

DETERMINANT FUNCTIONS AND APPLICATIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS

A. G. LADDE¹ AND G. S. LADDE²

¹Chesapeake Capital Corporation
Federal Reserve Building, 17th Floor
Richmond, VA 23219 USA

²Department of Mathematics & Statistics
University of South Florida
Tampa, FL 33620 USA
E-mail: gladde@cas.usf.edu

ABSTRACT. A few fundamental results regarding the of calculus of determinant functions of associated with $n \times n$ matrix functions are developed. Furthermore, stochastic versions of Liouville–Jacobi [8, 9, 11, 17], Abel–Liouville [1, 2, 17]. Lagrange formula [4, 8, 11, 15–17], Green’s formula [4, 8] and many fundamental results are established for the Itô–Doob type higher order and system of linear stochastic differential equations [6, 10, 12–14].

1. INTRODUCTION

To the best of our understanding and research, the study of determinant functions associated with $n \times n$ matrix functions is not well exposed in literature. However, determinant functions [4, 5, 7, 8, 11] are used both in the study of higher order or systems of ordinary differential equations.

This work attempts to outline certain basic results concerning a determinant function associated with an $n \times n$ matrix function. Furthermore, by deriving formulas for first and second derivatives of determinant function, the Generalized Mean-Value Theorem and Taylor’s Formula for determinant functions are also systematically established.

Finally, by employing the Taylor’s formula for determinant functions, Liouville–Jacobi [8, 9, 11, 17], the Abel–Liouville [1, 2, 4, 8, 17] type deterministic results are extended to the Itô–Doob type higher order and system of linear stochastic differential equations. These results play a very significant role in understanding the fundamental properties of these equations [6, 10, 12–14].

2. DETERMINANT FUNCTION

It is very well-known [4, 5, 7, 8, 11] that the determinant function associated with an $n \times n$ matrix function plays a significant role in the study of finding the complete set of solutions to higher order or linear system of differential equations. Moreover, these results provide an insight into the fundamental properties of solutions of differential equations.

In this section, by reviewing the definition of determinant function associated with an $n \times n$ matrix, a few nontraditional and basic results concerning the determinant function associated matrix functions are presented. These results play a very important role in motivating one to study the fundamental properties of solutions of higher order or systems of Itô-Doob type stochastic differential equations in a systematic and unified way [12].

Definition 2.1 ([3]). For $n > 1$, let $A = (a_{ij})_{n \times n}$ be an $n \times n$ matrix. Let $C = (C_{ij})_{n \times n}$, where C_{ij} is the cofactor of a_{ij} . The determinant of the matrix A is defined by:

- (1) for any $i = 1, 2, \dots, n$,

$$\det(A) = \sum_{j=1}^n a_{ij} C_{ij} = \sum_{j=1}^n a_{ij} (-1)^{i+j} \det(A_{ij}).$$

This is called the expansion of “ $\det(\mathbf{A})$ ” by the \mathbf{i} -th row of the matrix A ;

- (2) for any $j = 1, 2, \dots, n$,

$$\det(A) = \sum_{i=1}^n a_{ij} C_{ij} = \sum_{i=1}^n a_{ij} (-1)^{i+j} \det(A_{ij}).$$

This is called the expansion of “ $\det(\mathbf{A})$ ” by the \mathbf{j} -th column of the matrix A .

Observation 2.2. The following conclusions are based on the Principle of Mathematical Induction (PMI) and Definition 2.1.

1. The value of the determinant of any $n \times n$ matrix A is independent of a row or a column expansion of matrix A . Thus given an arbitrary $n \times n$ matrix $A = (a_{ij})_{n \times n}$, its determinant is uniquely determined by either any one of the rows or columns expansions. This is due to the fact that for either each i or each j , $\det(A)$ in Definition 2.1 (1) or (2) is a finite sum of a well-defined scalar multiple of values of determinants $\det(A_{ij})$ of $(n-1) \times (n-1)$ sub-matrices A_{ij} .
2. From conclusion 1, we can infer that the determinant of $n \times n$ matrix is a function defined on a collection of $n \times n$ matrices with values in a set of real/complex numbers.
3. We note that $\det(A)$ of any $n \times n$ matrix A has $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ terms. Each term is the product of n distinct entries of the matrix. Moreover, it is

the n -th degree homogeneous polynomial function of n^2 entries of the matrix of independent variables. It is denoted by

$$\det(A) = W(a_{11}, \dots, a_{1n}, \dots, a_{i1}, \dots, a_{in}, \dots, a_{n1}, \dots, a_{nn}).$$

4. We note that the size of any (i, j) -th sub-matrix A_{ij} of any $n \times n$ matrix $A = (a_{ij})_{n \times n}$ is $(n-1) \times (n-1)$. In this case, the (i, j) -th minor M_{ij} is the determinant $\det(A_{ij})$ of the $(n-1) \times (n-1)$ sub-matrix A_{ij} of matrix A corresponding to the i -th row and the j -th column of matrix A . M_{ij} is independent of all the entries of the i -th row and j -th column of matrix A . Moreover, $C_{ij} = (-1)^{i+j} M_{ij}$ can be considered either a function of $(n-1)$ row $A_1(1j), \dots, A_{i-1}(i-1j), A_{i+1}(i+1j), \dots, A_n(nj)$, or $(n-1)$ column $A^1(i1), \dots, A^{j-1}(ij-1), A^{j+1}(ij+1), \dots, A^n(in)$ vectors. This is due to the fact that $A_1(1j), \dots, A_{i-1}(i-1j), A_{i+1}(i+1j), \dots, A_n(nj)$ and $A^1(i1), \dots, A^{j-1}(ij-1), A^{j+1}(ij+1), \dots, A^n(in)$ are row and column vectors of the sub-matrix $A_{ij} = (a_{kl})_{(n-1) \times (n-1)}$ for $k \neq i$ and $\ell \neq j$, respectively. For any $k \neq i, i, k = 1, 2, \dots, n$, $\det(A_{ij})$ can be computed by using the k -th row A_k expansion of the original matrix A , that is, the k -th row vector $A_k(ij)$ of the sub-matrix A_{ij} obtained from A_k after deleting the i -th row and the j -th column of matrix A . In short, A_k and $A_k(ij)$ are n and $(n-1)$ -dimensional k -th row vectors of matrix A and sub-matrix A_{ij} , respectively. For $k \neq i$ and $\ell \neq j, i, k = 1, 2, \dots, n$, a_{kl} is the entry at the k -th row and the ℓ -th column of matrix A , and it is the component of the row vector $A_k(ij)$ of the sub-matrix A_{ij} . Hence, by using the definition of the determinant of $(n-1) \times (n-1)$, $M_{k\ell}(ij)$ is determined by this k -th row expansion. For simplicity, for $k \neq i$ and $\ell \neq j, M_{kl}(ij)$ is referred as the (k, ℓ) -th minor of sub-matrix A_{ij} corresponding to the k -th row and the ℓ -th column entry $a_{k\ell}$ of the original matrix in the context of (k, ℓ) -th entry $a_{k\ell}$ of the sub-matrix A_{ij} of matrix A . However, its exact representation is not essential to our discussion. We just need to know an information about $M_{kl}(ij)$ and its corresponding cofactor $C_{k\ell}(ij)$. From Definition 2.1, for any $k = 1, 2, \dots, i-1, i+1, \dots, n$, we have

$$M_{ij} = \det(A_{ij}) = \sum_{l \neq j}^n a_{kl} C_{kl}(ij) = A_k(ij) C_k^T(ij) = C_k(ij) A_k^T(ij)$$

where $M_{kl}(ij)$ is the (k, l) -th minor of sub-matrix A_{ij} corresponding to the k -th row and the l -th column entry $a_{k\ell}$ of the original matrix in the context of the entry $a_{k\ell}$ of sub-matrix A_{ij} of matrix A , for any $k \neq i$ and $l \neq j, i, k = 1, 2, 3, \dots, n$.

