Communications in Applied Analysis **14** (2010), no. 4, 325–336

MONOTONE ITERATIVE TECHNIQUE FOR INTEGRO DIFFERENTIAL EQUATIONS WITH RETARDATION AND ANTICIPATION

J. VASUNDHARA DEVI[1*], CH. V. SREEDHAR[1], AND S. NAGAMANI[2]

[1]Department of Mathematics,
GVP-Prof. V. Lakshmikantham Institute for Advanced Studies,
GVP College of Engineering, Visakhapatnam, AP, India
E-mail: jvdevi@gmail.com, chaduvulasreedhar@gmail.com

[2]Department of Mathematics, Government Junior College for Women,
Narsipatnam AP, India

ABSTRACT. In this paper, the monotone iterative technique is developed to study the existence of solutions of integro differential equations with retardation and anticipation.

AMS (MOS) Subject Classification. 45J05, 45J99

1. INTRODUCTION

Recently there have been special situations in decision making, organizational transformation, chaotic equations, wavelet theory and so on, where specific equations with anticipation as well as retardation and anticipation appear in modeling [2,5,6]. This lead to the initiation of the study of the general theory of differential equations involving anticipation as well as retardation and anticipation in [15] and continued in [7,8].

Integro differential equations [13] appear to be suitable models to investigate problems in biology and social sciences. The monotone iterative technique is an effective and a flexible mechanism to obtain theoretical as well as constructive existence results in a sector, which isa closed set. Using the lower and upper solutions of a given system, the sector is determined. These lower and upper solutions form lower and upper bounds for the solutions of the considered differential equations [2,9,10,11]. Hence in this paper we develop the monotone iterative technique for integro differential equations with retardation and anticipation, to study the existence of coupled minimal and maximal solutions of the considered equations.

---

*Corresponding author: J. Vasundhara Devi, Email address: jvdevi@gmail.com.

Received November 30, 2009

2. PRELIMINARIES

We begin with the integro differential equation given by,

$$x' = f(t, x, Sx), \quad t \in J,$$

$$x(t_0) = x_0$$

where $Sx(t) = \int\limits_{t_0}^{t} K(t,s)x(s)ds$, $t \in J$ $K(t,s) \in C[J \times J, R]$, and $f \in C[J \times R \times R, R]$. There has been an extensive study of these equations in [13]. The monotone iterative technique for initial value problem and periodic boundary value problems was developed in [3,13].

The initial value problem for delay differential equation is described by

$$x' = f(t, x_t), \quad t \in J,$$

$$x_{t_0} = \phi_0,$$

$\phi_0 \in C_1 = C[[-h_1, 0], R]$ and $f \in C[J \times C_1, R]$. Much literature exists for the above equations in [14] and the monotone iterative technique for delay differential equations was developed in [11]. The differential equation with anticipation is given by

$$x' = f(t, x, x^t), \quad t \in J,$$

$$x^T = \psi_0,$$

$\psi_0 \in C_2 = C[[0, h_2], R]$ and $f \in C[J \times R \times C_2, R]$. Combining all these types of equations we describe the following differential equation with retardation and anticipation.

Consider the integro differential equation with retardation and anticipation given by

$$x' = f(t, x, Sx, x_t, x^t), \quad t \in J, \tag{2.1}$$

$$x_{t_0} = \phi_0, \quad x^T = \psi_0 \tag{2.2}$$

where $\phi_0 \in C_1$, $\psi_0 \in C_2$ and $f \in C[J \times R \times R \times C_1 \times C_2, R]$.

The following definitions are a prerequisite to introduce the monotone method.

Definition 2.1 (Coupled Lower and Upper Solutions). (i) A function $v_0 \in C^1[J, R]$ is said to be coupled lower solution of (2.1) and (2.2) if $v_0' \leq f(t, v_0, Sv_0, v_{0t}, w_0^t)$ where $v_{0t_0} = \phi_1$, ${v_0}^T = \psi_1$;

(ii) A function $w_0 \in C^1[J, R]$ is said to be coupled upper solution of (2.1) and (2.2) if $w_0' \geq f(t, w_0, Sw_0, w_{0t}, v_0^t)$ where $w_{0t_0} = \phi_2$, ${w_0}^T = \psi_2$, and $\phi_1, \phi_2 \in C_1$, $\psi_1, \psi_2 \in C_2$ such that $\phi_1 \leq \phi_0 \leq \phi_2$, $\psi_1 \leq \psi_0 \leq \psi_2$.

