Communications in Applied Analysis **14** (2010), no. 4, 393–408

EVEN NUMBER OF POSITIVE SOLUTIONS FOR $3n^{th}$ ORDER THREE-POINT BOUNDARY VALUE PROBLEM

K. R. PRASAD[1] AND K. L. SARASWATHI DEVI[2]

[1]Department of Applied Mathematics, Andhra University
Visakhapatnam, 530 003, India
E-mail: rajendra92@rediffmail.com

[2]Department of Applied Mathematics, Andhra University
Visakhapatnam, 530 003, India
Department of Mathematics, Ch. S. D. St. Theresa's Degree College for Women
Eluru, 534 003, India
E-mail: saraswathikatneni@gmail.com

ABSTRACT. We are concerned with the existence of even number of positive solutions for the $3n^{th}$ order three-point boundary value problem

$$(-1)^n y^{(3n)} = f(y(t), y^{(3)}(t), y^{(6)}(t), \dots, y^{(3(n-1))}(t)), \quad t \in [t_1, t_3],$$

satisfying the boundary conditions

$$\alpha_{3i-2,1} y^{(3i-3)}(t_1) + \alpha_{3i-2,2} y^{(3i-2)}(t_1) + \alpha_{3i-2,3} y^{(3i-1)}(t_1) = 0,$$
$$\alpha_{3i-1,1} y^{(3i-3)}(t_2) + \alpha_{3i-1,2} y^{(3i-2)}(t_2) + \alpha_{3i-1,3} y^{(3i-1)}(t_2) = 0,$$
$$\alpha_{3i,1} y^{(3i-3)}(t_3) + \alpha_{3i,2} y^{(3i-2)}(t_3) + \alpha_{3i,3} y^{(3i-1)}(t_3) = 0,$$

for $1 \le i \le n$, where $n \ge 1$, $t_1 < t_2 < t_3$ and $f : \mathbb{R}^n \to \mathbb{R}^+$ is continuous. We establish the existence of at least two and then $2m$ positive solutions for an arbitrary positive integer m, by using the Avery and Henderson functional fixed point theorem.

AMS (MOS) Subject Classification. 39A 10, 34B 05

1. INTRODUCTION

The general theory of differential equations is emerging as an important area of investigation due to its powerful and versatile applications to almost all areas of science, engineering and technology. Much interest has been developed since last decade regarding the study of existence of positive solutions to the third order boundary value problems as they are arising in a variety of applied mathematics and physics problems, such as fluid flow problems in which surface tension forces are important. The study of the existence of positive solutions for higher order boundary value problems have been studied by Eloe and Henderson [5], Anderson [2], Anderson and Davis [3], Li [14], Guo, Sun and Zhao [11], Greaf and Yang [10], Shahed [19].

Received April 25, 2010 1083-2564 $15.00 ©Dynamic Publishers, Inc.

In this paper, we consider the $3n^{th}$ order differential equation

$$(-1)^n y^{(3n)} = f(y(t), y^{(3)}(t), y^{(6)}(t), \ldots, y^{(3(n-1))}(t)), \quad t \in [t_1, t_3], \tag{1.1}$$

satisfying the general three-point boundary conditions

$$\begin{aligned} \alpha_{3i-2,1}y^{(3i-3)}(t_1) + \alpha_{3i-2,2}y^{(3i-2)}(t_1) + \alpha_{3i-2,3}y^{(3i-1)}(t_1) &= 0, \\ \alpha_{3i-1,1}y^{(3i-3)}(t_2) + \alpha_{3i-1,2}y^{(3i-2)}(t_2) + \alpha_{3i-1,3}y^{(3i-1)}(t_2) &= 0, \\ \alpha_{3i,1}y^{(3i-3)}(t_3) + \alpha_{3i,2}y^{(3i-2)}(t_3) + \alpha_{3i,3}y^{(3i-1)}(t_3) &= 0, \end{aligned} \tag{1.2}$$

for $n \geq 1$, and $1 \leq i \leq n$, $t_1 < t_2 < t_3$. Our interest here is to establish even number of positive solutions for the boundary value problem (1.1)–(1.2) by using the Avery and Henderson functional fixed point theorem.

For convenience we adopt the following notation:
$\beta_j = \alpha_{3i-3+j,1}t_j + \alpha_{3i-3+j,2}$, $\gamma_j = \alpha_{3i-3+j,1}t_j^2 + 2\alpha_{3i-3+j,2}t_j + 2\alpha_{3i-3+j,3}$, $l_j = \alpha_{3i-3+j,1}s^2 - 2\beta_j s + \gamma_j$ and define

$$m_{kj} = \frac{\alpha_{3i-3+k,1}\gamma_j - \alpha_{3i-3+j,1}\gamma_k}{2(\alpha_{3i-3+k,1}\beta_j - \alpha_{3i-3+j,1}\beta_k)},$$

$$M_{kj} = \frac{\beta_{3i-3+k,1}\gamma_j - \beta_j\gamma_k}{(\alpha_{3i-3+k,1}\beta_j - \alpha_{3i-3+j,1}\beta_k)}$$

for $k, j = 1, 2, 3$ and also let $m = \max\{m_{12}, m_{13}, m_{23}\}$,

$$M = \min\left\{m_{23} + \sqrt{m_{23}^2 - M_{23}}, \quad m_{13} + \sqrt{m_{13}^2 - M_{13}}\right\}$$

and

$$d_i = [\alpha_{3i-2,1}(\beta_2\gamma_3 - \beta_3\gamma_2) - \beta_1(\alpha_{3i-1,1}\gamma_3 - \alpha_{3i,1}\gamma_2) + \gamma_1(\alpha_{3i-1,1}\beta_3 - \alpha_{3i,1}\beta_2)].$$

We assume the following conditions throughout this paper:

(A1) $f : \mathbb{R}^n \to \mathbb{R}^+$ is continuous;
(A2) $\alpha_{3i-2,1} > 0$, $\alpha_{3i-1,1} > 0$ and $\alpha_{3i,1} > 0$ for $1 \leq i \leq n$ are real constants, such that $\frac{\alpha_{3i-2,2}}{\alpha_{3i-2,1}} < \frac{\alpha_{3i-1,2}}{\alpha_{3i-1,1}} < \frac{\alpha_{3i,2}}{\alpha_{3i,1}}$.
(A3) $m \leq t_1 \leq t_2 \leq t_3 \leq M$, $2\alpha_{3i-1,3}\alpha_{3i-1,1} > \alpha_{3i-1,2}^2$,
$2\alpha_{3i-2,3}\alpha_{3i-2,1} < \alpha_{3i-2,2}^2$, $2\alpha_{3i,3}\alpha_{3i,3} > \alpha_{3i,2}^2$.
(A4) $m_{23}^2 > M_{23}$, $m_{12}^2 < M_{12}$, $m_{13}^2 > M_{13}$ and $d_i > 0$.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous boundary value problem corresponding to (1.1)–(1.2) and estimate the bounds for the Green's function. In Section 3, we establish a criteria for the existence of at least two positive solutions of the boundary value problem (1.1)–(1.2) by using the Avery-Henderson functional fixed point theorem. We also establish the existence of $2m$ positive solutions for any arbitrary positive integer m .

2. THE GREEN'S FUNCTION AND BOUNDS

In this section, we construct the Green's function for the homogeneous boundary value problem corresponding to (1.1)–(1.2) and estimate the bounds of the Greens function.

Let $G_i(t,s)$ be the Green's function for the homogeneous problem

$$-y''' = 0, \quad t \in [t_1, t_3] \tag{2.1}$$

satisfying the general three point boundary conditions (1.2). First we need few results on the related third order homogeneous boundary value problem (2.1) and (1.2).

Lemma 2.1. *The homogeneous boundary value problem* (2.1) *and* (1.2) *has only the trivial solution if and only if* $d_i = [\alpha_{3i-2,1}(\beta_2\gamma_3 - \beta_3\gamma_2) - \beta_1(\alpha_{3i-1,1}\gamma_3 - \alpha_{3i,1}\gamma_2) + \gamma_1(\alpha_{3i-1,1}\beta_3 - \alpha_{3i,1}\beta_2)] \neq 0$ *for* $1 \leq i \leq n$

Proof. On application of boundary conditions (1.2) to the general solution of (2.1), it can be established. □

Lemma 2.2. *For* $1 \leq i \leq n$, *the Green's function* $G_i(t,s)$ *for the homogeneous boundary value problem* (2.1) *and* (1.2) *is given by*

$$G_i(t,s) = \begin{cases} G_{i_1}(t,s), & t_1 < s < t \leq t_2 < t_3 \\ G_{i_2}(t,s), & t_1 \leq t < s < t_2 < t_3 \\ G_{i_3}(t,s), & t_1 \leq t < t_2 < s < t_3 \\ G_{i_4}(t,s), & t_1 < t_2 < s < t \leq t_3 \\ G_{i_5}(t,s), & t_1 < t_2 \leq t < s < t_3 \\ G_{i_6}(t,s), & t_1 \leq s < t_2 < t < t_3 \end{cases} \tag{2.2}$$

where

$$G_{i_1}(t,s) = \frac{1}{2d_i}[-(\beta_2\gamma_3 - \beta_3\gamma_2) + t(\alpha_{3i-1,1}\gamma_3 - \alpha_{3i,1}\gamma_2) - t^2(\alpha_{3i-1,1}\beta_3 - \alpha_{3i,1}\beta_2)] \times l_1,$$

$$\begin{aligned} G_{i_2}(t,s) = &\frac{1}{2d_i}[-(\beta_1\gamma_3 - \beta_3\gamma_1) + t(\alpha_{3i-2,1}\gamma_3 - \alpha_{3i,1}\gamma_1) \\ &- t^2(\alpha_{3i-2,1}\beta_3 - \alpha_{3i,1}\beta_1)] \times l_2 \\ &+ \frac{1}{2d_i}[(\beta_1\gamma_2 - \beta_2\gamma_1) - t(\alpha_{3i-2,1}\gamma_2 - \alpha_{3i-1,1}\gamma_1) \\ &+ t^2(\alpha_{3i-2,1}\beta_2 - \alpha_{3i-1,1}\beta_1] \times l_3, \end{aligned}$$

$$G_{i_3}(t,s) = \frac{1}{2d_i}[(\beta_1\gamma_2 - \beta_2\gamma_1) - t(\alpha_{3i-2,1}\gamma_2 - \alpha_{3i-1,1}\gamma_1) + t^2(\alpha_{3i-2,1}\beta_2 - \alpha_{3i-1,1}\beta_1)] \times l_3,$$

$$\begin{aligned}G_{i_4}(t,s) =&\frac{1}{2d_i}[-(\beta_2\gamma_3-\beta_3\gamma_2)+t(\alpha_{3i-1,2,1}\gamma_3-\alpha_{3i,1}\gamma_2)\\&-t^2(\alpha_{3i-1,1}\beta_3-\alpha_{3i,1}\beta_2)]\times l_1\\&+\frac{1}{2d_i}[(\beta_1\gamma_3-\beta_3\gamma_1)-t(\alpha_{3i-2,1}\gamma_3-\alpha_{3i,1}\gamma_1)\\&+t^2(\alpha_{3i-2,1}\beta_3-\alpha_{3i,1}\beta_1]\times l_2,\\G_{i_5}(t,s) =&\frac{1}{2d_i}[(\beta_1\gamma_2-\beta_2\gamma_1)-t(\alpha_{3i-2,1}\gamma_2-\alpha_{3i-1,1}\gamma_1)\\&+t^2(\alpha_{3i-2,1}\beta_2-\alpha_{3i-1,1}\beta_1)]\times l_3,\\G_{i_6}(t,s) =&\frac{1}{2d_i}[-(\beta_2\gamma_3-\beta_3\gamma_2)+t(\alpha_{3i-1,1}\gamma_3-\alpha_{3i,1}\gamma_2)\\&-t^2(\alpha_{3i-1,1}\beta_3-\alpha_{3i,1}\beta_2)]\times l_1.\end{aligned}$$

Proof. $G_i(t,s)$ is constructed by using standard methods [18]. □

Lemma 2.3. *Assume the conditions* $(A1)$–$(A4)$ *are satisfied. Then, for* $1 \le i \le n$, *the Green's function* $G_i(t,s)$ *of the boundary value problem* (2.1) *and* (1.2) *satisfies* $G_i(t,s) > 0$, *for* $(t,s) \in [t_1,t_3]\times[t_1,t_3]$.