5. M_{ij} 's and C_{ij} 's differ by a constant factor $(-1)^{i+j}$. Therefore both M_{ij} 's and C_{ij} 's are independent of all the entries of i -th row and j -th column of matrix A . Of course, both $\det(A_{ij})$ and C_{ij} depend on the remaining entries $(n-1)^2$. In fact, the i -th row expansion of $\det(A)$ depends on all rows, and M_{ij}/C_{ij} depends

on all rows of A except the i -th row. The j -th column expansion of $\det(A)$ depends on all columns, and M_{ij}/C_{ij} depends on all columns of A except the j -th column. An analogous statement can be made with regard to $M_{kl}(ij)$'s as defined in Conclusion 4. In fact, the k -th row expansion of $\det(A_{ij}) = M_{ij}$, corresponding to the k -th row of the original matrix A , depends on all rows of matrix A except the i -th row, and $M_{k\ell}(ij)/C_{k\ell}(ij)$ depends on all rows except i -th and k -th rows. The l -column expansion of $\det(A_{ij}) = M_{ij}$, corresponding to the l -column of the original matrix A , depends on all columns except the j -th column of A , and $M_{k\ell}(ij)/C_{kl}(ij)$ depends on all columns of A except the j -th and the ℓ -th columns of matrix A . We further note that $M_{kl}(ij)$ and $C_{k\ell}(ij)$ defined in Conclusion 4 are independent of i -th row, k -th row, j -th column and ℓ -th column of matrix A .

6. In addition, we note that

$$\det(A) = W(a_{11}, \dots, a_{1n}, \dots, a_{i1}, \dots, a_{in}, \dots, a_{n1}, \dots, a_{nn})$$

is a real/complex number valued function of n^2 variables. Furthermore, $\det(A)$ can be considered to be the function of n n -dimensional row vectors $A_1, A_2, \dots, A_i, \dots, A_n$, or n -dimensional column vectors $A^1, A^2, \dots, A^j, \dots, A^n$. In this setup, it is represented by

$$\det(A) = W(A_1, \dots, A_j, \dots, A_n) = W(A^1, \dots, A^i, \dots, A^n).$$

Moreover, any $i = 1, 2, \dots, n$, and from Definition 2.1, $\det(A)$ can be considered to be the product of two matrices A_i and C_i^T (A_i^T and C_i) that are i -th row of matrix A and j -th column of its adjoint matrix C^T corresponding to A . In conclusion, $\det(A) = A_i C_i^T = C_i A_i^T$. From the definition of the determinant of any $n \times n$ matrix A and Conclusion 4, we have another representation of $\det(A)$:

$$\det(A) = \sum_{i=1}^n a_{ij} (-1)^{i+j} C_k(ij) A_k^T(ij)$$

where $A_k(ij)$ is the k -th row of matrix A obtained by deleting the i -th row and j -th column vectors of matrix A for any $i, k = 1, 2, \dots, n$ and $k \neq i$.

7. In summary, for any positive integer $n > 1$, $\det(A)$ has the following row representation

$$\begin{aligned} W(A_1, A_2, \dots, A_i, \dots, A_n) &= \det(A) = \sum_{j=1}^n a_{ij} C_{ij} = A_i C_i^T = C_i A_i^T \\ &= \sum_{j=1}^n a_{ij} (-1)^{i+j} C_k(ij) A_k^T(ij) \end{aligned}$$

for any i and k , $i, k = 1, 2, \dots, n$ and $k \neq i$. Moreover, the C_{ij} 's are independent of the i -th row and j -th column of matrix A , and the $C_k^T(ij)$'s are independent

of the i -th row, k -th row, and j -th row and l -th column of matrix A . A similar comment can be made regarding the column representations for $\det(A)$.

8. In the case of $n = 1$, $A = (a_{ij})_{1 \times 1}$, the cofactor C_{ij} of a_{ij} is defined to be 1. Hence $\det(A) = (-1)^{1+1}a_{11}C_{11} = a_{11}$.

If matrix size is large, the computation of the determinant of the $n \times n$ matrix A by Definition 2.1 is not efficient. However, the concept of the determinant possesses several properties that are useful to increase the efficiency of the computational procedure. For easy reference, we state the following.

Theorem 2.3 ([3]). *Let $A = (a_{ij})_{n \times n}$ be an $n \times n$ matrix.*

- P₁. $\det(A^T) = \det(A)$, where A^T is the transpose of A .
- P₂. $\det(A') = -\det(A)$, where A' is obtained from A by interchanging two adjacent rows (or columns) of the matrix. In fact, this property remains valid for interchanging any two rows (or columns).
- P₃. If A has two identical rows (or columns), then $\det(A) = 0$.
- P₄. $\det(A') = c\det(A)$, where A' is obtained from A by multiplying a row (or column) by a scalar c quantity.
- P₅. $\det(A) = \det(A') + \det(A'')$, where each entry of the i -th row (or j -th column) of the matrix A is a sum ($a'_{ij} + a''_{ij}$), and when A' and A'' have the same entries as A , except in their either i -th row (or j -th column), in which they have entries a'_{ij} and a''_{ij} , respectively, $j = 1, 2, \dots, n$ (or $i = 1, 2, \dots, n$).
- P₆. $\det(A') = \det(A)$, where A' is obtained from A by adding a scalar multiple of a k -th row to an i -th row (or an l -th column to a j -th column with $l \neq j$) with $k \neq i$.
- P₇. If A is a triangular matrix (upper/lower), then $\det(A) = a_{11}a_{22} \cdots a_{jj} \cdots a_{nn}$, the product of its diagonal entries.

Observation 2.4. Let $A(t) = (a_{ij}(t))_{m \times n}$ be an $m \times n$ differentiable matrix valued function defined on J , and let Δt be an increment in t . The differential of the $m \times n$ matrix $A(t)$ is defined by: $dA(t) = (da_{ij}(t)dt)_{m \times n} = A(t)dt = dt(a_{ij}(t))_{m \times n}$, where $\Delta t = dt$. We note that the operation of differential is linear [4, 7, 11]. In fact,

- (i) $d(A(t) + B(t)) = dA(t) + dB(t)$ (Addition Rule).
- (ii) $d((cA)(t)) = c(t)dA(t) + A(t)dc(t)$, where c is scalar function (Product Rule for scalar multiplication).
- (iii) $d(A(t)B(t)) = dA(t)B(t) + A(t)dB(t)$ (Product Rule).

The following result provides expressions for the first derivative and differential of the determinant function associated with an $n \times n$ differentiable matrix function.