Definition 2.2 (Maximal and Minimal Solutions). (i) Let $r(t)$ be a solution of (2.1) and (2.2), then $r(t)$ is said to be maximal solution if for every solution $x(t)$ of (2.1) and (2.2) existing on $[t_0, T]$, the inequality $x(t) \leq r(t)$ holds for $t \in [t_0, T]$.

(ii) Let $\rho(t)$ be a solution of (2.1) and (2.2), then $\rho(t)$ is said to be minimal solution if for every solution $x(t)$ of (2.1) and (2.2) existing on $[t_0, T]$, the inequality $\rho(t) \leq x(t)$ holds for $t \in [t_0, T]$

3. MAIN RESULTS

We first give the necessary notation. Let $I = [t_0 - h_1, T + h_2]$ with $h_1 > 0,\ h_2 > 0$, $J = [t_0, T]$ and $x \in C[I, R]$, We Define the integral operator $S \in C[C[J, R], C[J, R]]$. by $Sx(t) = \int\limits_{t_0}^{t} K(t, s)x(s)ds,\ t \in J$ where $K(t, s) \in C[J \times J, R]$, and $\max K(t, s) = k_1\ (t \in J)$. Next consider the integro differential equation with retardation and anticipation given by

$$x' = f(t, x, Sx, x_t, x^t), \quad t \in J \tag{3.1}$$

$$x_{t_0} = \phi_0, \quad x^T = \psi_0 \tag{3.2}$$

where $f \in C[J \times R \times R \times C_1 \times C_2, R], C_1 = C[[-h_1, 0], R], C_2 = [[0, h_2], R]$, and $\phi_0 \in C_1,\ \psi_0 \in C_2$. Furthermore, $x_t : [-h_1, 0] \to R$ is such that $x_t(s) = x(t + s)$, $-h_1 \leq s \leq 0$ and, $x^t : [0, h_2] \to R$ is such that $x^t(\sigma) = x(t + \sigma)$, $0 \leq \sigma \leq h_2$. We need the following lemma concerning the linear integro differential equations involving retardation and anticipation to develop the monotone iterative technique.

Lemma 3.1. *Let $p \in C[[t_0 - h_1, T] \bigcap C^1[J, R]]$ where $J = [t_0, T]$ and*

$$p'(t) \leq -N_1 p(t) - N_2 Sp(t) + \int_{-h_1}^{0} p_t(s)ds \qquad \text{on } J \tag{3.3}$$

where $Sp(t) = \int\limits_{t_0}^{t} K(t, s)p(s)ds$. Suppose further that either

A) *$p(t_0) \leq p_{t_0}(s) \leq 0,\ s \in [-h_1, 0]$ and,*

$$[N_1 + N_2 k_1(T - t_0) + N_3 h_1](T - t_0) \leq 1. \tag{3.4}$$

or

B) *$p_{t_0}(s) \leq 0,\ s \in [-h_1, 0],\ p \in C^1[[t_0 - h_1, T], R]$, and $p'(t) \leq \frac{\lambda}{T - t_0 + h_1}$, where $t \in [t_0 - h_1, t_0]$, $\min p(s) = -\lambda,\ \lambda \geq 0$ and $[t_0 - h_1, t_0]$, $\max K(t, s) = k_1,\ t \in J$*

$$[N_1 + N_2 k_1(T - t_0) + N_3 h_1](T - t_0 + h_1) \leq 1 \tag{3.5}$$

then, $p(t) \leq 0$ on J.