Proof. For $(t,s) \in [t_1,t_3]\times[t_1,t_3]$, $G_i(t,s)$ stated as in (2.2), if we consider sequentially, from $(A2)$–$(A4)$,

$$G_i(t,s) > 0, \quad \text{for } (t,s) \in [t_1,t_3]\times[t_1,t_3]. \tag{2.3}$$

□

Lemma 2.4. *Assume the conditions* $(A1)$–$(A4)$ *are satisfied. Then, for* $1 \le i \le n$, *the Green's function* $G_i(t,s)$ *given by* (2.2) *satisfies that*

$$G_i(t,s) \le \max\left\{G_i(t_1,s), G_i(s,s), G_i(t_3,s)\right\}.$$

Proof. This can be proved by proceeding sequentially with the branches of $G_i(t,s)$ in (2.2).

Case 1. For $t_1 < s < t < t_2 < t_3$.

$$\begin{aligned}G_i(t,s) = G_{i_1}(t,s) = \frac{1}{2d_i}&[-(\beta_2\gamma_3-\beta_3\gamma_2)+t(\alpha_{3i-1,1}\gamma_3-\alpha_{3i,1}\gamma_2)\\&-t^2(\alpha_{3i-1,1}\beta_3-\alpha_{3i,1}\beta_2]\times l_1\end{aligned}$$

which is decreasing in t from $(A2)$–$(A4)$. Therefore $G_{i_1}(t,s) \le G_{i_1}(s,s) \le G_{i_1}(t_1,s)$. Hence $G_i(t,s) \le G_i(t_1,s)$.

Case 2. For $t_1 \le t < t_2 < s < t_3$.

$$\begin{aligned}G_i(t,s) = G_{i_3}(t,s) = \frac{1}{2d_i}&[(\beta_1\gamma_2-\beta_2\gamma_1)-t(\alpha_{3i-2,1}\gamma_2-\alpha_{3i-1,1}\gamma_1)\\&+t^2(\alpha_{3i-2,1}\beta_2-\alpha_{3i-1,1}\beta_1)]\times l_3\end{aligned}$$

which is increasing in t from $(A2)$–$(A4)$. Therefore $G_{i_3}(t,s) \le G_{i_3}(s,s) \le G_{i_3}(t_3,s)$. Hence $G_i(t,s) \le G_i(t_3,s)$.

Case 3. For $t_1 \le t < s < t_2 < t_3$.

$$\begin{aligned} G_i(t,s) &= G_{i_2}(t,s) \\ &= \frac{1}{2d_i}[-(\beta_1\gamma_3 - \beta_3\gamma_1) + t(\alpha_{3i-2,1}\gamma_3 - \alpha_{3i,1}\gamma_1) \\ &\quad - t^2(\alpha_{3i-2,1}\beta_3 - \alpha_{3i,1}\beta_1)] \times l_2 \\ &\quad + \frac{1}{2d_i}[(\beta_1\gamma_2 - \beta_2\gamma_1) - t(\alpha_{3i-2,1}\gamma_2 - \alpha_{3i-1,1}\gamma_1) \\ &\quad + t^2(\alpha_{3i-2,1}\beta_2 - \alpha_{3i-1,1}\beta_1] \times l_3 \end{aligned}$$

which is increasing in t by $(A2)$–$(A4)$ and case 2. Therefore $G_{i_2}(t,s) \le G_{i_2}(s,s)$. Hence $G_i(t,s) \le G_i(s,s)$.

Case 4. For $t_1 < t < t_2 < s < t < t_3$.

$$\begin{aligned} G_i(t,s) &= G_{i_4}(t,s) \\ &= \frac{1}{2d_i}[-(\beta_2\gamma_3 - \beta_3\gamma_2) + t(\alpha_{3i-1,1}\gamma_3 - \alpha_{3i,1}\gamma_2) \\ &\quad - t^2(\alpha_{3i-1,1}\beta_3 - \alpha_{3i,1}\beta_2)] \times l_1 \\ &\quad + \frac{1}{2d_i}[(\beta_1\gamma_3 - \beta_3\gamma_1) - t(\alpha_{3i-2,1}\gamma_3 - \alpha_{3i,1}\gamma_1) \\ &\quad + t^2(\alpha_{3i-2,1}\beta_3 - \alpha_{3i,1}\beta_1)] \times l_2 \end{aligned}$$

which is decreasing in t from case 1 and case 2. Therefore $G_{i_4}(t,s) \le G_{i_4}(s,s)$. Hence $G_i(t,s) \le G_i(s,s)$.

Similarly we can prove when the Green's function $G_i(t,s) = G_{i_5}(t,s)$ and $G_i(t,s) = G_{i_6}(t,s)$ as in case 2 and case 1 respectively, where $G_{i_5}(t,s)$, $G_{i_6}(t,s)$ are given as in (2.2). From all above cases

$$G_i(t,s) \le \max\{G_i(t_1,s), G_i(s,s), G_i(t_3,s)\}.$$

□

Lemma 2.5. *Assume that the conditions* $(A1)$–$(A4)$ *holds. For* $1 \le i \le n$, *and fixed* $s \in [t_1,t_3]$, *the Green's function* $G_i(t,s)$ *in* (2.2) *satisfies*

$$\min_{t\in[t_2,t_3]} G_i(t,s) \ge m_i \|G_i(.,s)\|,$$

where

$$m_i = \min\left\{\frac{G_{i_1}(t_3,s)}{G_{i_1}(t_2,s)}, \frac{G_{i_4}(t_3,s)}{G_{i_4}(t_2,s)}, \frac{G_{i_5}(t_2,s)}{G_{i_5}(t_3,s)}\right\}$$

and $\|\cdot\|$ *is defined by* $\|x\| = \max\{x(t) : t \in [t_1,t_3]\}$.