Lemma 2.5. *Let A be an $n \times n$ differentiable matrix function defined on J , and let $\det(A) = W(A_1, \dots, A_k, \dots, A_n) = W(A^1, \dots, A^k, \dots, A^n)$ be the determinant function defined in Definition 2.1. Then,*

$$\begin{aligned} \frac{d}{dt} \det(A) &= \frac{d}{dt} W(A_1, \dots, A_k, \dots, A_n) = \sum_{i=1}^n C_i \frac{d}{dt} A_i^T \\ &= \sum_{k=1}^n W\left(A_1, \dots, \frac{d}{dt} A_k, \dots, A_n\right) \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} d \det(A(t)) &= d W(A_1, \dots, A_k, \dots, A_n) \\ &= \sum_{k=1}^n dA_k \frac{\partial}{\partial A^k} W(A_1, \dots, A_k, \dots, A_n) \\ &= \sum_{k=1}^n W(A_1, \dots, dA_k, \dots, A_n). \end{aligned} \quad (2.2)$$

Proof. Based on Definition 2.1 and Observation 2.2, we have

$$\frac{\partial}{\partial A_i} W(A_1, \dots, A_i, \dots, A_n) = C_i \text{ and } \frac{\partial}{\partial A_i} C_i = 0, \text{ for any } i, \quad 1 \leq i \leq n. \quad (2.3)$$

This occurs because the co-factor vector C_i is independent of the i -th row vector A_i of the matrix A and because $W(A_1, \dots, A_i, \dots, A_n)$ is independent of any row expansion of A . Hence,

$$\begin{aligned} \frac{d}{dt} \det(A) &= \sum_{i=1}^n \frac{\partial}{\partial A_i} W(A_1, \dots, A_i, \dots, A_n) \frac{d}{dt} A_i^T \\ &= \sum_{k=1}^n C_k \frac{d}{dt} A_k^T \\ &= \sum_{i=1}^n W\left(A_1, \dots, \frac{d}{dt} A_i, \dots, A_n\right). \end{aligned} \quad (2.4)$$

This establishes the validity of (2.1). The validity of (2.2) follows from the above argument and the concept of the differential. \square

The following result provides expressions for the second derivative and differential of the determinant function associated with an $n \times n$ differentiable matrix function. It is useful in studying linear systems of differential equations [4, 5, 7, 8, 10–12].

Lemma 2.6. *Let A be an $n \times n$ twice differentiable matrix function defined on J , and let $\det(A) = W(A_1, \dots, A_k, \dots, A_n) = W(A^1, \dots, A^k, \dots, A^n)$ be the determinant*

function defined in Definition 2.1. Then,

$$\begin{aligned}
 \frac{d^2}{dt^2} \det(A) &= \frac{d^2}{dt^2} W(A_1, \dots, A_k, \dots, A_n) \\
 &= \sum_{i=1}^n C_i \frac{d^2}{dt^2} A_i^T + \sum_{i=1}^n \sum_{k \neq i}^n \frac{d}{dt} A_k \frac{\partial}{\partial A_k} C_i \frac{d}{dt} A_i^T \\
 &= \sum_{i=1}^n W\left(A_1, \dots, \frac{d^2}{dt^2} A_i, \dots, A_n\right) \\
 &\quad + \sum_{i=1}^n \sum_{k \neq i}^n W\left(A_1, \dots, \frac{d}{dt} A_i, \dots, \frac{d}{dt} A_k, \dots, A_n\right). \tag{2.5}
 \end{aligned}$$

$$\begin{aligned}
 d^2 \det(A) &= d^2 W(A_1, \dots, A_k, \dots, A_n) \\
 &= \sum_{i=1}^n C_i d^2 A_i^T + \sum_{i=1}^n \sum_{k \neq i}^n dA_k \frac{\partial}{\partial A_k} C_i dA_i^T \\
 &= \sum_{i=1}^n W(A_1, \dots, d^2 A_i, \dots, A_n) \\
 &\quad + \sum_{i=1}^n \sum_{k \neq i}^n W(A_1, \dots, dA_i, \dots, dA_k, \dots, A_n). \tag{2.6}
 \end{aligned}$$

Proof. Based on the argument used in the proof of Lemma 2.5, Observation 2.2, and (2.1), we obtain expressions as

$$\begin{aligned}
 \frac{d^2}{dt^2} \det(A) &= \frac{d^2}{dt^2} W(A_1, \dots, A_k, \dots, A_n) \\
 &= \sum_{i=1}^n C_i \frac{d^2}{dt^2} A_i^T + \sum_{i=1}^3 \frac{d}{dt} C_i \frac{d}{dt} A_i^T, \tag{2.7}
 \end{aligned}$$

and

$$\frac{d}{dt} C_i = \sum_{k=1}^3 \frac{d}{dt} A_k \frac{\partial}{\partial A_k} C_i, \text{ for any } i, k = 1, 2, \dots, n, \tag{2.8}$$

where,

$$\frac{\partial}{\partial A_k} C_i = \begin{cases} \left(\frac{\partial}{\partial a_{kl}} C_{ij}\right)_{n \times n}, & \text{for any } k \neq i \\ 0, & \text{for } k = i \end{cases} \tag{2.9}$$

$$\frac{\partial}{\partial A_k} C_{kl}(ij) = 0, \quad \frac{\partial}{\partial a_{kl}} C_{ij} = \begin{cases} (-1)^{i+j} C_{kl}(ij), & \text{for } l \neq j \\ 0, & \text{for } l = j. \end{cases}$$

We substitute the expression in (2.9) into (2.8) and then into (2.7); we derive the following formula:

$$\begin{aligned} \frac{d^2}{dt^2} \det(A) &= \frac{d^2}{dt^2} W(A_1, \dots, A_k, \dots, A_n) \\ &= \sum_{i=1}^n C_i \frac{d^2}{dt^2} A_i + \sum_{i=1}^n \sum_{k \neq i}^n \frac{d}{dt} A_k \frac{\partial}{\partial A_k} C_i \frac{d}{dt} A_i^T, \end{aligned} \tag{2.10}$$

for any $k \neq i, i, k = 1, 2, \dots, n$. By recalling the representation of C_i as

$$C_i = ((-1)^{i+j} A_k(ij) C_k^T(ij))_{1 \times n}, \text{ for any } k = 1, 2, \dots, i - 1, i + 1, \dots, n, \tag{2.11}$$

and using (2.11), we have

$$\frac{\partial}{\partial A_k} C_i^T \left(\frac{d}{dt} A_k \right)^T = \left((-1)^{i+j} \sum_{k \neq i}^n \frac{d}{dt} A_k(ij) C_k^T(ij) \right)_{1 \times n}. \tag{2.12}$$

From (2.1), (2.12), and the definition of the determinant and its variants, we obtain

$$\begin{aligned} \frac{d^2}{dt^2} \det(A) &= \frac{d}{dt} \left(\sum_{i=1}^n \frac{d}{dt} A_i \frac{\partial}{\partial A_i} W \right) = \sum_{i=1}^n C_i \frac{d^2}{dt^2} A_i^T + \sum_{i=1}^n \sum_{k \neq i}^n \frac{d}{dt} A_k \frac{\partial}{\partial A_k} C_i \frac{d}{dt} A_i^T \\ &= \sum_{i=1}^n C_i \frac{d^2}{dt^2} A^T + \sum_{j=1}^n \frac{d}{dt} a_{ij} \left[(-1)^{i+j} \sum_{k \neq i}^n C_k(ij) \frac{d}{dt} A_k^T(ij) \right] \\ &= \sum_{i=1}^n W \left(A_1, \dots, \frac{d^2}{dt^2} A_i, \dots, A_n \right) \\ &\quad + \sum_{i=1}^n \sum_{k \neq i}^n W \left(A_1, \dots, \frac{d}{dt} A_i, \dots, \frac{d}{dt} A_k, \dots, A_n \right). \end{aligned} \tag{2.13}$$

This completes the proof of (2.5), and the proof of (2.6) can be constructed analogously. □

In the following, we present a Generalized Mean-Value Theorem for differential calculus and the Taylor Polynomial Theorem of degree 2 for a determinant function. These results play a very important role in the study of stochastic differential equations of Itô-Doob type [12].

