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FRACTIONAL DIFFERENCE INEQUALITIES OF BIHARI TYPE

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ABSTRACT. In this paper, using the properties of $\nabla^{-\alpha}$, some discrete fractional inequalities of Bihari-type are established.

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1. INTRODUCTION

In the description of observations of evolution phenomena, difference equations play an important role, because most of the observations are measurements of time and are discrete. The theory of difference equations has been developed as a natural discrete analogue of corresponding theory of differential equations.

Though the idea of fractional calculus has been in existence for more than three centuries, much is not done in the theory of nonlinear fractional dynamical systems. The discrete counterpart of the fractional differential equations is discussed in [4].

In J. B. Diaz and T. J. Osler [5], the fractional difference operator is defined by the index of differencing to be any real or complex number, in the expression of n^{th} difference of a function. Later, R. Hirota [9], defined the difference operator ∇^{α} as the first n terms of the Taylor series of $[\frac{1-B}{\varepsilon}]^{\alpha}$ where ε is interval length. G. V. S. R. Deekshitulu and J. Jagan Mohan [4] modified the definition of Atsushi Nagai [1] for $0 < \alpha \leq 1$ in such a way that the expression for ∇^{α} does not involve any difference operator and using which some basic difference inequalities have been established. In this paper using the definition given in [4], discrete fractional Bihari inequality and more general inequalities have been obtained.

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2. PRELIMINARIES

Let u_n be any function defined for $n \in \mathbb{N}_0^+$ where $\mathbb{N}_a^+ = \{a, a+1, a+2, \dots\}$ for $a \in \mathbb{Z}$. Hirota [9] took the first n terms of Taylor series of $\Delta_{-n}^{\alpha} = \varepsilon^{-\alpha}(1-B)^{\alpha}$ and gave the following definition.

Definition 2.1. Let $\alpha \in \mathbb{R}$. Then difference operator of order α is defined by

$$\Delta_{-n}^{\alpha} u_n = \begin{cases} \varepsilon^{-\alpha} \sum_{j=0}^{n-1} \binom{\alpha}{j} (-1)^j u_{n-j}, & \alpha \neq 1,2,\dots \\ \varepsilon^{-m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j u_{n-j}, & \alpha = m \in \mathbb{Z}_{>0}. \end{cases} \tag{2.1}$$

Here $\binom{a}{n}$, $(a \in \mathbb{R}, n \in \mathbb{Z})$ stands for a binomial coefficient defined by

$$\binom{a}{n} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)} & n > 0 \\ 1 & n = 0 \\ 0 & n < 0. \end{cases} \tag{2.2}$$

In 2002, Atsushi Nagai [1] introduced another definition of fractional difference which is a slight modification of Hirota's fractional difference operator.

Definition 2.2. Let $\alpha \in \mathbb{R}$ and m be an integer such that $m-1 < \alpha \leq m$. The difference operator $\Delta_{*,-n}$ of order α is defined as

$$\Delta_{*,-n}^{\alpha} u_n = \Delta_{-n}^{\alpha-m} \Delta_{-n}^{m} u_n = \varepsilon^{m-\alpha} \sum_{j=0}^{n-1} \binom{\alpha-m}{j} (-1)^j \Delta_{-(n-j)}^{m} u_{n-j}. \tag{2.3}$$

G. V. S. R. Deekshitulu and J. Jagan Mohan [4] gave the following definition.

Definition 2.3. Let $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq 1$. The difference operator ∇ of order α is defined as

$$\nabla^{\alpha} u_n = \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla u_{n-j}. \tag{2.4}$$

Remark 1. For any $\alpha \in \mathbb{R}$ $(0 < \alpha \leq 1)$,

$$\nabla^{-\alpha} u_n = \sum_{j=0}^{n-1} \binom{j+\alpha}{j} \nabla u_{n-j}. \tag{2.5}$$

