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OSCILLATION CRITERIA FOR CERTAIN FOURTH ORDER NONLINEAR DIFFERENCE EQUATIONS

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ABSTRACT. We shall establish some new criteria for the oscillation of the fourth order nonlinear difference equation

$$\Delta^2\left(a(k)\left(\Delta^2 x(k)\right)^{\alpha}\right)+q(k)\,f\left(x\left(g(k)\right)\right)=0$$

via comparison with some difference equations of less order whose oscillatory characters are known.

1. INTRODUCTION

Consider the fourth order nonlinear difference equation

$$\Delta^2\left(a(k)\left(\Delta^2 x(k)\right)^{\alpha}\right)+q(k)\,f\left(x\left(g(k)\right)\right)=0 \tag{1}$$

where Δ is the forward difference operator defined by $\Delta x(k)=x(k+1)-x(k)$ and α is the ratio of positive odd integers. We shall assume that $g,a:\mathbb{N}(k)\to\mathbb{R}^+=(0,\infty)$ for some $k\in\mathbb{N}=\{0,1,\dots\}$ and $\mathbb{N}(n_0)=\{n_0,n_0+1,\dots\}$ where $n_0\in\mathbb{N}$, $g\in\bar{G}:=\{g:\mathbb{N}(k)\to\mathbb{N}$ for some $k\in\mathbb{N}: g(k)\leqslant k$, $g(k)$ is non-decreasing and $\lim_{k\to\infty}g(k)=\infty\}$, and $f:\mathbb{R}\to\mathbb{R}$ is continuous satisfying $xf(x)>0$ for $x\neq 0$ and f is non-decreasing.

By a solution of equation (1), we mean a nontrivial sequence $\{x(k)\}$ satisfying equation (1) for all $k\in\mathbb{N}(K)$ where K is some nonnegative integer. A solution $\{x(k)\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative and it is nonoscillatory otherwise. Equation (1) is said to be oscillatory if all its solutions are oscillatory. Determining oscillation criteria for difference equations has received a great deal of attention in the last two decades, see for example the Monographs of Agarwal et. al. [1]–[3]. This interest is motivated by the importance

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of difference equations in the numerical solutions of differential equations. Compared to equations of order less than or equal to two, the study of higher order equations and in particular fourth order equations, has received considerably less attention see [4]–[9]. In this paper, we shall establish some new criteria for the oscillation of equation (1) via comparison with some equations of less order, whose oscillatory characters are known.

2. MAIN RESULTS

For $k \geqslant n_0 \in \mathbb{N}$, we let

$$A[k, n_0] = \sum_{s=n_0}^{k-1} \sum_{i=n_0}^{s-1} \left(\frac{i}{a(i)} \right)^{1/\alpha}.$$

In the following results, we assume

$$\sum_{i=n_0}^{\infty} a^{-1/\alpha}(i) = \infty \tag{2}$$

and

$$-f(-xy) \geqslant f(xy) \geqslant f(x) f(y) \text{ for } xy > 0. \tag{3}$$

Now, we prove the following result:

Theorem 1. *Let conditions* (2) *and* (3) *hold and assume that there exists a non-decreasing sequence* $\{\xi(k)\}$ *such that* $g(k) < \xi(k) < k$ *for* $k \geqslant n_0$. *If the first order difference equation*

$$\Delta y(k) + c_1 q(k) f(A[g(k), n_0]) f\left(y^{1/\alpha}(g(k))\right) = 0 \tag{4}$$

for some constant c_1, $0 < c_1 < 1$ *is oscillatory and there exists a constant* c_2, $0 < c_2 < 1$, *such that all bounded solutions of the second order difference equation*

$$\Delta^2 z(k) - c_2 q(k) f(g(k)) f\left(\frac{\xi(k) - g(k)}{a^{1/\alpha}(\xi(k))} \right) f\left(z^{1/\alpha}(\xi(k))\right) = 0 \tag{5}$$

is oscillatory, then equation (1) *is oscillatory.*

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (1), say $x(k) > 0$ for $k \geqslant n_0 \in \mathbb{N}$. There exists an $n_1 \geqslant n_0$ such that the following two possibilities are considered:

(I) $\Delta\left(a(k)(\Delta^2 x(k))^\alpha\right) > 0$,

$$a(k)\left(\Delta^2 x(k)\right)^\alpha > 0 \text{ and } \Delta x(k) > 0 \text{ for } k \geqslant n_1, \tag{6}$$

(II) $\Delta\left(a(k)(\Delta^2 x(k))^\alpha\right) > 0$,

$$a(k)\left(\Delta^2 x(k)\right)^\alpha < 0 \text{ and } \Delta x(k) > 0 \text{ for } k \geqslant n_1. \tag{7}$$

Case (I). There exists an $n_2 \geqslant n_1$ and a constant b_1, $0 < b_1 < 1$ such that

$$y(k) \geqslant b_1 k \Delta y(k) \text{ for } k \geqslant n_2, \tag{8}$$

where $y(k) = a(k)\left(\Delta^2 x(k)\right)^\alpha$ for $k \geqslant n_2$. Thus,

$$\Delta^2 x(k) \geqslant b \left(\frac{k}{a(k)}\right)^{1/\alpha} (\Delta y(k))^{1/\alpha} \text{ for } k \geqslant n_2, \tag{9}$$

where $b = b_1^{1/\alpha}$. Summing this inequality twice, one can easily get

$$x(k) \geqslant bA[k, n_2] (\Delta y(k))^{1/\alpha} \text{ for } k \geqslant n_2. \tag{10}$$

Now, there exists an $n_3 \geqslant n_2$ such that $g(k) > n_2$ for $k \geqslant n_3$ and

$$x(g(k)) \geqslant bA[g(k), n_2] (\Delta y(g(k)))^{1/\alpha} \text{ for } k \geqslant n_3. \tag{11}$$

Using (3) and (11) in equation (1), we get

$$\Delta z(k) + f(b) q(k) f(A[g(k), n_2]) f\left(z^{1/\alpha}(g(k))\right) \leqslant 0 \text{ for } k \geqslant n_3, \tag{12}$$

where $z(k) = \Delta y(k)$ for $k \geqslant n_3$. Summing both sides of (12) from $k+1 \geqslant n_0$ to u and letting $u \to \infty$, we have

$$z(k) \geqslant f(b) \sum_{j=k+1}^{\infty} q(j) f(A[g(j), n_2]) f\left(z^{1/\alpha}(g(k))\right).$$

The sequence $\{z(k)\}$ is obviously non-increasing for $k \geqslant n_3$. Hence by a result in [3], we conclude that there exists a positive solution $\{y(k)\}$ of equation (4) with $\lim_{k\to\infty} y(k) = 0$, which is a contradiction.

Case (II). There exist a constant c, $0 < c < 1$ and an $n_2 \geqslant n_1$ such that

$$x(g(k)) \geqslant c g(k) \Delta x(g(k)) \text{ for } k \geqslant n_2. \tag{13}$$

Using (3) and (13) in equation (1), we obtain

$$\Delta^2 \left(a(k) (\Delta y(k))^\alpha\right) + \bar{c} f(g(k)) f(y(g(k))) \leqslant 0 \text{ for } k \geqslant n_2, \tag{14}$$

where $y(k) = \Delta x(k)$ for $k \geqslant n_2$, $\bar{c} = f(c)$. Clearly, we see that $y(k) > 0$, $\Delta y(k) < 0$ and $\Delta\left(a(k)(\Delta y(k))^\alpha\right) > 0$ for $k \geqslant n_2$. Now, for $t \geqslant s \geqslant n_2$ we have

$$y(s) \geqslant (t - s)(-\Delta y(t)).$$

Replacing s and t by $g(k)$ and $\xi(k)$ respectively, we find

$$\begin{aligned} y(g(k)) &\geqslant (\xi(k) - g(k))(-\Delta y(\xi(k))) \\ &:= \frac{\xi(k) - g(k)}{a^{1/\alpha}(\xi(k))} \left(-a(\xi(k)) (\Delta y(\xi(k)))^\alpha\right)^{1/\alpha} \text{ for } k \geqslant n_3 \geqslant n_2. \end{aligned} \tag{15}$$