Theorem 2.7 (Generalized Mean-Value Theorem). *Let A and X be $n \times n$ matrices. Let $\det(A) = W(A) = W(A_1, \dots, A_k, \dots, A_n)$ and $\det(X) = W(X) = W(X_1, \dots, X_k, \dots, X_n)$ be the determinant of A and X , respectively. Let L be an $n \times n$ matrix function defined on the interval $[0, 1]$ by $L(t) = A + t(X - A)$ for t in*

$[0, 1]$. Then,

$$\det(X) - \det(A) = \sum_{i=1}^n (X_i - A_i) C_i^T(A) + \int_0^1 (X_i - A_i) [C_i^T(L(t)) - C_i^T(A)] dt, \tag{2.14}$$

where $C_i(L(t))$ and $C_i(A)$ are the i -th rows of cofactor matrices $C(L(t))$ and $C(A)$ of corresponding matrices $L(t)$ and A , respectively.

Proof. From Observation 2.2, we note that a determinant of an $n \times n$ matrix is the n -th degree homogeneous polynomial function of n^2 entries of the matrix as the independent variables. We know that every polynomial function in several independent variables is continuously differentiable. Hence, the determinant of any $n \times n$ matrix is continuously differentiable with respects to its independent variables. $L(t) = A + t(X - A)$ (a line segment joining A and X) for t in $[0, 1]$ is also continuously differentiable on $[0, 1]$. We define a function h defined on $[0, 1]$ into R as follows:

$$h(t) = \det(L(t)) = W(L(t)) = W(L(t)_1, \dots, L_i(t), \dots, L_n(t)).$$

h is a composite function of continuously differentiable functions, namely, “det” and L . Now, Lemma 2.5 is applicable to $\det(L(t))$, and hence we have

$$\begin{aligned} h'(t) &= \frac{d}{dt} \det(L(t)) = \sum_{i=1}^n \frac{d}{dt} L_i C_i^T \\ &= \sum_{i=1}^n (X_i - A_i) C_i^T(L(t)), \text{ (by } \frac{d}{dt} L_i = (X_i - A_i)) \end{aligned}$$

Now, we integrate the above expression both sides from $t = 0$ to $t = 1$. This is possible because the expression is a continuous function on the interval $[0, 1]$. Hence,

$$\begin{aligned} \int_0^1 h'(t) dt &= \int_0^1 \left[\sum_{i=1}^n (X_i - A_i) C_i^T(L(t)) \right] dt \\ h(1) - h(0) &= \int_0^1 \left[\sum_{i=1}^n (X_i - A_i) C_i^T(L(t)) \right] dt \\ &= \sum_{i=1}^n (X_i - A_i) \int_0^1 C_i^T(L(t)) dt. \end{aligned}$$

Along with the definition of the function h , one gets:

$$\begin{aligned} h(1) &= \det(L(1)) = \det(X) \text{ (by } L(1) = A + 1(X - A) = X), \\ h(0) &= \det(L(0)) = \det(A) \text{ (by } L(0) = A + 0(X - A) = A). \end{aligned}$$

Hence

$$h(1) - h(0) = \det(X) - \det(A) = \sum_{i=1}^n (X_i - A_i) \int_0^1 C_i^T(L(t)) dt. \tag{2.15}$$

By adding and subtracting $(X_i - A_i) C_i^T(A)$ to (2.15), the right-hand side of (2.14) follows immediately. This completes the proof. \square

Theorem 2.8 (Taylor's Formula). *Let us assume that all the hypotheses of Theorem 2.7 are satisfied. Let N be an $n \times n$ matrix function defined on $[0, 1] \times [0, 1]$ by $N(t, s) = A + st(X - A)$ for (t, s) in $[0, 1] \times [0, 1]$. Then,*

$$\begin{aligned} \det(X) &= \det(A) + \sum_{i=1}^n (X_i - A_i) C_i^T(A) \\ &+ \frac{1}{2!} \sum_{i=1}^n \sum_{k \neq i}^n (X_i - A_i) \frac{\partial}{\partial A_k} C_i^T(A) (X_k - A_k)^T \\ &+ \sum_{i=1}^n \sum_{k \neq i}^n (X_i - A_i) \int_0^1 \left[\int_0^1 t [0^{ik}(X - A)] ds (X_k - A_k)^T \right] dt, \end{aligned} \quad (2.16)$$

where

$$\frac{\partial}{\partial A_k} C_i^T(N(t, s)) - \frac{\partial}{\partial A_k} C_i^T(A) = 0^{ik}(X - A),$$

$0^{ik}(X - A)$ is independent of the i -th and k -th row of the matrix $(X - A)$, and it is bounded by the magnitude of $(X - A)$.

Proof. From (2.14), we have

$$\begin{aligned} \det(X) &= \det(A) + \sum_{i=1}^n (X_i - A_i) C_i^T(A) \\ &+ \sum_{i=1}^n (X_i - A_i) \int_0^1 [C_i^T(L(t)) - C_i^T(A)] dt. \end{aligned} \quad (2.17)$$

Moreover, under the hypotheses of the theorem and by following the argument used in Theorem 2.7, $C_i^T(L(t)) - C_i^T(A)$ in (2.17) can be represented by

$$C_i^T(L(t)) - C_i^T(A) = \sum_{k \neq i}^n \int_0^1 \left[\frac{\partial}{\partial A_k} C_i^T(N(t, s)) ds \right] t (X_k - A_k), \quad (2.18)$$

where N is an $n \times n$ matrix function defined on $[0, 1] \times [0, 1]$ by $N(t, s) = A + st(X - A)$ for (t, s) in $[0, 1] \times [0, 1]$. We substitute the expression in (2.18) into (2.17) and we

have

$$\begin{aligned}
 \det(X) &= \det(A) + \sum_{i=1}^n (X_i - A_i) C_i^T(A) \\
 &\quad + \sum_{k=1}^n (X_i - A_i) \int_0^1 \left[\sum_{k \neq i}^n \int_0^1 \left[\frac{\partial}{\partial A_k} C_i^T(N(t, s)) ds \right] t (X_k - A_k)^T \right] dt \\
 &= \det(A) + \sum_{i=1}^n (X_i - A_i) C_i^T(A) \\
 &\quad + \sum_{i=1}^n \sum_{k \neq i}^n (X_i - A_i) \int_0^1 \left[\int_0^1 \left[\frac{\partial}{\partial A_k} C_i^T(N(t, s)) ds \right] t (X_k - A_k)^T \right] dt.
 \end{aligned} \tag{2.19}$$