Further $\nabla^{\alpha} u_n$ and $\nabla^{-\alpha} u_n$ can be expressed in the terms of the arguments of u_n as for any $\alpha \in \mathbb{R}$ $(0 < \alpha \leq 1)$,

$$\nabla^{\alpha} u_n = u_n - \binom{n-1-\alpha}{n-1} u_0 - \alpha \sum_{j=1}^{n-1} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} u_{n-j} \tag{2.6}$$

and

$$\nabla^{-\alpha} u_n = u_n - \binom{n-1+\alpha}{n-1} u_0 + \alpha \sum_{j=1}^{n-1} \frac{1}{(j+\alpha)} \binom{j+\alpha}{j} u_{n-j}. \tag{2.7}$$

Remark 2. The difference operator ∇ of order α satisfies the following properties.

i For any real numbers α and β, $\nabla^{\alpha}\nabla^{\beta}u_n = \nabla^{\alpha+\beta}u_n$.

ii For any constant 'c', $\nabla^{\alpha}[cu_n + v_n] = c\nabla^{\alpha}u_n + \nabla^{\alpha}v_n$ where v_n be any function defined for $n \in \mathbb{N}_0^+$.

iii For $\alpha \in \mathbb{R}$, $\nabla^{\alpha}(u_n v_n) = \sum_{m=0}^{n-1} \binom{\alpha}{m}[\nabla^{\alpha-m}u_{n-m}][\nabla^{\alpha}v_n]$.

iv For $\alpha \in \mathbb{R}$, $\nabla^{\alpha}u_0 = 0$ and $\nabla^{\alpha}u_1 = u_1 - u_0 = \nabla u_1$.

v For $\alpha \in \mathbb{R}$, $\nabla^{\alpha}\nabla^{-\alpha}u_n = \nabla^{-\alpha}\nabla^{\alpha}u_n = u_n - u_0$.

vi For $\alpha \in \mathbb{R}$, $\nabla^{\alpha}\nabla^{-\alpha}(u_n - u_0) = \nabla^{-\alpha}\nabla^{\alpha}(u_n - u_0) = u_n - u_0$.

Using the Definition 2.3 the following basic fractional difference inequalities have been established [4] and are useful in the study of the main results.

Theorem 2.1. *Let $n \in \mathbb{N}_0^+$, $0 \le r < \infty$ and f(n,r) be a non decreasing function in r for any fixed n. Let v_n and w_n be two functions defined for $n \in \mathbb{N}_0^+$. Suppose that for $n \ge 0$ and $0 < \alpha \le 1$ the inequalities*

$$\nabla^{\alpha}v_{n+1} \le f(n, v_n), \tag{2.8}$$

$$\nabla^{\alpha}w_{n+1} \ge f(n, w_n) \tag{2.9}$$

hold. Then $v_0 \le w_0$ implies

$$v_n \le w_n \quad \text{for all } n \ge 0. \tag{2.10}$$

Theorem 2.2. *Let $m_1(n, r)$ and $m_2(n, r)$ be two non negative functions defined for $n \in \mathbb{N}_0^+$, $0 \le r < \infty$ and non decreasing with respect to r for any fixed $n \in \mathbb{N}_0^+$. Let y_n be a function defined for $n \in \mathbb{N}_0^+$ and that*

$$m_1(n, y_n) \le \nabla^{\alpha}y_{n+1} \le m_2(n, y_n) \tag{2.11}$$

for all $n \in \mathbb{N}_0^+$ and $0 < \alpha \le 1$. Let v_n and w_n be the solutions of the difference equations

$$\nabla^{\alpha}v_{n+1} = m_1(n, v_n), \quad v(0) = v_0, \tag{2.12}$$

$$\nabla^{\alpha}w_{n+1} = m_2(n, w_n), \quad w(0) = w_0 \tag{2.13}$$

and suppose that $v_0 \le y_0 \le w_0$. Then

$$v_n \le y_n \le w_n \quad \text{for all } n \in \mathbb{N}_0^+. \tag{2.14}$$

Definition 2.4. Let f be a function defined from N_0^+ to $\mathbb{R}^+$. Then f is said to be subadditive if, for all $m, n \in N_0^+$ we have