Proof. For $s \in [t_1, t_2]$, $G_i(t,s) = G_{i_1}(t,s)$ which is decreasing in t by $(A2)$–$(A4)$. Therefore

$$\frac{G_i(t,s)}{G_i(s,s)} = \frac{G_{i_1}(t,s)}{G_{i_1}(s,s)} \geq \frac{G_{i_1}(t_3,s)}{G_{i_1}(t_2,s)}.$$

For $s \in [t_2, t_3]$ and $t_1 < t_2 \leq t < s < t_3$. $G_i(t,s) = G_{i_5}(t,s)$ which is increasing in t on $[t_1, t_3]$ by $(A2)$–$(A4)$. Therefore

$$\frac{G_i(t,s)}{G_i(s,s)} = \frac{G_{i_5}(t,s)}{G_{i_5}(s,s)} \leq \frac{G_{i_5}(t_2,s)}{G_{i_5}(t_3,s)}.$$

For $s \in [t_2, t_3]$ and $t_1 < t_2 < s < t < t_3$. $G_i(t,s) = G_{i_4}(t,s)$ which is decreasing in t on $[t_1, t_3]$ by $(A2)$–$(A4)$. Therefore

$$\frac{G_i(t,s)}{G_i(s,s)} = \frac{G_{i_4}(t,s)}{G_{i_4}(s,s)} \geq \frac{G_{i_4}(t,s)}{G_{i_4}(t_2,s)} \geq \frac{G_{i_4}(t_3,s)}{G_{i_4}(t_2,s)}.$$

Therefore from Lemma 2.4 and by all the above cases we have

$$\min_{t \in [t_2,t_3]} G_i(t,s) \geq m_i \|G(.,s)\|,$$

where

$$m_i = \min \left\{ \frac{G_{i_1}(t_3,s)}{G_{i_1}(t_2,s)}, \frac{G_{i_4}(t_3,s)}{G_{i_4}(t_2,s)}, \frac{G_{i_5}(t_2,s)}{G_{i_5}(t_3,s)} \right\}.$$

□

Lemma 2.6. *Assume the conditions* $(A1)$–$(A4)$ *are satisfied and* $G_i(t,s)$ *as in* (2.2). *Let us define* $H_1(t,s) = G_1(t,s)$ *and recursively define*

$$H_j(t,s) = \int_{t_1}^{t_3} H_{j-1}(t,r) G_j(r,s) dr$$

for $2 \leq j \leq n$, *then* $H_n(t,s)$ *is the Green's function for the homogeneous problem corresponding to* (1.1)–(1.2).

Lemma 2.7. *Assume the conditions* $(A1)$–$(A4)$ *holds. If we define*

$$K = \prod_{j=1}^{n-1} K_j, \quad L = \prod_{j=1}^{n-1} m_j L_j,$$

then the Green's function $H_n(t,s)$ *in Lemma* 2.6 *satisfies*

$$0 \leq H_n(t,s) \leq K \|G_n(s,s)\|, \quad (t,s) \in [t_1,t_3] \times [t_1,t_3] \tag{2.4}$$

and

$$H_n(t,s) \geq m_n L \|G_n(s,s)\|, \quad (t,s) \in [t_2,t_3] \times [t_1,t_3] \tag{2.5}$$

where m_n *is given as in Lemma* 2.5,

$$K_j = \int_{t_1}^{t_3} \|G_j(s,s)\| ds > 0, \quad \textit{for } 1 \leq j \leq n,$$

and

$$L_j = \int_{t_2}^{t_3} \|G_j(s,s)\| ds > 0, \quad \textit{for } 1 \leq j \leq n.$$

Proof. By using Lemma 2.5 and induction on n, we can easily establish the Proof. □

Let $C = \{v|v : [t_1, t_3] \to \mathbb{R}$ is continuous function$\}$. For each $1 \leq j \leq n-1$, define the operator $T_j : C \to C$ by

$$(T_j v)(t) = \int_{t_1}^{t_3} H_j(t,s)v(s)ds, \quad t \in [t_1, t_3].$$

By the construction of T_j, and the properties of $H_j(t,s)$, it is clear that

$$\begin{gathered}(-1)^j (T_j v)^{(3j)}(t) = v(t), \quad t \in [t_1, t_3],\\ \alpha_{3i-2,1} T_j v^{(3i-3)}(t_1) + \alpha_{3i-2,2} T_j v^{(3i-2)}(t_1) + \alpha_{3i-2,3} T_j v^{(3i-1)}(t_1) = 0,\\ \alpha_{3i-1,1} T_j v^{(3i-3)}(t_2) + \alpha_{3i-1,2} T_j v^{(3i-2)}(t_2) + \alpha_{3i-1,3} T_j v^{(3i-1)}(t_2) = 0,\\ \alpha_{3i,1} T_j v^{(3i-3)}(t_3) + \alpha_{3i,2} T_j v^{(3i-2)}(t_3) + \alpha_{3i,3} T_j v^{(3i-1)}(t_3) = 0.\end{gathered}$$