Again by adding and subtracting

$$\frac{1}{2!} \sum_{i=1}^n \sum_{k \neq i}^n (X_i - A_i) \frac{\partial}{\partial A_k} C_i^T(A) (X_k - A_k)^T$$

in the right hand side of (2.19) and noting the fact

$$\begin{aligned}
 &\frac{1}{2!} \sum_{i=1}^n \sum_{k \neq i}^n (X_i - A_i) \frac{\partial}{\partial A_k} C_i^T(A) (X_k - A_k)^T \\
 &= \sum_{i=1}^n \sum_{k \neq i}^n (X_i - A_i) \int_0^1 \left[\int_0^1 \left[\frac{\partial}{\partial A_k} C_i^T(A) \right] t (X_k - A_k)^T \right] ds dt,
 \end{aligned}$$

we finally get:

$$\begin{aligned}
 \det(X) &= \det(A) + \sum_{i=1}^n (X_i - A_i) C_i^T(A) + \frac{1}{2!} \sum_{i=1}^n \sum_{k \neq i}^n (X_i - A_i) \frac{\partial}{\partial A_k} C_i^T(A) (X_k - A_k)^T \\
 &\quad + \sum_{i=1}^n \sum_{k \neq i}^n (X_i - A_i) \int_0^1 \left[\int_0^1 [0^{ik}(X - A)] t (X_k - A_k)^T \right] ds dt,
 \end{aligned}$$

where

$$\frac{\partial}{\partial A_k} C_i^T(N(t, s)) - \frac{\partial}{\partial A_k} C_i^T(A) = 0^{ik}(X - A).$$

□

3. APPLICATIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS

In this section, we present a few fundamental results that play a very important role in the study of both the computational and conceptual aspects of solutions of linear systems of stochastic differential equations. Let us consider the following system of linear homogeneous system of stochastic differential equations of Itô-Doob type:

$$dx = A(t)x dt + B(t)x dw(t). \tag{3.1}$$

Its corresponding nonhomogeneous case and initial value problems are

$$dy = [A(t)y + p(t)] dt + [B(t)y + q(t)] dw(t) \quad (3.2)$$

and

$$dx = A(t)x dt + B(t)x dw(t), \quad x(t_0) = x_0, \quad (3.3)$$

respectively.

Here dx stands for the stochastic differential the of Itô-Doob type. A and B are $n \times n$ continuous matrix functions defined on an interval $J = [a, b](J \subseteq R)$; p and q are n -dimensional continuous vector functions defined on an interval $J = [a, b](J \subseteq R)$. w is a scalar normalized Wiener process. Initial data/conditions $(t_0, x_0) \in J \times R^n$; x_0 is an R^n -valued random vector defined on a complete probability space $(\Omega, \mathfrak{F}, P)$ independent of $w(t)$ for all t in J and $E[\|x_0\|^2] < \infty$. Moreover, we remark that under these conditions, the initial value problem has a unique solutions [6, 13].

Now we present a stochastic version of Abel-Jacobi-Liouville [4, 8, 9, 11, 17]. This result provides an alternative analytic test for the complete set of solutions of both the time-invariant and time-varying linear homogeneous systems of the Itô-Doob type stochastic differential equations. This test is easy to verify and computationally attractive.

Theorem 3.1 (Stochastic Version of Abel-Jacobi-Liouville [8, 9, 17]). *Let $x_1(t), x_2(t), \dots, x_k(t), \dots, x_n(t)$ be any n solutions process of (3.1) on J . Let*

$$\begin{aligned} \Phi(t, w(t)) &\equiv \Phi(t) = [x_1(t), x_2(t), \dots, x_k(t), \dots, x_n(t)] \\ &= [\Phi_1(t), \dots, \Phi_i(t), \dots, \Phi_n(t)]^T \end{aligned}$$

be the $n \times n$ matrix function defined on J , where $\Phi_i(t) = [x_{i1}, x_{i2}, \dots, x_{ik}, \dots, x_{im}]$ is the i -th row of matrix process $\Phi(t)$. Let

$$\begin{aligned} \mathbb{L}(B(t)) &= \frac{1}{2} \sum_{i=1}^n \sum_{k \neq i}^n (b_{ii}(t)b_{kk}(t) - b_{ik}b_{ki}), \\ \text{tr}(A(t)) &= \sum_{i=1}^n a_{ii}(t), \end{aligned}$$

and

$$\text{tr}(B(t)) = \sum_{i=1}^n b_{ii}(t).$$

Then,

$$d \det(\Phi(t)) = [\text{tr}(A(t)) + \mathbb{L}(B(t))] \det(\Phi(t)) dt + \text{tr}(B(t)) \det(\Phi(t)) dw(t), \quad (3.4)$$

and

$$\begin{aligned} & \exp \left[- \int_{t_0}^t \left[\text{tr}(A(s)) + \mathbb{L}(B(s)) - \frac{1}{2} (\text{tr}(B(s)))^2 \right] ds \right. \\ & \left. - \int_{t_0}^t \text{tr}(B(s)) dw(s) \right] \det(\Phi(t)) = C \text{ (constant)}. \end{aligned} \tag{3.5}$$

Moreover, $\det(\Phi(t)) \neq 0$ (a **general fundamental matrix solution process** of (3.1)) if and only if $C \neq 0$.

Proof. Due to the nature of (3.1) and definition of $\Phi(t)$, it is obvious that $\Phi(t)$ satisfies (3.1), that is

$$\begin{aligned} d\Phi(t) &= (dx_{ik}(t))_{n \times n} = \left(\sum_{j=1}^n a_{ij} x_{jk}(t) dw(t) + \sum_{j=1}^n b_{ij} x_{jk}(t) dw(t) \right)_{n \times n} \\ &= A(t)\Phi(t) dt + B\Phi(t) dw(t). \end{aligned} \tag{3.6}$$

Applying Theorem 2.8 to $\det(\Phi(t))$, we have

$$\begin{aligned} d \det(\Phi(t)) &= \sum_{i=1}^n d\Phi_i(t) C_i^T(\Phi(t)) + \frac{1}{2!} \sum_{i=1}^n \sum_{k \neq i}^n d\Phi_i(t) \frac{\partial}{\partial \Phi_k} C_i^T(\Phi(t)) (d\Phi_k(t))^T \\ &= \sum_{i=1}^n W(\Phi_1(t), \dots, d\Phi_i(t), \dots, \Phi_n(t)) \quad (\text{by Observation 2.2}) \\ &\quad + \frac{1}{2!} \sum_{i=1}^n \sum_{k \neq i}^n W(\Phi_1(t), \dots, d\Phi_i(t), \dots, d\Phi_k(t), \dots, \Phi_n(t)). \end{aligned} \tag{3.7}$$

Now, we compute the expressions for the terms in the right-hand side of (3.7). For each i , $1 \leq i \leq n$, we first rewrite $W(\Phi_1(t), \dots, d\Phi_i(t), \dots, \Phi_n(t))$:

$$\begin{aligned} W(\Phi_1(t), \dots, d\Phi_i(t), \dots, \Phi_n(t)) &= \begin{vmatrix} x_{11} & \cdots & x_{1k} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ dx_{i1} & \cdots & dx_{ik} & \cdots & dx_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{nk} & \cdots & x_{nn} \end{vmatrix} \\ &= \begin{vmatrix} x_{11} & \cdots & x_{1k} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{j=1}^n [a_{ij}(t)x_{j1}(t)dt \\ + b_{ij}(t)x_{j1}dw(t)] & \cdots & \sum_{j=1}^n [a_{ij}(t)x_{jk}(t)dt \\ + b_{ij}(t)x_{jk}dw(t)] & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{nk} & \cdots & x_{nn} \end{vmatrix}. \end{aligned}$$