$$f(m + n) \le f(m) + f(n).$$

Definition 2.5. Let f be a function defined from N_0^+ to $\mathbb{R}^+$. Then f is said to be submultiplicative if, for all $m, n \in N_0^+$ we have

$$f(mn) \le f(m)f(n).$$

3. MAIN RESULTS

In this section, the definition given in (2.5) is expressed in a more convenient form by rearranging the coefficients. Using this, discrete Bihari inequality and more general inequalities are established. For any $\alpha \in \mathbb{R}$ $(0 < \alpha \leq 1)$, we have

$$\begin{aligned}
\nabla^{-\alpha} u_n &= \sum_{j=0}^{n-1} \binom{j+\alpha}{j} \nabla u_{n-j} \\
&= u_n - \binom{n-1+\alpha}{n-1} u_0 + \alpha \sum_{j=1}^{n-1} \frac{1}{(j+\alpha)} \binom{j+\alpha}{j} u_{n-j} \\
&= \alpha \sum_{j=0}^{n-1} \frac{1}{(j+\alpha)} \binom{j+\alpha}{j} u_{n-j} - \binom{n-1+\alpha}{n-1} u_0 \\
&= \sum_{j=0}^{n-1} \frac{\alpha}{(j+\alpha)} \frac{(j+\alpha)\Gamma(j+\alpha)}{\Gamma(j+1)\alpha\Gamma\alpha} u_{n-j} - \binom{n-1+\alpha}{n-1} u_0 \\
&= \sum_{j=0}^{n-1} \frac{\Gamma(j+\alpha)}{\Gamma(j+1)\Gamma\alpha} u_{n-j} - \binom{n-1+\alpha}{n-1} u_0 \\
&= \sum_{j=0}^{n-1} \binom{j+\alpha-1}{j} u_{n-j} - \binom{n-1+\alpha}{n-1} u_0 \\
\text{or} \quad \nabla^{-\alpha} u_n &= \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} u_j - \binom{n-1+\alpha}{n-1} u_0. \qquad (3.1)
\end{aligned}$$

If $u_0 = 0$ then

$$\nabla^{-\alpha} u_n = \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} u_j \qquad (3.2)$$

$$\text{and} \quad \nabla^{\alpha}\nabla^{-\alpha} u_n = \nabla^{-\alpha}\nabla^{\alpha} u_n = u_n.$$

Theorem 3.1. *Let u_n and v_n be non negative functions defined for $n \in \mathbb{N}_0^+$ such that $u_0 = v_0$. For $0 < \alpha \leq 1$, if $u_n \leq v_n$ for each $n \in \mathbb{N}_1^+$ then $\nabla^{-\alpha} u_n \leq \nabla^{-\alpha} v_n$.*

Proof. Since $u_j \leq v_j$ and $\binom{n-j+\alpha-1}{n-j} = \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)\Gamma(\alpha)}$ is nonnegative for $j = 1, 2, \ldots, n$,

$$\sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} u_j \leq \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} v_j$$

implies

$$\sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} u_j - \binom{n-1+\alpha}{n-1} u_0 \leq \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} v_j - \binom{n-1+\alpha}{n-1} v_0$$

then from (3.1), $\nabla^{-\alpha} u_n \leq \nabla^{-\alpha} v_n$. With this basic idea now we proceed to prove the monotone property of $\nabla^{-\alpha}$. ∎

Theorem 3.2. *Let u_n be non negative and non decreasing function defined for $n \in \mathbb{N}_0^+$ then $\nabla^{-\alpha} u_n$ is also non negative and non decreasing function defined for $n \in \mathbb{N}_0^+$.*