Using (3) and (15) in (14) we get

$$\Delta^2 z(k) \geqslant \bar{c} q(k) f(g(k)) f\left(\frac{\xi(k) - g(k)}{a^{1/\alpha}(\xi(k))}\right) f\left(z^{1/\alpha}(\xi(k))\right) \text{ for } k \geqslant n_3, \tag{16}$$

where $z(k) = -a(k)(\Delta y(k))^{\alpha}$ for $k \geqslant n_3$. The rest of the proof is similar to that in [3] and hence is omitted. □

Next, we let

$$Q(k) = \left(\frac{1}{a(k)} \sum_{s=k+1}^{\infty} \sum_{j=s+1}^{\infty} q(j) \right)^{1/\alpha} \text{ for } k \geqslant n_0 \in \mathbb{N}$$

and establish the following result.

Theorem 2. *Let the hypothesis of Theorem* 1 *hold except that we replace "all bounded solutions of equation* (5) *are oscillatory" with the equation*

$$\Delta^2 z(k) + Q(k) f^{1/\alpha}(z(g(k))) = 0 \tag{17}$$

is oscillatory. Then the conclusion of Theorem 1 *holds.*

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (1), say $x(k) > 0$ for $k \geqslant n_0 \in \mathbb{N}$. As in the proof of Theorem 1, there are two cases to consider (I) and (II). The proof of Case (I) is similar to that of Theorem 1 - Case (I) and hence is omitted.

Case (II). Summing equation (1) twice from $k+1 > n_0$ to u and letting $u \to \infty$, we get

$$\begin{aligned} -\Delta^2 x(k) &\geqslant \left(\frac{1}{a(k)} \sum_{s=k+1}^{\infty} \sum_{j=s+1}^{\infty} q(j) f(x(g(k))) \right)^{1/\alpha} \\ &\geqslant Q(k) f^{1/\alpha}(x(g(k))) \text{ for } k \geqslant n_0. \end{aligned} \tag{18}$$

Summing both sides of (5) from $k+1 \geqslant n_0$ to u and letting $u \to \infty$, we find

$$\Delta x(k) \geqslant \sum_{j=k+1}^{\infty} Q(j) f^{1/\alpha}(x(g(j))). \tag{19}$$

Summing both sides of (19) from n_0 to $k-1 \geqslant n_0$, we have

$$x(k) \geqslant x(n_0) + \sum_{s=n_0}^{k-1} \sum_{j=s+1}^{\infty} Q(j) f^{1/\alpha}(x(g(j))).$$

Now, we define a sequence $\{y_m(k)\}$ by

$$\begin{aligned} y_0(k) &= x(k) \\ y_{m+1}(k) &= x(n_0) + \sum_{s=n_0}^{k-1} \sum_{j=s+1}^{\infty} Q(j) f^{1/\alpha}(y_m(g(j))), \; m = 0, 1, \ldots, \; k \geqslant n_0. \end{aligned}$$

It is easy to check that the sequence $\{y_m(k)\}$ is well-defined as an increasing sequence and satisfies

$$x(n_0) \leqslant y_m(k) \leqslant x(k) \text{ for } k \geqslant n_0 \text{ and } m = 0, 1, \ldots.$$

Hence, there exists a sequence $\{y(k)\}$ for $k \geqslant n_0$ such that

$$\lim_{m\to\infty} y_m(k) = y(k),$$

and

$$x(n_0) \leqslant y(k) \leqslant x(k) \text{ for } k \geqslant n_0.$$

From the Lebesge Convergence Theorem, it follows that

$$x(k) = x(n_0) + \sum_{s=n_0}^{k-1} \sum_{j=s+1}^{\infty} Q(j) f^{1/\alpha}(x(g(j))), \text{ for } k \geqslant n_0.$$

Taking the difference twice, we conclude that $\{x(k)\}$ is a nonoscillatory solution of equation (17), a contradiction. This completes the proof. □

Next, we establish the following comparison result:

Theorem 3. *Let the hypotheses of Theorem* 2 *hold except that we replace "the equation* (17) *is oscillatory" with the first order difference equation*

$$\Delta w(k) + \theta Q(k) f^{1/\alpha}(g(k)) f^{1/\alpha}(w(g(k))) = 0 \tag{20}$$

is oscillatory for every θ, $0 < \theta < 1$. Then the conclusion of the Theorem 2 *holds.*

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (1), say $x(k) > 0$ for $k \geqslant n_0 \in \mathbb{N}$. As in the proof of Theorem 1, we consider the two Cases (I) and (II). The proof of Case (I) is similar to that of Theorem 1 - Case (I) and hence is omitted.