Now, by applying the property of Theorem 2.3: P_6 of the determinant, we have

$$W(\Phi_1(t), \dots, d\Phi_1(t), \dots, \Phi_n(t)) = \begin{vmatrix} x_{11} & \cdots & x_{1k} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{ii}(t)x_{i1}(t)dt + b_{ii}(t)x_{i1}dw(t) & \cdots & a_{ii}(t)x_{ik}(t)dt + b_{ii}(t)x_{ik}dw(t) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{nk} & \cdots & x_{nn} \end{vmatrix}.$$

This, together with Theorem 2.3, P_4 and P_5 , one gets

$$W(\Phi_1(t), \dots, d\Phi_i(t), \dots, \Phi_n(t)) = a_{ii}(t) \det(\Phi(t))dt + b_{ii}(t) \det(\Phi(t)) dw(t),$$

for each i , $1 \leq i \leq n$. Hence the expression for the first term in (3.7) is

$$\begin{aligned} \sum_{i=1}^n W(\Phi_1(t), \dots, d\Phi_i(t), \dots, \Phi_n(t)) &= \sum_{i=1}^n [a_{ii}(t) dt + b_{ii}(t) dw(t)] \det(\Phi(t)) \\ &= \text{tr}(A(t)) \det(\Phi(t)) dt + \text{tr}(B(t)) \det(\Phi(t)) dw(t). \end{aligned} \tag{3.8}$$

Now, for each $i, k, k \neq i, i, k = 1, 2, \dots, n$, we rewrite $W(\Phi_1(t), \dots, d\Phi_i(t), \dots, d\Phi_r(t), \dots, \Phi_n(t))$:

$$W(\Phi_1(t), \dots, d\Phi_i(t), \dots, d\Phi_r(t), \dots, \Phi_n(t)) = \begin{vmatrix} x_{11} & \cdots & x_{1\ell} & \cdots & x_{1k} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ dx_{i1} & \cdots & dx_{i\ell} & \cdots & dx_{ik} & \cdots & dx_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ dx_{r1} & \cdots & dx_{r\ell} & \cdots & dx_{rk} & \cdots & dx_{rn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{n\ell} & \cdots & x_{nk} & \cdots & x_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} x_{11} & \cdots & x_{1\ell} & \cdots & x_{1k} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \sum_{j=1}^n [a_{ij}(t)x_{j\ell}(t)dt + b_{ij}(t)x_{j\ell}dw(t)] & \cdots & \sum_{j=1}^n [a_{ij}(t)x_{jk}(t)dt + b_{ij}(t)x_{jk}dw(t)] & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \sum_{j=1}^n [a_{rj}(t)x_{j\ell}(t)dt + b_{rj}(t)x_{j\ell}dw(t)] & \cdots & \sum_{j=1}^n [a_{rj}(t)x_{jk}(t)dt + b_{rj}(t)x_{jk}dw(t)] & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{ni} & \cdots & x_{nk} & \cdots & x_{nn} \end{vmatrix}.$$

First, by the repeated application of the property in Theorem 2.3: P_6 of the determinant and by then repeatedly applying the property in Theorem 2.3: P_5 , we have

By applying Theorem 2.3: P_4 and using the fact that the $E[(dw(t))^2] = dt$, the above repression reduces to:

$$W(\Phi_1(t), \dots, d\Phi_i(t), \dots, d\Phi_r(t), \dots, \Phi_n(t)) =$$

$$\begin{aligned} & \begin{vmatrix} x_{11} & \cdots & x_{1\ell} & \cdots & x_{1k} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & x_{i\ell} & \cdots & x_{ik} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & x_{r\ell} & \cdots & x_{rk} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{n\ell} & \cdots & x_{nk} & \cdots & x_{nn} \end{vmatrix} b_{ii}(t)b_{rr}(t)dt \\ & + \begin{vmatrix} x_{11} & \cdots & x_{1\ell} & \cdots & x_{1k} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & x_{i\ell} & \cdots & x_{ik} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & x_{i\ell} & \cdots & x_{ik} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{n\ell} & \cdots & x_{nk} & \cdots & x_{nn} \end{vmatrix} b_{ii}(t)b_{ri}(t)dt \\ & + \begin{vmatrix} x_{11} & \cdots & x_{1\ell} & \cdots & x_{1k} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & x_{r\ell} & \cdots & x_{rk} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & x_{r\ell} & \cdots & x_{rk} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{n\ell} & \cdots & x_{nk} & \cdots & x_{nn} \end{vmatrix} b_{ir}(t)b_{rr}(t)dt \\ & + \begin{vmatrix} x_{11} & \cdots & x_{1\ell} & \cdots & x_{1k} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & x_{r\ell} & \cdots & x_{rk} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & x_{i\ell} & \cdots & x_{ik} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{n\ell} & \cdots & x_{nk} & \cdots & x_{nn} \end{vmatrix} b_{ir}(t)b_{ri}(t)dt. \end{aligned} \tag{3.9}$$

By observing the fact that the second and third terms in (3.9) are the determinants with the i -th and r -th rows being identical, and by applying the property of the determinant (Theorem 2.3 P_3), (3.9) reduces to:

$$\begin{aligned}
 W(\Phi_1(t), \dots, d\Phi_i(t), \dots, d\Phi_r(t), \dots, \Phi_n(t)) = & \\
 & \begin{vmatrix} x_{11} & \cdots & x_{1\ell} & \cdots & x_{1k} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & x_{i\ell} & \cdots & x_{ik} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & x_{r\ell} & \cdots & x_{rk} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{n\ell} & \cdots & x_{nk} & \cdots & x_{nn} \end{vmatrix} b_{ii}(t)b_{rr}(t)dt \\
 & \\
 + & \begin{vmatrix} x_{11} & \cdots & x_{1\ell} & \cdots & x_{1k} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & x_{r\ell} & \cdots & x_{rk} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & x_{i\ell} & \cdots & x_{ik} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{n\ell} & \cdots & x_{nk} & \cdots & x_{nn} \end{vmatrix} b_{ir}(t)b_{ri}(t)dt.
 \end{aligned} \tag{3.10}$$

This, along with the property of the determinant (Theorem 2.3: P_2) and the notation of the determinant of $\Phi(t)$, yields

$$\begin{aligned}
 W(\Phi_1(t), \dots, d\Phi_i(t), \dots, d\Phi_r(t), \dots, \Phi_n(t)) & \\
 = b_{ii}(t)b_{rr}(t) \det(\Phi(t)) dt - b_{ir}(t)b_{ri}(t) \det(\Phi(t)) dt & \\
 = (b_{ii}(t)b_{rr}(t) - b_{ir}(t)b_{ri}(t)) \det(\Phi(t)) dt & \tag{3.11}
 \end{aligned}$$

for each $ir, r \neq i, i, r = 1, 2, \dots, n$ and hence the expression for the second term in (3.7) is

$$\begin{aligned}
 \frac{1}{2!} \sum_{i=1}^n \sum_{k \neq i}^n W(\Phi_1(t), \dots, d\Phi_i(t), \dots, d\Phi_k(t), \dots, \Phi_n(t)) & \\
 = \frac{1}{2} \sum_{i=1}^n \sum_{k \neq i}^n (b_{ii}(t)b_{kk}(t) - b_{ik}(t)b_{ki}(t)) \det(\Phi(t)) dt & \\
 = \mathbb{L}(B) \det(\Phi(t)) dt. & \tag{3.12}
 \end{aligned}$$