Proof. From Remark 1 we have

$$\nabla^{-\alpha} u_n = \sum_{j=0}^{n-1} \binom{j+\alpha}{j} \nabla u_{n-j}.$$

For $j = 0, 1, \ldots, (n-1)$, $\binom{j+\alpha}{j} = \frac{\Gamma(j+\alpha+1)}{\Gamma(j+1)\Gamma(\alpha+1)}$ is nonnegative. Since u_n is non negative and non decreasing function defined for $n \in \mathbb{N}_0^+$, ∇u_{n-j} is also nonnegative for $j = 0, 1, \ldots, (n-1)$. Thus $\nabla^{-\alpha} u_n$ is nonnegative for $n \in \mathbb{N}_0^+$. Now

$$\begin{aligned} \nabla^{-\alpha} u_n - \nabla^{-\alpha} u_{n-1} &= \nabla^{-\alpha}(u_n - u_{n-1}) \\ &= \nabla^{-\alpha}(\nabla u_n) \\ &= \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} \nabla u_j - \binom{n-1+\alpha}{n-1} \nabla u_0 \\ &= \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} \nabla u_j \quad (\text{since } \nabla u_0 = 0). \end{aligned}$$

For $j = 1, 2, \ldots, n$, $\binom{n-j+\alpha-1}{n-j} = \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)\Gamma(\alpha)}$ is nonnegative. Since u_n is non negative and non decreasing function defined for $n \in \mathbb{N}_0^+$, ∇u_j is also nonnegative for $j = 1, 2, \ldots, n$. So $(\nabla^{-\alpha} u_n - \nabla^{-\alpha} u_{n-1})$ is nonnegative for $n \in \mathbb{N}_0^+$. Thus $\nabla^{-\alpha} u_n - \nabla^{-\alpha} u_{n-1} \geq 0$ i.e., $\nabla^{-\alpha} u_n \geq \nabla^{-\alpha} u_{n-1}$. Therefore, for $n \in \mathbb{N}_0^+$,

$$u_{n-1} \leq u_n \Rightarrow \nabla^{-\alpha} u_{n-1} \leq \nabla^{-\alpha} u_n.$$

Hence $\nabla^{-\alpha} u_n$ is non negative and non decreasing function defined for $n \in \mathbb{N}_0^+$. □

Remark 3. Let u_n be non negative and non decreasing function defined for $n \in \mathbb{N}_0^+$ then $\nabla^{\alpha} u_n$ is non negative function defined for $n \in \mathbb{N}_0^+$.

Proof. From (2.4) we have

$$\nabla^{\alpha} u_n = \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla u_{n-j}.$$

For $j = 0, 1, \ldots, (n-1)$, $\binom{j-\alpha}{j} = \frac{\Gamma(j-\alpha+1)}{\Gamma(j+1)\Gamma(1-\alpha)}$ is nonnegative. Since u_n is non negative and non decreasing function defined for $n \in \mathbb{N}_0^+$, ∇u_{n-j} is also nonnegative for $j = 0, 1, \ldots, (n-1)$. Thus $\nabla^{\alpha} u_n$ is nonnegative for $n \in \mathbb{N}_0^+$. □

Theorem 3.3. *Let u_n, a_n, b_n and y_n be nonnegative functions defined for $n \in \mathbb{N}_0^+$ such that $y_0 = 0$ and c be a nonnegative constant. Let $f(n, r)$ be a nonnegative function defined for $n \in \mathbb{N}_0^+$, $0 \leq r < \infty$ and non decreasing in r for any fixed $n \in \mathbb{N}_0^+$. If*

$$u_n \leq a_n + b_n \left[c + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j f(j, u_j) \right] \tag{3.3}$$

for $n \in \mathbb{N}_0^+$, *then*

$$u_n \leq a_n + b_n r_n \tag{3.4}$$

for $n \in \mathbb{N}_0^+$, *where* r_n *is the solution of the difference equation*

$$\nabla^\alpha r_{n+1} = y_n f(n, a_n + b_n r_n), \quad r(0) = c \tag{3.5}$$

for $n \in \mathbb{N}_0^+$ *and* $0 < \alpha \leq 1$.