Case (II). Proceeding as in the proof of Theorem 2 - Case (II) and obtain the inequality (18). Now, there exist a constant c, $0 < c < 1$ and an $n_1 \geqslant n_0$ such that

$$x(g(k)) \geqslant c g(k) (\Delta x(g(k))) \text{ for } k \geqslant n_1. \tag{21}$$

Using (21) in (18) we get

$$\Delta w(k) + \bar{c} Q(k) f^{1/\alpha}(g(k)) f^{1/\alpha}(w(g(k))) \leqslant 0 \text{ for } k \geqslant n_1,$$

where $\bar{c} = f^{1/\alpha}(c)$ and $w(k) = \Delta x(k)$ for $k \geqslant n_1$. The rest of the proof is similar to that of Theorem 1- Case (I) and hence is omitted. □

We may combine equations (4) and (20) in one by letting

$$\tilde{Q}(k) \geqslant \min\left\{Q(k) f^{1/\alpha}(g(k)), q(k) f(A[g(k), n_0])\right\} \text{ for } k \geqslant n_0,$$

and

$$f^{1/\alpha}(u) = f\left(u^{1/\alpha}\right) \text{ for } u \neq 0.$$

Thus, we get

Theorem 4. *Let conditions* (2) *and* (3) *hold. If the equation*

$$\Delta v(k) + \theta \tilde{Q}(k) f\left(v^{1/\alpha}(g(k))\right) = 0 \tag{22}$$

is oscillatory for every constant θ, $0 < \theta < 1$, then equation (1) *is oscillatory.*

As an example, we consider a special case of equation (1), namely the equation

$$\Delta^2 \left(a\left(k\right)\left(\Delta^2 x\left(k\right)\right)^{\alpha}\right) + q\left(k\right) x^{\beta}\left(k-\tau+1\right) = 0 \tag{23}$$

where $\tau \geqslant 1$ is a real number and β is the ratio of positive odd integers. Clearly

$$\tilde{Q}\left(k\right) \geqslant \min\left\{ g^{\beta/\alpha}\left(k\right)\left(\frac{1}{a\left(k\right)}\sum_{s=k+1}^{\infty}\sum_{j=s+1}^{\infty} q\left(j\right)\right)^{1/\alpha}, q\left(k\right)\left(\sum_{s=n_0}^{k-\tau-2}\sum_{i=n_0}^{s-1}\left(\frac{i}{a\left(i\right)}\right)^{1/\alpha}\right)^{\beta}\right\}.$$

Now, we have the following immediate result.

Corollary 5. *Let condition* (2) *hold. Equation* (23) *is oscillatory if one of the following conditions holds:*

(O_1) $\alpha = \beta$ *and* $\lim\limits_{k\to\infty} \sum_{i=k-\tau}^{k-1} \tilde{Q}\left(i\right) > \left(\frac{\tau}{\tau+1}\right)^{\tau-1}$;

(O_2) $0 < \beta < \alpha$ *and* $\sum_{i=n_0\in\mathbb{N}}^{\infty} \tilde{Q}\left(i\right) = \infty$.

Remark 6. We note that the results presented in this paper are not applicable to equations of type (1) when $g\left(k\right) = k$.

Remark 7. The results of this paper can be extended easily to dynamic equations on time-scales of the type

$$\left(a\left(t\right)\left(x^{\Delta\Delta}\left(t\right)\right)^{\alpha}\right)^{\Delta\Delta} + q\left(t\right) f\left(x\left(g\left(t\right)\right)\right) = 0.$$

The details are left to the reader.

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