begin{bmatrix} -1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}, & A_{(2)} &= \begin{bmatrix} 11 & 0 & -2 \\ 0 & -0.1 & 0 \\ 2 & 0 & 11 \end{bmatrix}, \\ B_{(1)} &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -\sqrt{3} & -1 \end{bmatrix}, & B_{(2)} &= \begin{bmatrix} -1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \end{aligned}$$

$$\Gamma_{(1)} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = \sqrt{3}x_2, x_1 > 0\},$$

$$\Gamma_{(2)} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = \sqrt{3}x_3, x_1 > 0\},$$

$$f_{(1)}(x) = \begin{bmatrix} -x_1 \sqrt{x_1^2 + x_2^2} \\ -x_2 \sqrt{x_1^2 + x_2^2} \\ -x_3 \sqrt{x_1^2 + x_2^2} \end{bmatrix}, \quad f_{(2)}(x) = \begin{bmatrix} -x_1 \sqrt{x_1^2 + x_3^2} \\ x_2 \sqrt{x_1^2 + x_3^2} \\ -x_3 \sqrt{x_1^2 + x_2^2} \end{bmatrix},$$

and

$$\tilde{f}_{(1)}(x, \mu) = \begin{bmatrix} x_1 + x_2 x_3^2 + \mu x_1 x_3 \\ x_2 + x_1 x_3 + \mu x_1 x_2 \\ x_3 \end{bmatrix}, \quad \tilde{f}_{(2)}(x, \mu) = \begin{bmatrix} x_1 - \mu x_2 x_3 \\ x_2 - \mu x_1 x_3 \\ x_3 - \mu x_1 x_2 \end{bmatrix}.$$

Now, we find $\tilde{\Gamma}_{(1)}$ and $\tilde{\Gamma}_{(2)}$ as

$$\begin{aligned} \tilde{\Gamma}_{(1)} &= (I + B_{(1)})\Gamma_{(1)} \\ &= \{(x_2, x_3, x_1 - \sqrt{3}x_2) : (x_1, x_2, x_3) \in \Gamma_{(1)}\} \\ &= \{(x_1, x_2, x_3) : x_3 = 0, x_1 > 0\} \end{aligned}$$

and

$$\begin{aligned} \tilde{\Gamma}_{(2)} &= (I + B_{(2)})\Gamma_{(2)} \\ &= \{(2x_3, 2x_1, x_2) : (x_1, x_2, x_3) \in \Gamma_{(2)}\} \\ &= \{(x_1, x_2, x_3) : x_2 = \sqrt{3}x_1, x_1 > 0\}. \end{aligned}$$

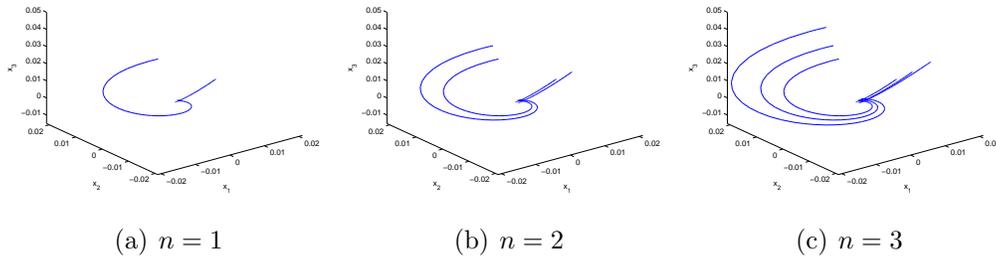


FIGURE 1. The solution of (5.1) starting at $(x_{10}, x_{20}, x_{30}) = (0.002, 0, 0.001)$ with $\mu = 0.1$ after n rotations.

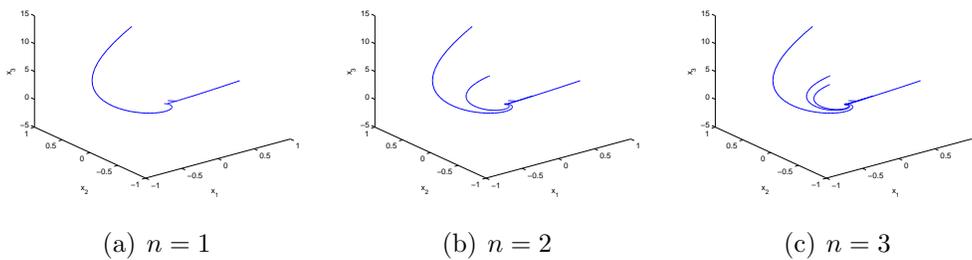


FIGURE 2. The solution of (5.1) starting at $(x_{10}, x_{20}, x_{30}) = (0.2, 0, 0.01)$ with $\mu = 0.1$ after n rotations.

Thus, we have $\gamma_{(1)} = \pi/6, \tilde{\gamma}_{(1)} = 0, \gamma_{(2)} = \pi/6, \tilde{\gamma}_{(2)} = \pi/3$, and hence $\theta_{(1)} = \pi/6, \theta_{(2)} = 11\pi/6$. Therefore, using (4.8) and (4.9), we obtain $q_1(\mu) = e^{\mu\pi}$ and $q_2(\mu) = e^{\mu-\pi/10}$. Clearly, $q_1(0) = 1, q_1'(0) \neq 0$ and $|q_2(0)| < 1$. Hence, by Theorem 4.1, system (5.1) has a periodic solution with period $\approx \pi$. In Figure 1, an inner solution of (5.1) is seen. Figure 2 shows an outer solution of (5.1).

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