Proof. Define a function z_n by

$$z_n = c + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j f(j, u_j)$$

i.e.

$$z_{n+1} = c + \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} y_j f(j, u_j)$$
$$= \nabla^{-\alpha} y_j f(j, u_j) + c$$

Using Remark 2(v), $\nabla^\alpha z_{n+1} = y_n f(n, u_n) - y_0 f(0, u_0) = y_n f(n, u_n)$, $z_0 = c$. Further (3.3) reduces to $u_n \leq a_n + b_n z_n$. Since $f(n, r)$ is non decreasing in r, we get

$$\nabla^\alpha z_{n+1} = y_n f(n, u_n) \leq y_n f(n, a_n + b_n z_n). \tag{3.6}$$

By using Theorem 2.2 for (3.5) and (3.6) we have $z_n \leq r_n$. Then $u_n \leq a_n + b_n z_n \leq a_n + b_n r_n$. Hence the proof. □

Theorem 3.4 (Discrete Bihari Inequality)**.** *Let* u_n *and* y_n *be non negative functions defined for* $n \in \mathbb{N}_0^+$ *such that* $y_0 = 0$ *and* c *be a non negative constant. Let* g *be a non decreasing positive function defined on* R^+. *If*

$$u_n \leq c + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(u_j) \tag{3.7}$$

for $n \in \mathbb{N}_0^+$ *then for* $0 \leq n \leq n_1$; $n, n_1 \in \mathbb{N}_0^+$,

$$u_n \leq G^{-1}\left(G(c) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j\right) \tag{3.8}$$

where G *is the solution of*

$$\nabla^\alpha G(V_{n+1}) = \frac{\nabla^\alpha V_{n+1}}{g(V_n)} \tag{3.9}$$

for any non negative function V_n *defined for* $n \in \mathbb{N}_0^+$ *and* G^{-1} *is the inverse of* G *and* $n_1 \in \mathbb{N}_0^+$ *be chosen so that*

$$G(c) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j \in Dom(G^{-1}) \tag{3.10}$$

for all $n \in \mathbb{N}_0^+$ *such that* $0 \leq n \leq n_1$.

Proof. Define a function v_n by

$$v_n = c + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(u_j)$$

$$\text{i.e.} \quad v_{n+1} = c + \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} y_j g(u_j).$$

Then $v_0 = c$ and by using Remark 2(v), $\nabla^\alpha v_{n+1} = y_n g(u_n) - y_0 g(u_0) = y_n g(u_n)$. By the monotonicity of g and $u_n \leq v_n$ imply $\nabla^\alpha v_{n+1} \leq y_n g(v_n)$. Now from (3.9),

$$\nabla^\alpha G(v_{n+1}) = \frac{\nabla^\alpha v_{n+1}}{g(v_n)} \leq y_n.$$

Let $w_n = \nabla^\alpha G(v_{n+1})$. For $n = 0$, $w_0 = \nabla^\alpha G(v_1) = \frac{\nabla^\alpha v_1}{g(v_0)} = \frac{\nabla^\alpha c}{g(c)} = 0$. Since $w_0 = y_0 = 0$ and $w_n \leq y_n$, by using Theorem 3.1

$$\begin{aligned}
\nabla^{-\alpha} w_n &\leq \nabla^{-\alpha} y_n \\
\text{or} \quad \nabla^{-\alpha}\nabla^\alpha G(v_{n+1}) &\leq \nabla^{-\alpha} y_n \\
\text{i.e.} \quad G(v_{n+1}) - G(v_0) &\leq \nabla^{-\alpha} y_n \quad \text{(from Remark 2(v))} \\
\text{or} \quad G(v_{n+1}) &\leq G(v_0) + \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} y_j.
\end{aligned}$$

Hence, for $0 \leq n \leq n_1$; $n, n_1 \in \mathbb{N}_0^+$,

$$v_{n+1} \leq G^{-1}\left(G(c) + \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} y_j\right)$$

$$\text{i.e.} \quad v_n \leq G^{-1}\left(G(c) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j\right)$$

$$\text{i.e.} \quad u_n \leq G^{-1}\left(G(c) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j\right)$$

where G^{-1} is the inverse of G and $n_1 \in \mathbb{N}_0^+$ be chosen so that

$$G(c) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j \in Dom(G^{-1})$$

for all $n \in \mathbb{N}_0^+$ such that $0 \leq n \leq n_1$. ☐

Remark 4. If $\alpha = 1$, the above discrete fractional Bihari inequality coincides with discrete Bihari inequality [6].