Proof. Assume that the conclusion of the Lemma does not hold, A) There exists $t_1, t_2 \in J$ such that $t_1 < t_2$, $p(t_2) > 0$, and $\min p(t) = -\lambda = p(t_1) \leq 0$ with $\lambda \geq 0$. $[t_0 - h_1, t_2]$. Let us first consider the case $\lambda > 0$. By mean value theorem there exists $\bar{t} \in [t_1, t_2]$ such that,

$$p'(\bar{t}) = \frac{p(t_2) - p(t_1)}{t_2 - t_1} > \frac{\lambda}{T - t_0} \tag{3.6}$$

on the other hand, we have,

$$\begin{aligned}
p'(\bar{t}) &\leq -N_1 p(\bar{t}) - N_2 Sp(\bar{t}) - N_3 \int_{-h_1}^{0} p_{\bar{t}}(s)ds \\
&\leq N_1\lambda - N_2 \int_{t_0}^{\bar{t}} K(\bar{t}, s)p(s)ds - N_3 \int_{-h_1}^{0} p(\bar{t} + s)ds \\
&\leq N_1\lambda + N_2 k_1 \lambda(T - t_0) + N_3\lambda h_1 \\
&\leq [N_1 + N_2 k_1(T - t_0) + N_3 h_1]\lambda
\end{aligned}$$

$$p'(\bar{t}) \leq \frac{\lambda}{T - t_0} \tag{3.7}$$

by hypothesis (A) this is contradiction to (3.6). Hence $p(t) \leq 0$ on J. If $\lambda = 0$ then $p(t) \geq 0$ on $[t_0 - h_1, t_2]$ but from (3.7), $p'(t) \leq 0$. As a consequence $p(t_2) \leq p(0) \leq 0$, which contradicts the fact that $p(t_2) > 0$. Hence in this case also $p(t) \leq 0$ on J. B) The minimal value of p(t) on $[t_0 - h_1, t_2]$ lies on $[t_0 - h_1, t_0]$, $\min p(t) = \min p(t) = -\lambda = p(t\,') \leq 0$ where $t' \in [t_0 - h_1, t_0]$, $[t_0 - h_1, t_2]$, $[t_0 - h_1, t_0]$. Hence using the mean value theorem on $[t', t_2]$, there exists $\widehat{t} \in [t', t_2]$ such that

$$p'(\widehat{t}) = \frac{p(t_2) - p(t')}{t_2 - t'} > \frac{\lambda}{T - t_0 + h_1} \tag{3.8}$$

Now $\widehat{t}$ may belong to either $[t_0 - h_1, t_0]$ or $[t_0, t_2]$. We consider the two possibilities. If $\widehat{t} \in [t_0, t_2]$ then (3.5) yields,

$$\begin{aligned}
p'(\widehat{t}) &\leq -N_1 p(\widehat{t}) - N_2 Sp(\widehat{t}) - N_3 \int_{-h_1}^{0} p_{\widehat{t}}(s)ds \\
&\leq N_1\lambda - N_2 \int_{t_0}^{\widehat{t}} K(\widehat{t}, s)p(s)ds - N_3 \int_{-h_1}^{0} p(\widehat{t} + s)ds \\
&\leq N_1\lambda + N_2 k_1 \lambda(T - t_0) + N_3\lambda h_1 \\
&\leq [N_1 + N_2 k_1(T - t_0) + N_3 h_1]\lambda \\
&\leq \frac{\lambda}{T - t_0 + h_1}
\end{aligned}$$

which is a contradiction to (3.8) and thus we conclude that $p(t) \leq 0$ on J. If $\widehat{t} \in [t_0 - h_1, t_0]$ then from (3.8) we have $p'(\widehat{t}) > \frac{\lambda}{T - t_0 + h_1}$ which is a contradiction to hypothesis (B). Hence $p(t) \leq 0$ on J and the proof is complete. □

We now state the following hypothesis which is needed.

H_1 : $v_0, w_0 \in C^1[J, R]$ satisfying, $v_0' \leq f(t, v_0, Sv_0, v_{0t}, w_0^t), v_{0t_0} = \phi_1, v_0^T = \psi_1$, and $w_0' \geq f(t, w_0, Sw_0, w_{0t}, v_0^t), w_{0t_0} = \phi_2, w_0^T = \psi_2$, where $\phi_1, \phi_2 \in C_1, \psi_1, \psi_2 \in C_2$, such that $\phi_1 \leq \phi_0 \leq \phi_2$ and $\psi_1 \leq \psi_0 \leq \psi_2$, and $v_0(t) \leq w_0(t)$ on J;

H_2 : $f(t, x, \overline{x}, \phi, \xi)$ is non increasing in ξ for each $(t, x, \overline{x}, \phi)$;