We substitute the expressions for the first and second terms in (3.8) and (3.12) into (3.7) and obtain equation (3.4)

$$d \det(\Phi(t)) = [\text{tr}(A(t)) + \mathbb{L}(B(t))] \det(\Phi(t)) dt + \text{tr}(B(t)) \det(\Phi(t)) dw(t).$$

This establishes the linear first order scalar homogeneous Itô-Doob type stochastic differential equation (3.4). One can obtain an expression for a general solution of (3.4) by:

$$\det(\Phi(t)) = \exp \left[\int_{t_0}^t \left(\text{tr}(A(ts)) + \mathbb{L}(B(s)) - \frac{1}{2}(\text{tr}(B))^2 \right) ds + \int_{t_0}^t \text{tr}(B) dw(s) \right] C$$

where C is an arbitrary constant. The above equation is equivalent to the expression in (3.5). For $t = t_0$, if $\det(\Phi(t_0))$ is known, then $\det(\Phi(t_0)) = C$. Thus C is determined by the initial data $(t_0, \det(\Phi_0))$. Thus, the particular solution of (3.4) can be determined for any given value of Φ at $t = t_0$. \square

Remark 3.2. For some $t^* \in J$, $\det(\Phi(t^*)) = 0$ if and only if $\det(\Phi(t)) \equiv 0$. This is equivalent to the statement that the set of solutions of (3.1) in Theorem 3.1 do not form a complete set of independent solutions of (3.1).

In the following, we present a very significant byproduct of Theorem 3.1.

Theorem 3.3. *Let Φ be a fundamental matrix solution process of (3.1). Then, Φ is invertible, and its inverse Φ^{-1} satisfies the following matrix stochastic differential equations of Itô-Doob type:*

$$d\Phi^{-1}(t) = \Phi^{-1}(t) [-A(t) + B^2(t)] dt - \Phi^{-1}(t)B(t) dw(t). \tag{3.13}$$

Moreover,

$$d(\Phi^{-1})^T = [-A^T(t) + (B^T(t))^2] (\Phi^{-1}(t))^T dt - B^T(t) (\Phi^{-1}(t))^T dw(t). \tag{3.14}$$

Proof. From Theorem 3.1, we conclude that $\det(\Phi(t)) \neq 0$. This implies that the fundamental matrix solution process of (3.1) Φ is invertible. Its inverse is denoted by Φ^{-1} .

In order to establish (3.13), we compute the Itô-Doob differential of both sides of the processes in $\Phi\Phi^{-1} = I$, and we obtain

$$d(\Phi\Phi^{-1}) = d\Phi\Phi^{-1} + \Phi d\Phi^{-1} + d\Phi d\Phi^{-1} = dI = 0,$$

and hence

$$d\Phi\Phi^{-1} + \Phi d\Phi^{-1} + d\Phi d\Phi^{-1} = 0.$$

Along with (3.6), this yields

$$\begin{aligned} d\Phi^{-1}(t) &= -\Phi^{-1}(t)d\Phi(t)\Phi^{-1}(t) - \Phi^{-1}(t)d\Phi(t)d\Phi^{-1}(t) \\ &= -\Phi^{-1}(t)A(t) dt - \Phi^{-1}(t)B(t) dw(t) \\ &\quad - \Phi^{-1}(t) [A(t)\Phi(t) dt + B(t)\Phi(t) dw(t)] d\Phi^{-1}(t). \end{aligned} \tag{3.15}$$

By using the unknown but explicit nature of $d\Phi^{-1}(t)$ in (3.15) and the nature of the Itô-Doob differential calculus, (3.15) can be simplified:

$$\begin{aligned} d\Phi^{-1}(t) &= -\Phi^{-1}(t)A(t)dt - \Phi^{-1}(t)B(t)dw(t) \\ &\quad - \Phi^{-1}(t)[A(t)\Phi(t)dt + B(t)\Phi(t)dw(t)][-\Phi^{-1}(t)B(t)dw(t)] \\ &= -\Phi^{-1}(t)A(t)dt - \Phi^{-1}(t)B(t)dw(t) + \Phi^{-1}(t)B^2dt \\ &= \Phi^{-1}[-A(t) + B^2(t)]dt - \Phi^{-1}(t)B(t)dw(t). \end{aligned}$$

This concludes the derivation of equation (3.13). The proof of the expression in (3.14) follows from the properties of transposition and inverse of matrices. \square

Observation 3.4. We note that $(\Phi^{-1})^T(t)$ is the fundamental matrix solution of an Itô-Doob type linear system of stochastic differential equations:

$$dy = \left[-A^T(t) + (B^T(t))^2\right]ydt - B^T(t)ydw(t), \quad y(t_0) = x_0. \quad (3.16)$$

This statement is proved later in this section. The system in (3.16) is called the **adjoint** to the system in (3.1). System (3.16) is equivalent to the following system:

$$dy = y[-A(t) + B^2(t)]dt - yB(t)dw(t), \quad y(t_0) = x_0^T. \quad (3.17)$$

We note that y in (3.17) is a row vector, and y in (3.16) is a column vector. In light of this notational understand, $\Phi^{-1}(t)$ is a fundamental matrix solution of (3.17). This can be justified by (3.13).

As a byproduct of Theorems 3.1 and 3.3, we now present a few algebraic properties of the normalized fundamental solution process of (3.1). These algebraic properties are stochastic versions of the deterministic results [4, 7, 8, 11]. Moreover, a method of solving linear nonhomogeneous stochastic differential equation (3.2) is outlined.

Lemma 3.5. *Let $\Phi(t, w(t), t_0) \equiv \Phi(t, t_0)$ and $\Psi(t, w(t), t_0) \equiv \Psi(t, t_0)$ be normalized fundamental matrix solution processes of (3.1) and (3.17) at $t = t_0$, respectively. Let $\Phi_1(t, t_1)$ be a normalized fundamental matrix solution of (3.1) at $t = t_1$. Then for all t_0, t_1, s , and t in I :*

$$(a) \quad \Psi(t, t_0)\Phi(t, t_0) = I_{n \times n} = \Phi(t_0, t)\Phi(t, t_0), \quad \text{for } t_0 \leq t; \quad (3.18)$$

where $I_{n \times n}$ is an $n \times n$ identity matrix, and

$$\Psi(t, t_0) = \Phi^{-1}(t, t_0) = \Phi(t_0, t), \quad \text{for } t_0 \leq t; \quad (3.19)$$

$$(b) \quad \Phi_1(t, t_1) = \Phi(t, t_0)\Phi_1(t_0, t_1), \quad \text{for } t_0 \leq t; \quad (3.20)$$

$$(c) \quad \Phi(t, t_0) = \Phi(t, s)\Phi(s, t_0), \quad \text{for } t_0 \leq t; \quad (3.21)$$

$$(d) \quad \Phi(t, s) = \Phi(t, t_0)\Psi(s, t_0) = \Phi(t, t_0)\Phi(t_0, s), \quad \text{for } t_0 \leq t; \quad (3.22)$$

$$(e) \quad \partial_s \Phi(t, s) = \Phi(t, s)[-A(s) + B^2(s)]ds - \Phi(t, s)B(s)dw(s), \quad (3.23)$$

where $\partial_s \Phi(t, s)$ is the Itô-Doob stochastic partial differential of $\Phi(t, s)$ with respect to s for fixed t .