Hence, we see that the boundary value problem (1.1)–(1.2) has a solution if and only if the following boundary value problem has a solution

$$v^{(3)}(t) + f(T_{n-1}v(t), T_{n-2}v(t), \ldots, T_1 v(t), v(t)) = 0, \quad t \in [t_1, t_3] \tag{2.6}$$

$$\begin{gathered}\alpha_{3i-2,1} v^{(3i-3)}(t_1) + \alpha_{3i-2,2} v^{(3i-2)}(t_1) + \alpha_{3i-2,3} v^{(3i-1)}(t_1) = 0,\\ \alpha_{3i-1,1} v^{(3i-3)}(t_2) + \alpha_{3i-1,2} v^{(3i-2)}(t_2) + \alpha_{3i-1,3} v^{(3i-1)}(t_2) = 0,.\\ \alpha_{3i,1} v^{(3i-3)}(t_3) + \alpha_{3i,2} v^{(3i-2)}(t_3) + \alpha_{3i,3} v^{(3i-1)}(t_3) = 0.\end{gathered} \tag{2.7}$$

Indeed, if y is a solution of the boundary value problem (1.1)–(1.2), then $v(t) = y^{3(n-1)}(t)$ is a solution of the boundary value problem (2.6)–(2.7). Conversely, if v is a solution of the boundary value problem (2.6)–(2.7), then $y(t) = T_{n-1}v(t)$ is a solution of the boundary value problem (1.1)–(1.2).

In fact, $y(t)$ represented as

$$y(t) = \int_{t_1}^{t_3} H_n(t,s)v(s)ds,$$

where

$$v(s) = \int_{t_1}^{t_3} G(s,\tau) f(T_{n-1}v(\tau), T_{n-2}v(\tau), \ldots, T_1 v(\tau), v(\tau))d\tau.$$

is a solution of the boundary value problem (1.1)–(1.2).

3. MULTIPLE POSITIVE SOLUTIONS

In this section, we establish the existence of at least two positive solutions of the boundary value problem (1.1)–(1.2) using Avery-Henderson functional fixed point theorem.

Let B be a real Banach space. Every cone $P \subset B$ induces an ordering in B given by

$$x \leq y \quad \text{if and only if } y - x \in P.$$

We say

$$x < y \quad \text{whenever } x \leq y \text{ and } x \neq y.$$

A functional ψ is said to be an increasing functional on a cone P of a real Banach space B provided $\psi(x) \leq \psi(y)$ for all $x, y \in P$ with $x \leq y$.

Let ψ be a nonnegative continuous functional on a cone P of the real Banach space B. Then for a positive real number c' we define the set

$$P(\psi, c') = \{y \in P : \psi(y) < c'\}.$$

In obtaining multiple positive solutions of the boundary value problem (1.1)–(1.2), the following Avery and Henderson's functional fixed point theorem will be the fundamental tool.

Theorem 3.1. *Let P be a cone in a real Banach space B. Suppose α and γ are increasing nonnegative continuous functionals on P and θ is nonnegative continuous functional on P with $\theta(0) = 0$ such that for some positive numbers c' and k,*

$$\gamma(y) \leq \theta(y) \leq \alpha(y) \text{ and } \|y\| \leq k\gamma(y) \text{ for all } y \in \overline{P(\gamma, c')}.$$

Suppose there exists positive numbers a' and b' with $a' < b' < c'$ such that

$$\theta(\lambda y) \leq \lambda\theta(y), \quad 0 \leq \lambda \leq 1 \text{ and } y \in \partial P(\theta, b').$$

Further, let $T : \overline{P(\gamma, c')} \to P$ is completely continuous operator such that $(B1)\gamma(Ty) > c'$ for all $y \in \partial\overline{P(\gamma, c')}$, $(B2)\theta(Ty) < b'$ for all $y \in \partial\overline{P(\theta, b')}$, $(B3)P(\alpha, a') \neq \emptyset$ and $\alpha(Ty) > a'$ for all $y \in \partial\overline{P(\alpha, a')}$ with $\theta(Ty) > b'$. Then, T has at least two fixed points $y_1, y_2 \in \overline{P(\gamma, c')}$ such that

$$\theta(y_1) < b', \text{ with } \alpha(y_1) > a',$$

and

$$\gamma(y_2) < c' \text{ with } \theta(y_2) > b'.$$

Let $B = \{v | v \in C[t_1, t_3]$ be the Banach space equipped with the norm

$$\|v\| = \max_{t \in [t_1, t_3]} |v(t)|.$$

Define the cone $P \subset B$ by

$$P = \Big\{v \in B : v(t) \geq 0, \text{ and } \min_{t \in [t_2, t_3]} v(t) \geq M\|v\|\Big\},$$

where $M = \frac{m_j L}{K}$ and m_j, L, K are as in Lemma 2.7.

Define the nonnegative continuous increasing functionals γ, θ and α on the cone P by

$$\gamma(v) = \min_{t\in[t_2,t_3]} |v(t)|,$$

$$\theta(v) = \max_{t\in[t_1,t_3]} |v(t)|,$$

$$\alpha(v) = \max_{t\in[t_2,t_3]} |v(t)|.$$

We observe that for any $v \in P$,

$$\gamma(v) = \theta(v) \leq \max_{t\in[t_2.t_3]} |v(t)| = \alpha(v), \tag{3.1}$$

$$\|v\| \leq \frac{1}{M} \min_{t\in[t_2,t_3]} |v(t)| \leq \frac{1}{M} \max_{t\in[t_1,t_3]} |v(t)| \leq \frac{1}{M}\theta(v) = \frac{1}{M}\gamma(v), \tag{3.2}$$

and also

$$\|v\| \leq \min_{t\in[t_2,t_3]} v(t) \leq \max_{t\in[t_2,t_3]} v(t) = \frac{1}{M}\alpha(v).$$

Let

$$G(t,s) = \min\{G_1(t,s), G_2(t,s), \ldots, G_n(t,s)\},$$

and

$$\overline{L} = \max\Big\{ \int_{t_2}^{t_3} G_1(s,s)ds, \int_{t_2}^{t_3} G_2(s,s)ds, \ldots, \int_{t_2}^{t_3} G_j(s,s)ds\Big\}.$$

We are now ready to present the main result of this section.