H_3 : $f(t, x, \overline{x}, \phi, \psi) - f(t, y, \overline{y}, \xi, \psi) \geq -M_1(x - y) - M_2(\overline{x} - \overline{y}) - M_3 \int_{-h_1}^{0} (\phi - \xi)(s)ds$, with $[M_1 + M_2k_0(T - t_0) + M_3h_1](T - t_0) \leq 1$, and $M(T - t_0) > \frac{1}{2}$, where $v_0(t) \leq y \leq x \leq w_0(t), Sv_0(t) \leq \overline{y} \leq \overline{x} \leq Sw_0(t)$, $M_1, M_2, M_3 \geq 0$, $v_{0t} \leq \xi \leq \phi \leq w_{0t}, \phi, \xi \in C_1, \psi \in C_2$;

H_4 : $v_{0t_0} - \phi_0, \phi_0 - w_{0t_0}$ satisfying the assumptions of Lemma 3.1.

We now proceed to present the main result.

Theorem 3.2. *Suppose that the assumptions H_1 to H_4 are satisfied. Then there exist monotone sequences $\{v_n(t)\}, \{w_n(t)\}$ such that $v_n(t) \to \rho(t)$ and $w_n(t) \to r(t)$ uniformly as, $n \to \infty$ in $[t_0 - h_1, T + h_2]$ and that ρ, r are coupled minimal and maximal solutions of* (3.1), (3.2).

Proof. For any $\eta, \nu \in [v_0, w_0]$, consider the linear integro differential equation with retardation and anticipation given by,

$$x' + M_1x = -M_2Sx - M_3 \int_{-h_1}^{0} x_t(s)ds + g(t) \tag{3.9}$$

where $g(t) = f(t, \eta(t), S\eta(t), \eta_t, \nu^t) + M_1\eta(t) + M_2S\eta(t) + M_3 \int_{-h_1}^{0} \eta_t(s)ds$. Now by using the variation of parameters and simplifying we get,

$$x(t) = x(t_0)e^{-M_1(t-t_0)} + e^{-M_1t} \int_{t_0}^{t} e^{M_1s}[-M_2Sx - M_3 \int_{-h_1}^{0} x_s(\sigma)d\sigma + g(s)]ds$$

which is a solution of the linear differential equation. Now define,

$$Fx(t) = x(t_0)e^{-M_1(t-t_0)} + e^{-M_1t} \int_{t_0}^{t} e^{M_1s}[-M_2Sx - M_3 \int_{-h_1}^{0} x_s(\sigma)d\sigma + g(s)]ds$$

We claim that F is a contraction and to show this consider

$$|Fx(t) - Fy(t)| = \Bigg| \Bigg[x(t_0)e^{-M_1(t-t_0)} + e^{-M_1t} \int_{t_0}^{t} e^{M_1s} \left[-M_2Sx(s) - M_3 \int_{-h_1}^{0} x_s(\sigma)d\sigma + g(s) \right] ds \Bigg]$$

$$
\begin{aligned}
&- \left[x(t_0)e^{-M_1(t-t_0)} + e^{-M_1 t} \int_{t_0}^{t} e^{M_1 s}[-M_2 Sy(s) - M_3 \int_{-h_1}^{0} y_s(\sigma)d\sigma + g(s)]ds \right] \Bigg| \\
&\leq e^{-M_1 t} \left[M_2 k_0 |x-y|_0 (T-t_0) \frac{e^{M_1 t}}{M_1} + M_3 |x-y|_0 h_1 \left(\frac{e^{M_1 t} - e^{M_1 t_0}}{M_1} \right) \right] \\
&\leq \frac{M_2 k_0 |x-y|_0 (T-t_0)}{M_1} + \frac{M_3 |x-y|_0 h_1}{M_1} (1 - e^{M_1(t_0 - t)}) \\
&\leq \frac{M_2 k_0 |x-y|_0 (T-t_0)}{M_1} + \frac{M_3 |x-y|_0 h_1}{M_1} \\
&\leq \left(\frac{M_2 k_0 (T-t_0) + M_3 h_1}{M_1} \right) |x-y|_0
\end{aligned}
$$