In the next Theorem, a more general Bihari type inequality is obtained.

Theorem 3.5. *Let u_n, a_n, b_n, c_n and y_n be non negative functions defined for $n \in \mathbb{N}_0^+$ such that $y_0 = 0$. Let g be a non decreasing subadditive submultiplicative positive function defined on R^+. If*

$$u_n \leq a_n + b_n \left[c_n + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(u_j) \right] \tag{3.11}$$

for $n \in \mathbb{N}_0^+$ then for $0 \leq n \leq n_2$; $n, n_2 \in \mathbb{N}_0^+$,

$$u_n \leq G^{-1} \left(G(c_0) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} \left[\frac{\nabla^\alpha c_{j+1} + y_j g(a_j)}{g(c_j)} + y_j g(b_j) \right] \right) \tag{3.12}$$

where G is the solution of

$$\nabla^\alpha G(V_{n+1}) = \frac{\nabla^\alpha V_{n+1}}{g(V_n)} \tag{3.13}$$

for any non negative function V_n defined for $n \in \mathbb{N}_0^+$ and G^{-1} is the inverse of G and $n_2 \in \mathbb{N}_0^+$ be chosen so that

$$G(c_0) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} \left[\frac{\nabla^\alpha c_{j+1} + y_j g(a_j)}{g(c_j)} + y_j g(b_j) \right] \in Dom(G^{-1}) \tag{3.14}$$

for all $n \in \mathbb{N}_0^+$ such that $0 \leq n \leq n_2$.

Proof. Define a function v_n by

$$v_n = a_n + b_n \left[c_n + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(u_j) \right]$$

Let

$$z_n = c_n + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(u_j)$$

$$\text{or} \qquad z_{n+1} = c_{n+1} + \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} y_j g(u_j).$$

Then by using Remark 2(v), $\nabla^\alpha z_{n+1} = \nabla^\alpha c_{n+1} + y_n g(u_n) - y_0 g(u_0) = \nabla^\alpha c_{n+1} + y_n g(u_n)$. By the monotonicity of g and $u_n \leq v_n$ imply $\nabla^\alpha z_{n+1} \leq \nabla^\alpha c_{n+1} + y_n g(v_n) \leq \nabla^\alpha c_{n+1} + y_n g(a_n + b_n z_n) \leq \nabla^\alpha c_{n+1} + y_n g(a_n) + y_n g(b_n) g(z_n)$. Now from (3.13),

$$\nabla^\alpha G(z_{n+1}) = \frac{\nabla^\alpha z_{n+1}}{g(z_n)} \leq \frac{\nabla^\alpha c_{n+1}}{g(z_n)} + \frac{y_n g(a_n)}{g(z_n)} + y_n g(b_n) \leq \frac{\nabla^\alpha c_{n+1} + y_n g(a_n)}{g(c_n)} + y_n g(b_n).$$

Let $p_n = \nabla^\alpha G(z_{n+1})$ and $q_n = \frac{\nabla^\alpha c_{n+1} + y_n g(a_n)}{g(c_n)} + y_n g(b_n)$. For $n = 0$, $p_0 = \nabla^\alpha G(z_1) = \frac{\nabla^\alpha z_1}{g(z_0)} = \frac{z_1 - z_0}{g(z_0)} = \frac{c_1 - c_0}{g(c_0)}$ and $q_0 = \frac{\nabla^\alpha c_1 + y_0 g(a_0)}{g(c_0)} + y_0 g(b_0) = \frac{c_1 - c_0}{g(c_0)}$. Since $p_0 = q_0 = \frac{c_1 - c_0}{g(c_0)}$