Theorem 3.2. *Suppose there exist* $0 < a' < b' < c'$ *such that* f *satisfies the following conditions:*

(D1) $f(u_{n-1}, u_{n-2}, \ldots, u_1, u_0) > \frac{c'}{\overline{L}}$ *for all* $(|u_{n-1}|, |u_{n-2}|, \ldots, |u_1|, |u_0|)$ *in* $\Pi_{j=n-1}^{1}[m_j c'(\Pi_{i=1}^{n-1} L_i)L_j,\ \frac{c'(\Pi_{i=1}^{n-1}K_i)K_j}{M}] \times [c', \frac{c'}{M}]$,

(D2) $f(u_{n-1}, u_{n-2}, \ldots, u_1, u_0) < \frac{b'}{K}$ *for all* $(|u_{n-1}|, |u_{n-2}|, \ldots, |u_1|, |u_0|)$ *in* $\Pi_{j=n-1}^{1}[m_j b'(\Pi_{i=1}^{n-1} L_i)L_j,\ \frac{b'(\Pi_{i=1}^{n-1}K_i)K_j}{M}] \times [0, \frac{b'}{M}]$,

(D3) $f(u_{n-1}, u_{n-2}, \ldots, u_1, u_0) > \frac{a'}{\overline{L}}$ *for all* $(|u_{n-1}|, |u_{n-2}|, \ldots, |u_1|, |u_0|)$ *in* $\Pi_{j=n-1}^{1}[m_j a'(\Pi_{i=1}^{n-1} L_i)L_j,\ \frac{a'(\Pi_{i=1}^{n-1}K_i)K_j}{M}] \times [Ma', a']$.

Then the boundary value problem (1.1)–(1.2) *has at least two positive solutions.*

Proof. Define the completely continuous operator $T: P \to B$ by

$$Tv(t) = \int_{t_1}^{t_3} G(t,s) f(T_{n-1}v(s), T_{n-2}v(s), \ldots, T_1 v(s), v(s))ds. \tag{3.3}$$

It is obvious that a fixed point of T is the solution of the boundary value problem (2.6)–(2.7). We seek two fixed points $v_1, v_2 \in P$ of T. First, we show that $T: P \to P$.

Let $v \in P$. Clearly, $Tv(t) \geq 0$ for $t \in [t_1, t_3]$, we have

$$\begin{aligned}\min_{t\in[t_2,t_3]} Tv(t) &= \min_{t\in[t_2,t_3]} \int_{t_1}^{t_3} G(t,s) f(T_{n-1}v, T_{n-2}v, \ldots, T_1 v, v) ds \\ &\geq M \int_{t_2}^{t_3} G(s,s) f(T_{n-1}v, T_{n-2}v, \ldots, T_1 v, v) ds \\ &= M\|Tv\|.\end{aligned}$$

Thus, $T : P \to P$.

Next, it is obvious that $\theta(0) = 0$. Further, for any $v \in P$, by (3.1)–(3.2), respectively, we have

$$\gamma(v) = \theta(v) \leq \alpha(v) quad \text{ and } \|v\| \leq \gamma(v).$$

Also, for any $0 \leq \lambda \leq 1$ and $v \in P$, we have

$$\theta(\lambda v) = \max_{t\in[t_1,t_3]} |\lambda v(t)| = \lambda \max_{t\in[t_1,t_3]} |v(t)| = \lambda\theta(v).$$

It remains to verify conditions $(B1)$–$(B3)$ of Theorem 3.1. To show that condition $(B1)$ holds, let $v \in \partial\overline{P(\gamma, c')}$, so

$$\gamma(v) = \min_{t\in[t_2,t_3]} |v(t)|.$$

For $t \in [t_2, t_3]$ it is clear from (3.2) that

$$c' = \min_{t\in[t_2,t_3]} |v(t)| \leq |v(t)| \leq \|v\| \leq \frac{1}{M}\alpha(v) \leq \frac{1}{M}c'.$$

For $1 \leq j \leq n-1$ and $t \in [t_2, t_3]$,

$$\begin{aligned}T_j v(t) &= \int_{t_1}^{t_3} H_j(t,s) v(s) ds \\ &\leq \frac{c'}{M} \int_{t_1}^{t_3} H_j(t,s) ds \\ &\leq \frac{c'K}{M} \int_{t_1}^{t_3} \|G_j(s,s)\| ds \\ &= \frac{c'KK_j}{M}.\end{aligned}$$

For $1 \leq j \leq n-1$ and $t \in [t_1, t_3]$,

$$\begin{aligned}T_j v(t) &= \int_{t_1}^{t_3} H_j(t,s) v(s) ds \\ &\geq c' \int_{t_2}^{t_3} H_j(t,s) ds \\ &\geq c' m_j L \int_{t_2}^{t_3} \|G_j(s,s)\| ds \\ &= c' m_j L L_j.\end{aligned}$$

We may now use condition $(D1)$ to obtain

$$\begin{aligned}\gamma(Tv) &= \min_{t\in[t_2,t_3]}\int_{t_1}^{t_3} G(t,s)f(T_{n-1}v, T_{n-2}v,\ldots,T_1v,v)ds\\ &> \frac{c'}{\overline{L}}\int_{t_2}^{t_3} G(s,s)ds > c'.\end{aligned}$$

Therefore, we have shown that $\gamma(Tv) > c'$ for all $v \in \partial\overline{P(\gamma,c')}$.