For any $x, y \in C[J, R]$ and the norm $|\cdot|$ is the supremum norm in $C[J, R]$. So we have that F is a contraction using hypothesis H_3. Thus by contraction principle F has a unique fixed point and the linear integro differential equation with retardation and anticipation has a unique solution. Now we proceed to build the iterates. Consider the following coupled linear problem for $n = 1, 2, 3, 4 \ldots$

$$
\begin{aligned}
v'_{n+1} &= f(t, v_n, Sv_n, v_{nt}, w_n^t) - M_1(v_{n+1} - v_n) - M_2[Sv_{n+1} - Sv_n] \\
&\quad - M_3 \int_{-h_1}^{0} (v_{n+1t} - v_{nt})(s)ds \\
w'_{n+1} &= f(t, w_n, Sw_n, w_{nt}, v_n^t) - M_1(w_{n+1} - w_n) - M_2[Sw_{n+1} - Sw_n] \\
&\quad - M_3 \int_{-h_1}^{0} (w_{n+1t} - w_{nt})(s)ds,
\end{aligned}
$$

where $v_{n+1_{t_0}} = \phi_0$, $w_{n+1_{t_0}} = \phi_0$ and ${v_{n+1}}^T, {w_{n+1}}^T$ are chosen such that ${v_0}^T \leq {v_1}^T \leq {v_2}^T \leq \cdots \leq {v_n}^T \leq {v_{n+1}}^T \leq {w_{n+1}}^T \leq {w_n}^T \leq \cdots \leq {w_1}^T \leq {w_0}^T$ and $\{{v_n}^T\}, \{{w_n}^T\}$ converges uniformly to ϕ_0 on $[0, h_2]$. Now clearly the linear problem has a unique solution on $[t_0 - h_1, T + h_2]$ we wish to show that $v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq v_{n+1} \leq w_{n+1} \leq w_n \leq \cdots \leq w_1 \leq w_0$ on J. We first claim that $v_0 \leq v_1$ on J. For this consider

$$
\begin{aligned}
p &= v_0 - v_1 \\
p' &= v'_0 - v'_1 \\
&\leq f(t, v_0, Sv_0, v_{0t}, w_0^t) - \{f(t, v_0, Sv_0, v_{0t}, w_0^t) - M_1(v_1 - v_0) - M_2[Sv_1 - Sv_0] \\
&\quad - M_3 \int_{-h_1}^{0} (v_{1t} - v_{0t})(s)ds\} \\
&\leq -M_1 p(t) - M_2 Sp(t) - M_3 \int_{-h_1}^{0} p_t(s)ds
\end{aligned}
$$

But $p_{t_0} \le v_{0_{t_0}} - v_{1_{t_0}} = \phi_0 - \phi_0 = 0$. Now M_1, M_2, M_3 are such that hypothesis of Lemma 3.1 is satisfied. Thus we can conclude that $p(t) \le 0$ on J. Hence $v_0 \le v_1$ on J. Similarly we can show that $w_1 \le w_0$ on J. Next we consider

$$\begin{aligned}
p &= v_1 - w_1 \\
p' &= v_1' - w_1' \\
&= \left\{ f(t, v_0, Sv_0, v_{0t}, w_0^t) - M_1(v_1 - v_0) - M_2[Sv_1 - Sv_0] - M_3 \int_{-h_1}^{0} (v_{1t} - v_{0t})(s)ds \right\} \\
&\quad - \left\{ f(t, w_0, Sw_0, w_{0t}, v_0^t) - M_1(w_1 - w_0) - M_2[Sw_1 - Sw_0] \right. \\
&\quad \left. - M_3 \int_{-h_1}^{0} (w_{1t} - w_{0t})(s)ds \right\} \\
&= [f(t, v_0, Sv_0, v_{0t}, w_0^t) - f(t, w_0, Sw_0, w_{0t}, v_0^t)] - M_1(v_1 - v_0) - M_2[Sv_1 - Sv_0] \\
&\quad - M_3 \int_{-h_1}^{0} (v_{1t} - v_{0t})(s)ds + M_1(w_1 - w_0) + M_2[Sw_1 - Sw_0] \\
&\quad + M_3 \int_{-h_1}^{0} (w_{1t} - w_{0t})(s)ds
\end{aligned}$$