and $p_n \le q_n$, by using Theorem 3.1

$$\nabla^{-\alpha} p_n \le \nabla^{-\alpha} q_n$$

$$\text{i.e.} \quad \nabla^{-\alpha}\nabla^{\alpha} G(z_{n+1}) \le \nabla^{-\alpha}\left[\frac{\nabla^{\alpha} c_{n+1} + y_n g(a_n)}{g(c_n)} + y_n g(b_n)\right]$$

$$\text{i.e.} \quad G(z_{n+1}) - G(z_0) \le \nabla^{-\alpha}\left[\frac{\nabla^{\alpha} c_{n+1} + y_n g(a_n)}{g(c_n)} + y_n g(b_n)\right] \ \text{(from Remark 2(v))}$$

$$\text{or} \quad G(z_{n+1}) \le G(z_0) + \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j}\left[\frac{\nabla^{\alpha} c_{j+1} + y_j g(a_j)}{g(c_j)} + y_j g(b_j)\right].$$

Hence for $0 \le n \le n_2$; $n, n_2 \in \mathbb{N}_0^+$,

$$z_{n+1} \le G^{-1}\left(G(z_0) + \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j}\left[\frac{\nabla^{\alpha} c_{j+1} + y_j g(a_j)}{g(c_j)} + y_j g(b_j)\right]\right)$$

$$\text{i.e.} \quad z_n \le G^{-1}\left(G(z_0) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1}\left[\frac{\nabla^{\alpha} c_{j+1} + y_j g(a_j)}{g(c_j)} + y_j g(b_j)\right]\right).$$

Now

$$\begin{aligned} u_n &\le a_n + b_n z_n \\ &\le G^{-1}\left(G(z_0) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1}\left[\frac{\nabla^{\alpha} c_{j+1} + y_j g(a_j)}{g(c_j)} + y_j g(b_j)\right]\right) \\ &\le G^{-1}\left(G(c_0) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1}\left[\frac{\nabla^{\alpha} c_{j+1} + y_j g(a_j)}{g(c_j)} + y_j g(b_j)\right]\right) \end{aligned}$$

where G^{-1} is the inverse of G and $n_2 \in \mathbb{N}_0^+$ be chosen so that

$$G(c_0) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1}\left[\frac{\nabla^{\alpha} c_{j+1} + y_j g(a_j)}{g(c_j)} + y_j g(b_j)\right] \in Dom(G^{-1})$$

for all $n \in \mathbb{N}_0^+$ such that $0 \le n \le n_2$. □

Corollary 1. *Let u_n, a_n and y_n be non negative functions defined for $n \in \mathbb{N}_0^+$ such that $y_0 = 0$. Let g be a non decreasing positive function defined on R^+. If*

$$u_n \le a_n + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(u_j) \tag{3.15}$$

for $n \in \mathbb{N}_0^+$ then for $0 \le n \le n_3$; $n, n_3 \in \mathbb{N}_0^+$,

$$u_n \le G^{-1}(G(a_0) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1}\left[\frac{\nabla^{\alpha} a_{j+1}}{g(a_j)} + y_j\right] \tag{3.16}$$

where G is the solution of

$$\nabla^{\alpha} G(V_{n+1}) = \frac{\nabla^{\alpha} V_{n+1}}{g(V_n)} \tag{3.17}$$

for any non negative function V_n defined for $n \in \mathbb{N}_0^+$ and G^{-1} is the inverse of G and $n_3 \in \mathbb{N}_0^+$ be chosen so that

$$G(a_0) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} \left[\frac{\nabla^\alpha a_{j+1}}{g(a_j)} + y_j \right] \in Dom(G^{-1}) \tag{3.18}$$

for all $n \in \mathbb{N}_0^+$ such that $0 \le n \le n_3$.