Next, we shall verify condition $(B2)$ holds, let $v \in \partial\overline{P(\theta,b')}$, so

$$\theta(v) = \max_{t\in[t_1,t_3]}|v(t)|.$$

For $t \in [t_1,t_3]$, it is clear from (3.2) that

$$b' = \max_{t\in[t_1,t_3]}|v(t)| \le \|v\| \le \frac{b'}{M}.$$

For $1 \le j \le n-1$ and $t \in [t_1,t_3]$,

$$\begin{aligned}T_jv(t) &= \int_{t_1}^{t_3} H_j(t,s)v(s)ds\\ &\le \frac{b'}{M}\int_{t_1}^{t_3} H_j(t,s)ds\\ &\le \frac{b'K}{M}\int_{t_1}^{t_3}\|G_j(s,s)\|ds\\ &= \frac{b'KK_j}{M}.\end{aligned}$$

For $1 \le j \le n-1$ and $t \in [t_2,t_3]$,

$$\begin{aligned}T_jv(t) &= \int_{t_1}^{t_3} H_j(t,s)v(s)ds\\ &\ge b'\int_{t_2}^{t_3} H_j(t,s)ds\\ &\ge b'm_jL\int_{t_2}^{t_3}\|G_j(s,s)\|ds\\ &= b'm_jLL_j.\end{aligned}$$

We may now use condition $(D2)$ to obtain

$$\begin{aligned}\theta(Tv) &= \max_{t\in[t_1,t_3]}\int_{t_1}^{t_3} G(t,s)f(T_{n-1}v,T_{n-2}v,\ldots,T_1v,v)ds\\ &< \frac{b'}{K}\int_{t_1}^{t_3} G(s,s)ds\\ &< b'.\end{aligned}$$

Therefore, we have shown that $\theta(Tv) < b'$ for all $v \in \partial\overline{P(\theta,b')}$.

Finally, we show that $(B3)$ holds. Clearly, $\frac{a'}{2} \in P(\alpha, a') \neq \emptyset$. Now, let $v \in \partial\overline{P(\alpha, a')}$, so

$$\alpha(v) = \max_{t\in[t_2,t_3]} |v(t)|.$$

For $t \in [t_2, t_3]$, it is clear from (3.2) that

$$a' = \max_{t\in[t_2,t_3]} |v(t)| \leq |v(t)| \leq \|v\| \leq \frac{a'}{M}.$$

For $1 \leq j \leq n-1$ and $t \in [t_2, t_3]$,

$$\begin{aligned} T_j v(t) &= \int_{t_1}^{t_3} H_j(t,s)v(s)ds \\ &\leq \frac{a'}{M}\int_{t_1}^{t_3} H_j(t,s)ds \\ &\leq \frac{a'K}{M}\int_{t_1}^{t_3} \|G_j(s,s)\|ds \\ &= \frac{a'KK_j}{M}. \end{aligned}$$

For $1 \leq j \leq n-1$ and $t \in [t_2, t_3]$,

$$\begin{aligned} T_j v(t) &= \int_{t_1}^{t_3} H_j(t,s)v(s)ds \\ &\geq a'\int_{t_2}^{t_3} H_j(t,s)ds \\ &\geq a' m_j L\int_{t_2}^{t_3} \|G_j(s,s)\|ds \\ &= a' m_j L L_j. \end{aligned}$$

We may now use condition $(D3)$ to obtain

$$\begin{aligned} \alpha(Tv) &= \max_{t\in[t_2,t_3]} \int_{t_1}^{t_3} G(t,s) f(T_{n-1}v, T_{n-2}v, \ldots, T_1 v, v)ds \\ &> \frac{a'}{\overline{L}}\int_{t_2}^{t_3} G(s,s)ds \\ &> a'. \end{aligned}$$

Therefore, we have shown that $\alpha(Tv) > a'$ for all $v \in \partial\overline{P(\alpha, a')}$.

We have proved that all the conditions of Theorem 3.1 are satisfied and so there exist at least two positive solutions $v_1, v_2 \in \overline{P(\gamma, c')}$ for the boundary value problem (2.6)–(2.7). Therefore the boundary value problem (1.1)–(1.2) has at least two positive solutions y_1, y_2 of the form

$$y_i(t) = T_{n-1}v_i(t) = \int_{t_1}^{t_3} G_{n-1}(t,s)v_i(s)ds, \quad i = 1, 2.$$

This completes the proof of the theorem. □

Now we prove the existence of at least $2m$ positive solutions for the boundary value problem (1.1)–(1.2) by using induction on m.

Theorem 3.3. *Let m be an arbitrary positive integer. Assume that there exist numbers $a_i (1 \le i \le m+1)$ and $b_j (1 \le j \le m)$ with*

$$0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m < a_{m+1}$$

such that

$$\left.\begin{array}{l} f(u_{n-1}, u_{n-2}, \ldots, u_1, u_0) > \dfrac{a_i}{\overline{L}} \text{ for all } (|u_{n-1}|, |u_{n-2}|, \ldots, |u_1|, |u_0|) \\ \text{in } \Pi_{j=n-1}^{1}[m_j a_i (\Pi_{i=1}^{n-1} L_i) L_j, \dfrac{a_i (\Pi_{i=1}^{n-1} K_i) K_j}{M}] \times [M a_i, a_i], \quad 1 \le i \le m+1, \end{array}\right\} \quad (3.4)$$

$$\left.\begin{array}{l} f(u_{n-1}, u_{n-2}, \ldots, u_1, u_0) < \frac{b_l}{K} \text{ for all } (|u_{n-1}|, |u_{n-2}|, \ldots, |u_1|, |u_0|) \\ \text{in } \Pi_{j=n-1}^{1}[m_j b_l (\Pi_{i=1}^{n-1} L_i) L_j, \frac{b_l (\Pi_{i=1}^{n-1} K_i) K_j}{M}] \times [0, \frac{b_l}{M}], \quad 1 \le l \le m. \end{array}\right\} \quad (3.5)$$

Then the boundary value problem (1.1)–(1.2) *has at least $2m$ positive solutions in $\overline{P}_{a_{m+1}}$.*