using the hypothesis H_2 and simplifying we get,

$$\le -M_1 p(t) - M_2 Sp(t) - M_3 \int_{-h_1}^{0} p_t(s)ds$$

But $p_{t_0} \le v_{1_{t_0}} - w_{1_{t_0}} = \phi_0 - \phi_0 = 0$. Now M_1, M_2, M_3 are such that hypothesis of Lemma 3.1 is satisfied. Thus we can conclude that $p(t) \le 0$ on J. Hence $v_1 \le w_1$ on J, and thus we proved that $v_0 \le v_1 \le w_1 \le w_0$ on J. Now assume that for some $n = k$ $v_0 \le v_1 \le v_2 \le \cdots \le v_k \le w_k \le \cdots w_1 \le w_0$ on J. Now we claim that $v_k \le v_{k+1} \le w_{k+1} \le w_k$ on J. Next we consider

$$\begin{aligned}
p &= v_k - v_{k+1} \\
p' &= v_k' - v_{k+1}' \\
&= [f(t, v_{k-1}, Sv_{k-1}, v_{k-1t}, {w_{k-1}}^t) - f(t, v_k, Sv_k, v_{kt}, {w_k}^t)] \\
&\quad - M_1(v_k - v_{k-1}) + M_1(v_{k+1} - v_k) \\
&\quad - M_2[Sv_k - Sv_{k-1}] + M_2[Sv_{k+1} - Sv_k] - M_3 \int_{-h_1}^{0} (v_{kt} - v_{k-1t})(s)ds
\end{aligned}$$

$$+ M_3 \int_{-h_1}^{0} (v_{k+1t} - v_{kt})(s)ds$$

using the hypothesis H_2 and simplifying we get,

$$\leq -M_1(v_k - v_{k+1}) - M_2[Sv_k - Sv_{k+1}] - M_3 \int_{-h_1}^{0} (v_{kt} - v_{k-1t})(s)ds$$

$$\leq -M_1 p(t) - M_2 Sp(t) - M_3 \int_{-h_1}^{0} p_t(s)ds$$

But $p_{t_0} \leq v_{k_{t_0}} - v_{k+1_{t_0}} = \phi_0 - \phi_0 = 0$. Now M_1, M_2, M_3 are such that hypothesis of Lemma 3.1 is satisfied. Thus we can conclude that $p(t) \leq 0$ on J. Hence $v_k \leq v_{k+1}$. Similarly we can show that $w_{k+1} \leq w_k$ on J. We next show that $v_{k+1} \leq w_{k+1}$ on J. For this consider

$$\begin{aligned}
p &= v_{k+1} - w_{k+1} \\
p' &= v'_{k+1} - w'_{k+1} \\
&= f(t, v_k, Sv_k, v_{kt}, {w_k}^t) - M_1(v_{k+1} - v_k) - M_2[Sv_{k+1} - Sv_k] \\
&\quad - M_3 \int_{-h_1}^{0} (v_{k+1t} - v_{kt})(s)ds \\
&\quad - \{f(t, w_k, Sw_k, w_{kt}, {v_k}^t) - M_1(w_{k+1} - w_k) - M_2[Sw_{k+1} - Sw_k] \\
&\quad - M_3 \int_{-h_1}^{0} (w_{k+1t} - w_{kt})(s)ds\}
\end{aligned}$$

using the hypothesis H_2 and simplifying we get,

$$\leq -M_1 p(t) - M_2 Sp(t) - M_3 \int_{-h_1}^{0} p_t(s)ds$$

But $p_{t_0} \leq v_{k+1_{t_0}} - w_{k+1_{t_0}} = \phi_0 - \phi_0 = 0$. Now M_1, M_2, M_3 are such that hypothesis of Lemma 3.1 is satisfied. Thus we can conclude that $p(t) \leq 0$ on J. $v_{k+1} \leq w_{k+1}$ on J. Now using mathematical induction,we observe that, $v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_k \leq v_{k+1} \leq w_{k+1} \leq w_k \leq \cdots w_1 \leq w_0$, The sequences $\{v_n\}$ and $\{w_n\}$ are uniformly bounded and equicontinuous. Hence from Ascoli's Theorem and Dini's Theorem we can conclude that the sequences $\{v_n\}$ and $\{w_n\}$ converges uniformly on $[t_0-h_1,\ T+h_2]$ i.e., $v_n \to \rho$ and $w_n \to r$ uniformly on $[t_0 - h_1, T + h_2]$. To show that ρ and r are coupled minimal and maximal solutions of (3.1) and (3.2). We have $v'_{n+1} =$