Theorem 3.6. *Let u_n, a_n, b_n and y_n be non negative functions defined for $n \in \mathbb{N}_0^+$ such that $y_0 = 0$. Let g be a non decreasing subadditive submultiplicative positive function defined on R^+. If*

$$u_n \le a_n + b_n \left[\sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(u_j) \right] \tag{3.19}$$

for $n \in \mathbb{N}_0^+$ then for $0 \le n \le n_4$; $n, n_4 \in \mathbb{N}_0^+$,

$$u_n \le a_n + b_n \left[G^{-1} \left(G(c) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(b_j) \right) \right] \tag{3.20}$$

where

$$c = \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(a_j) \tag{3.21}$$

and G is the solution of

$$\nabla^\alpha G(V_{n+1}) = \frac{\nabla^\alpha V_{n+1}}{g(V_n)} \tag{3.22}$$

for any non negative function V_n defined for $n \in \mathbb{N}_0^+$ and G^{-1} is the inverse of G and $n_4 \in \mathbb{N}_0^+$ be chosen so that

$$G(c) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(b_j) \in Dom(G^{-1}) \tag{3.23}$$

for all $n \in \mathbb{N}_0^+$ such that $0 \le n \le n_4$.

Proof. We assume without loss of generality that $u_n \ge a_n$. We have

$$u_n - a_n \le b_n \left[\sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(u_j) \right]$$

$$\text{i.e.} \quad u_n - a_n \le b_n \left[\sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(u_j - a_j) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(a_j) \right].$$

Let $w_n = u_n - a_n$ and define

$$v_n = \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(w_j) + c$$

$$\text{i.e.} \quad v_{n+1} = \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} y_j g(w_j) + c.$$

Then $v_0 = c$ and by using Remark 2(v), $\nabla^\alpha v_{n+1} = y_n g(w_n) - y_0 g(w_0) = y_n g(w_n)$. By the monotonicity of g and $w_n \le b_n v_n$ imply $\nabla^\alpha v_{n+1} \le y_n g(b_n) g(v_n)$. Now from (3.22),

$$\nabla^\alpha G(v_{n+1}) = \frac{\nabla^\alpha v_{n+1}}{g(v_n)} \le y_n g(b_n).$$

Let $p_n = \nabla^\alpha G(v_{n+1})$ and $q_n = y_n g(b_n)$. For $n = 0$, $p_0 = \nabla^\alpha G(v_1) = \frac{\nabla^\alpha v_1}{g(v_0)} = \frac{\nabla^\alpha c}{g(c)} = 0$ and $q_0 = y_0 g(b_0) = 0$. Since $p_0 = q_0 = 0$ and $p_n \le q_n$, by using Theorem 3.1

$$\nabla^{-\alpha} p_n \le \nabla^{-\alpha} q_n$$

$$\text{i.e.}\quad \nabla^{-\alpha}\nabla^\alpha G(v_{n+1}) \le \nabla^{-\alpha} y_n g(b_n)$$

$$\text{i.e.}\quad G(v_{n+1}) - G(v_0) \le \nabla^{-\alpha} y_n g(b_n) \text{ (from Remark 2(v))}$$

$$\text{or}\quad G(v_{n+1}) \le G(v_0) + \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} y_j g(b_j).$$

Hence for $0 \le n \le n_4$; $n, n_4 \in \mathbb{N}_0^+$,

$$v_{n+1} \le G^{-1}\left(G(c) + \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} y_j g(b_j)\right)$$

$$\text{i.e.}\quad v_n \le G^{-1}\left(G(c) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(b_j)\right)$$

Now

$$w_n \le b_n v_n$$

$$\text{i.e.}\quad u_n - a_n \le b_n\left[G^{-1}\left(G(c) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(b_j)\right)\right]$$

$$\text{i.e.}\quad u_n \le a_n + b_n\left[G^{-1}\left(G(c) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(b_j)\right)\right]$$

where G^{-1} is the inverse of G and $n_4 \in \mathbb{N}_0^+$ be chosen so that

$$G(c) + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} y_j g(b_j) \in Dom(G^{-1})$$

for all $n \in \mathbb{N}_0^+$ such that $0 \le n \le n_4$. □

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