Proof. We use induction on m. First, for $m = 1$, we know from (3.4) and (3.5) that $T : \overline{P}_{a_2} \to P_{a_2}$, then, it follows from Avery and Hendersons functional fixed point theorem that the boundary value problem (1.1)–(1.2) has at least two positive solutions in $\overline{P}_{a_2}$. Next, we assume that this conclusion holds for $m = k$. In order to prove that this conclusion holds for $m = k+1$, we suppose that there exist numbers

$$a_i (1 \le i \le k+2) \quad \text{and } b_j (1 \le j \le k+1)$$

with

$$0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_{k+1} < b_{k+1} < a_{k+2}$$

such that

$$\left.\begin{array}{l} f(u_{n-1}, u_{n-2}, \ldots, u_1, u_0) > \frac{a_i}{\overline{L}} \text{ for all } (|u_{n-1}|, |u_{n-2}|, \ldots, |u_1|, |u_0|) \\ \text{in } \Pi_{j=n-1}^{1}[m_j a_i (\Pi_{i=1}^{n-1} L_i) L_j, \frac{a_i (\Pi_{i=1}^{n-1} K_i) K_j}{M}] \times [M a_i, a_i], \quad 1 \le i \le k+2, \end{array}\right\} \quad (3.6)$$

$$\left.\begin{array}{l} f(u_{n-1}, u_{n-2}, \ldots, u_1, u_0) < \frac{b_l}{K} \text{ for all } (|u_{n-1}|, |u_{n-2}|, \ldots, |u_1|, |u_0|) \\ \text{in } \Pi_{j=n-1}^{1}[m_j b_l (\Pi_{i=1}^{n-1} L_i) L_j, \frac{b_l (\Pi_{i=1}^{n-1} K_i) K_j}{M}] \times [0, \frac{b_l}{M}], 1 \le l \le k+1. \end{array}\right\} \quad (3.7)$$

By assumption, the boundary value problem (1.1)–(1.2) has at least $2k$ positive solutions u_i $(i = 1, 2, \ldots, 2k)$ in $\overline{P}_{a_{k+1}}$. At the same time, it follows from Theorem 3.2, and (3.6) and (3.7) that the boundary value problem (1.1)–(1.2) has at least two positive solutions u, v in $\overline{P}_{a_{k+2}}$ such that,

$$\theta(u) < b_{k+1}, \quad \text{with } \alpha(u) > a_{k+1},$$

and

$$\gamma(v) < a_{k+2} \quad \text{with } \theta(v) > b_{k+1}.$$

Obviously, u and v are different from u_i ($i = 1, 2, \ldots, 2k$). Therefore, the boundary value problem (1.1)–(1.2) has at least $2k + 2$ positive solutions in $\overline{P}_{a_{k+2}}$ which shows that this conclusion also holds for $m = k + 1$. □

ACKNOWLEDGEMENTS

The authors thank the referee for his valuable suggestions. One of the authors K. L. Saraswathi Devi is grateful to the U. G. C. (India) and the management of Ch. S. D. St. Theresa's College for Women, Eluru, for selecting under Faculty Development Programme.

REFERENCES

[1] R. P. Agarwal, D. O' Regan and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Nethelands, 1999.

[2] D. R. Anderson, Multiple positive solutions for a three-point boundary value problem, *Math. Comput. Modelling.*, **27**(1998), 49–57.

[3] D. R. Anderson and J. M. Davis, Multiple solutions and eigenvalues for third order right focal boundary value problem, *J. Math. Anal. Appl.*, **267**(2002), 135–157.

[4] P. W. Eloe and J. Henderson, Positive solutions for (n-1,1) conjugate boundary value problems, *Nonlinear. Anal.*, **28**(1997), 1669–1680.

[5] P. W. Eloe and J. Henderson, Positive solutions and nonlinear multipoint conjugate eigenvalue problems, *Elec. J. Diff. Eqns.*, **1997**(1997), No.3, 1–11.

[6] P. W. Eloe and J. Henderson, Positive solutions for (k,n-k) conjugate eigenvalue problems, *Diff. Eqns. Dyn. Sys.*, **6**(1998), 309–317.

[7] L. H. Erbe, S. Hu and H. Wang, Multiple positive solutions of some boundary value problems, *J. Math. Anal. Appl.*, **184**(1994), 640–648.

[8] L. H. Erbe and H. Wang, On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.*, **120**(1994), 743–748.

[9] J. R. Greaf and B.Yang, Positive solutions to a multi-point higher order boundary value problem, *J. Math. Anal. Appl.*, **316**(2006), 409–421.

[10] J. R. Greaf and B. Yang, Positive solutions of nonlinear third order eigenvalue problem, *Dyn. Sys. Appl.* **15**(2006), 97–110.

[11] L. J. Guo, J. P. Sun, Y. H. Zhao, Multiple positive solutions for nonlinear third-order three-point boundary value problems, *Ele. J. Diff. Eqns.*, **2007**(2007), No. 112, 1–7.

[12] J. Henderson and E. R. Kaufmann, Multiple positive solutions for focal boundary value problems, *Comm. Appl. Anal.*, **1**(1997), 53–60.

[13] E. R. Kaufmann, Multiple positive solutions for higher order boundary value problems, *Rocky Mtn. J. Math.*, **28**(1998), No.3, 1017–1028.

[14] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[15] S. Li, Positive solutions of nonlinear singular third-order two-point boundary value problem, *J. Math. Anal. Appl.*, **323**(2006), 413–425.

[16] W. C. Lian, F. H. Wang and C. C. Yeh, On the existence of positive solution of nonlinear second order differential equations, *Proc. Amer. Math. Soc.*, **124**(1996), 1111–1126.

[17] K. R. Prasad and P. Murali, Multiple positive solutions for nonlinear third order general three point boundary value problems, *Diff. Eqns. Dyn. Sys.*, **16**(2008), 63–75.

[18] D. R. K. S. Rao, K. N. Murthy and M. S. N. Murthy, On three-point boundary value problems containing parameters, *Bull. Inst. Math. Academia Sinica.*, **10**(1982), No. 3, 265–275.

[19] M. E. Shahed, Positive solutions of boundary value problems for n-th order ordinary differential equtations, *Elec. J. Qual. Theory. Diff. Eqns.*, **2008**(2008), No.1, 1–9.

[20] B. Yang, Positive solutions of a third-order three-point boundary value problem, *Elec. J. Diff. Eqns.*, **2008**(2008), No. 99, 1–10.