$f(t, v_n, Sv_n, v_{nt}, {w_n}^t) - M_1(v_{n+1} - v_n) - M_2[Sv_{n+1} - Sv_n] - M_3 \int\limits_{-h_1}^{0} (v_{n+1t} - v_{nt})(s)ds$
where $v_{k+1t_0} = \phi_0$. The corresponding integral equation is given by,

$$v_{n+1}(t) = x_0 + \int_{t_0}^{t} \Bigg[f(t, v_n, Sv_n, v_{nt}, {w_n}^t) - M_1(v_{n+1} - v_n) \\ - M_2[Sv_{n+1} - Sv_n] - M_3 \int_{-h_1}^{0} (v_{n+1t} - v_{nt})(s)ds \Bigg] ds.$$

It is easy to show that ρ is a solution of (3.1) and (3.2) since by taking limit as $n \to \infty$. We conclude that $\rho(t) = x_0 + \int\limits_{t_0}^{t} f(t, \rho, Sv(\rho), \rho_t, r^t)ds$ Similarly we show that r(t) is also a solution of (3.1) and (3.2). To prove ρ and r are minimal and maximal solutions of (3.1) and (3.2) respectively, we need to show that if $x(t)$ is any solution of (3.1) and (3.2) with $x_{t_0} = \phi_0$, $x^T = \psi_0$ such that $v_0 \le x \le w_0$ on J then $v_n \le \rho \le r \le w_n$ on J. Next we consider

$$\begin{aligned}
p &= v_1 - x \\
p' &= v_1' - x' \\
&= \{f(t, v_0, Sv_0, v_0t, w_0^t) - f(t, x, Sx, x_t, x^t)\} - M_1(v_1 - v_0) \\
&\quad - M_2[Sv_1 - Sv_0] - M_3 \int_{-h_1}^{0} (v_{1t} - v_{0t})(s)ds \\
&\le \left[-M_1(v_0 - x) - M_2[Sv_0 - Sx] - M_3 \int_{-h_1}^{0} (v_{0t} - x_t)(s)ds \right] - M_1(v_1 - v_0) \\
&\quad - M_2[Sv_1 - Sv_0] - M_3 \int_{-h_1}^{0} (v_{1t} - v_{0t})(s)ds \\
&\le -M_1(v_0 - x + v_1 - v_0) - M_2[Sv_0 - Sx + Sv_1 - Sv_0] \\
&\quad - M_3 \int_{-h_1}^{0} (v_{0t} - x_t + v_1t - v_0t)(s)ds \\
&\le -M_1(v_1 - x) - M_2[Sv_1 - Sx] - M_3 \int_{-h_1}^{0} (v_1t - x_t)(s)ds \\
&\le -M_1p(t) - M_2Sp(t) - M_3 \int_{-h_1}^{0} p_t(s)ds
\end{aligned}$$

But $p_{t_0} \leq v_{1t_0} - x_{t_0} = \phi_0 - \phi_0 = 0$. Now M_1, M_2, M_3 are such that hypothesis of Lemma 3.1 is satisfied. Thus we can conclude that $p(t) \leq 0$ on J, $v_1 \leq x$ on J, similarly we show that $x \leq w_1$ on J. Therefore $v_1 \leq x \leq w_1$ on J. Assume the result is true for $n = k$, $v_0 \leq v_1 \leq \cdots \leq v_k \leq \cdots \leq \rho \leq x \leq r \leq w_k \leq \cdots \leq w_1 \leq w_0$ on J. Next we prove this for $n = k + 1$. For this consider

$$\begin{aligned}
p &= v_{k+1} - x \\
p' &= v'_{k+1} - x' \\
&= f(t, v_k, Sv_k, v_k t, w_k^t) - M_1(v_{k+1} - v_k) - M_2[Sv_{k+1} - Sv_k] \\
&\quad - M_3 \int_{-h_1}^{0} (v_{k+1t} - v_{kt})(s)ds - f(t, x, Sx, x_t, x^t)
\end{aligned}$$

using the hypothesis H_2 and simplifying we get,

$$\begin{aligned}
&\leq -M_1(v_{k+1} - x) - M_2[Sv_{k+1} - Sx] - M_3 \int_{-h_1}^{0} (v_{k+1t} - x)(s)ds \\
&\leq -M_1 p(t) - M_2 Sp(t) - M_3 \int_{-h_1}^{0} p_t(s)ds
\end{aligned}$$

But $p_{t_0} \leq v_{k+1_{t_0}} - x_{t_0} = \phi_0 - \phi_0 = 0$. Now M_1, M_2, M_3 are such that hypothesis of Lemma 3.1 is satisfied. Thus we can conclude that $p(t) \leq 0$ on J. $v_{k+1} \leq x$ on J. Similarly we show that $x \leq w_{k+1}$ on J. Therefore by mathematical induction we conclude that for all n, $v_n \leq x \leq w_n$, $t \in J$. By taking the limit as $n \to \infty$ we conclude that $\rho \leq x \leq r$, $t \in J$. Therefore $v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq \rho \leq x \leq r \leq \cdots \leq w_n \leq \cdots w_1 \leq w_0$ on J. Therefore ρ and r are coupled minimal and maximal solutions of (3.1) and (3.2). Hence the theorem. $\square$

REFERENCES

[1] Z. Drici, F. A. McRae, and J. Vasundhara Devi, quasilinearization for functional differential equations with retardation and anticipation, *Non-linear Analysis* (2008), Doi: 10.1016/j.na 2008.02.079.

[2] A. Augustynowiez, H. Lesceszynski, and W. Walter, Nonlinear Ordinary Differential Equations with Advanced arguments, *Non-linear Analysis* **53** (2008), 495–505.

[3] S. G. Deo, and V. Lakshmikantham, *Method of Variation of Parameters for Dynamic Systems*, Gordam and Breach Publishers, Amsterdam, 1998.

[4] S. G. Deo, and V. Lakshmikantham and V. Raghavendra, *Text book of Ordinary Differential Equations*, Tata McGraw-Hill Publishing Company Limited, New Delhi, 1997.

[5] D. M. Dubois, Extension of the Kaldo-Kalacki model of business cycle with computational anticipated capital stock, *Organisational and Social change*, **1** (2004), 63–80.

[6] D. M. Dubois, Theory of incursive synchronization and application to the anticipation of a chaotic epidemic, *International Tour of Computing Anticipatory System*, **10** (2001), 3–18.

[7] T. Gnana Bhaskar, and V. Lakshmikantham, Functional Differential Systems with Retardation and Anticipation, *Non Linear Analysis Real world problems* **8**(3) 865–871.

[8] T. Gnana Bhaskar, V. Lakshmikantham and J. Vasundhara Devi, Monotone Iterative Technique for functional differential equations with retardation and anticipation, *Non Linear Analysis*, **66** (2007), 2237–2242.

[9] S. K. Koul, and A. S. Vatsala, Monotone Method for Integro Differential Equation with periodic Boundary Conditions, *Applicable Analysis*, (1986), Vol.21, 297–305.

[10] G. S. Ladde, V. Lakshmikantham and A. S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman Publishing INC., 1985.

[11] V. Lakshmikantham, B. G. Zhang, Monotone Iterative Technique for Delay Differential Equation, *Applicable Analysis*, (1986), Vol. 22, 227–233.

[12] V. Lakshmikantham, S. Koksal, it Monotone Flows and Rapid Convergence for Nonlinear Partial Differential Equations, *Taylors and Francis*, New York, 2003.

[13] V. Lakshmikantham, M. Rama Mohana Rao, *Theory of Integro Differential Equations*, Gordon and Breach Science Publishers, S.A, 1995.

[14] V. Lakshmikantham, S.Leela, *Differential and Integral Inequalities*, Volume I and II. Academic Press, New York, 1969.

[15] V. Lakshmikantham, and J. Vasundhara Devi , Functional Differential Equations with Anticipation, it Nonlinear studies Vol. 14, **3** (2007), 235–240.