Proof. To prove (a), we use the Itô-Doob stochastic differential and we compute

$$\begin{aligned} d(\Psi(t, t_0) \Phi(t, t_0)) &= d\phi(t, t_0) \Phi(t, t_0) + \Psi(t, t_0) d\Phi(t, t_0) + d\Psi(t, t_0) d\Phi(t, t_0) \\ &= [\Psi(t, t_0) [-A(t) + B^2(t)] dt - \Psi(t, t_0) B(t) dw(t)] \Phi(t, t_0) \\ &\quad + \Psi(t, t_0) [A(t)\Phi(t, t_0) dt + B(t)\Phi(t, t_0) dw(t)] \\ &\quad + (-\Psi(t, t_0) B(t)dw(t)) (B(t)\Phi(t)dw(t)) = 0. \end{aligned}$$

This establishes the fact that $\Psi(t, t_0) \Phi(t, t_0) = C$, where C is an arbitrary constant random matrix. This concludes the fact that if the Itô-Doob differential of a stochastic process is zero on an interval J for $t \geq t_0$. Hence, the matrix process $\Psi(t, t_0) \Phi(t, t_0)$ is a constant random matrix on J . Moreover, since $\Psi(t, t_0)$ and $\Phi(t, t_0)$ are normalized solution processes (3.1) and (3.17) at $t = t_0$, respectively, then,

$$\Psi(t, t_0) \Phi(t, t_0) = C = \Psi(t_0, t_0) \Phi(t_0, t_0) = I_{n \times n}. \tag{3.24}$$

This shows that $\Psi(t, t_0)$ is the algebraic inverse of $\Phi(t, t_0)$, and it is denoted by $\Phi(t_0, t)$. This statement is equivalent to the other notations in (3.18). In view of these notations and (3.24), we have

$$\Psi(t, t_0) \Phi(t, t_0) = I_{n \times n} = \Phi(t_0, t) \Phi(t, t_0), \text{ for } t_0 \leq t.$$

This completes the proof of (3.19).

To prove (b), we consider $\Upsilon(t) = \Phi^{-1}(t, t_0) \Phi_1(t, t_1)$ and compute from it the Itô-Doob differential of $\Upsilon(t)$. In fact, by using (3.6) and (3.13), we have

$$\begin{aligned} d\Upsilon(t) &= d\Phi^{-1}(t, t_0) \Phi_1(t, t_1) + \Phi^{-1}(t, t_0) d\Phi_1(t, t_1) + d\Phi^{-1}(t, t_0) d\Phi_1(t, t_1) \\ &= [\Phi^{-1}(t) [-A(t) + B^2(t)] dt - \Phi^{-1}(t) B(t) dw(t)] \Phi_1(t, t_1) \\ &\quad + \Phi^{-1}(t, t_0) [A(t)\Phi_1(t, t_1) dt + B(t)\Phi_1(t, t_1) dw(t)] \\ &\quad - \Phi^{-1}(t) B^2(t) \Phi_1(t, t_1) dt \\ &= 0. \end{aligned}$$

This implies that $\Upsilon(t) = C = \Phi^{-1}(t, t_0) \Phi_1(t, t_1)$, where C is a nonsingular matrix. Moreover, $C = \Phi^{-1}(t_0, t_0) \Phi_1(t_0, t_1) = \Phi_1(t_0, t_1)$. From this discussion, we conclude that $\Phi_1(t_0, t_1) = \Phi^{-1}(t, t_0) \Phi_1(t, t_1)$, which leads to, $\Phi_1(t, t_1) = \Phi(t, t_0) \Phi_1(t_0, t_1)$. This completes the proof of (3.20).

To prove (c), we note that $\Phi(t, t_0)$ and $\Phi(t, s)$ are fundamental solutions of (3.1). The proof of (3.21) follows from the proof of (3.20). Furthermore, the proof of (3.22) is a direct consequence of (3.19) and (3.21).

For the proof of (e), we apply the Itô-Doob differential to both sides of the expression in (3.22) with respect to s for fixed (t, t_0) , and we obtain

$$\begin{aligned} \partial_s \Phi(t, s) &= \Phi(t, t_0) [\Psi(s, t_0) [-A(s) + B^2(s)] ds - \Psi(s, t_0) B(s) dw(s)] \\ &= [-\Phi(t, t_0) \Psi(s, t_0) A(s) + \Phi(t, t_0) \Psi(s, t_0) B^2(s)] ds \\ &\quad - \Phi(t, t_0) \Psi(s, t_0) B(s) dw(s) \\ &= [-\Phi(t, s) A(s) + \Phi(t, s) B^2(s)] ds - \Phi(t, s) B(s) dw(s) \\ &= \Phi(t, s) [-A(s) + B^2(s)] ds - \Phi(t, s) B(s) dw(s). \end{aligned}$$

This completes the proof of the lemma. \square

Theorem 3.6 (Stochastic Version of Lagrange Formula [4, 8, 11, 17]). *Let Φ be the fundamental matrix solution process of (3.1). Then the solutions of (3.2) are given by:*

$$y(t) = \Phi(t) \left[c + \int_{t_0}^t \Phi^{-1}(s) [p(s) - B(s)q(s)] ds + \int_{t_0}^t \Phi^{-1}(s) q(s) dw(s) \right]. \quad (3.25)$$

Proof. Let

$$y(t) = \Phi(t)z(t) \quad (3.26)$$

be a solution of (3.2), where z is an unknown process that depends on both $(t, w(t))$. The goal is to find an unknown process. We apply the Itô-Doob stochastic differential to $y(t)\Phi^{-1}(t)$ and obtain

$$dz(t) = d\Phi^{-1}(t)y(t) = d\Phi^{-1}(t)y(t) + \Phi^{-1}(t) dy(t) + d\Phi^{-1}(t) dy(t). \quad (3.27)$$

From (3.2), (3.13) and (3.27), we get

$$\begin{aligned} dz(t) &= \Phi^{-1}(t) dy(t) + d\Phi^{-1}(t)y(t) + d\Phi^{-1}(t) dy(t) \\ &= \Phi^{-1}(t) [[A(t)y + p(t)] dt + [B(t)y + q(t)] dw(t)] \\ &\quad + [\Phi^{-1}(t) [-A(t) + B^2(t)] dt - \Phi^{-1}(t) B(t) dw(t)] y(t) \\ &\quad + [-\Phi^{-1}(t) B(t) dw(t)] [B(t)y + q(t)] dw(t) \\ &= [\Phi^{-1}(t) A(t)y + \Phi^{-1}(t) p(t)] dt + [\Phi^{-1}(t) B(t)y + \Phi^{-1}(t) q(t)] dw(t) \\ &\quad + [-\Phi^{-1}(t) A(t)y(t) + \Phi^{-1}(t) B^2(t)y(t)] dt - \Phi^{-1}(t) B(t)y(t) dw(t) \\ &\quad + [-\Phi^{-1}(t) B^2(t)y - \Phi^{-1}(t) B(t)q(t)] dt \\ &= \Phi^{-1}(t) [p(t) - B(t)q(t)] dt + \Phi^{-1}(t) q(t) dw(t). \end{aligned} \quad (3.28)$$

From (3.28) and (3.26), we have

$$\begin{aligned} y(t) &= \Phi(t)z(t) \\ &= \Phi(t) \left[c + \int_{t_0}^t \Phi^{-1}(s) [p(s) - B(s)q(s)] ds + \int_{t_0}^t \Phi^{-1}(s) q(s) dw(s) \right] \end{aligned}$$

where $c \equiv z(t_0)$ is an arbitrary constant random variable, since z is an unknown process. The method of finding a solution process is called the **Method of Variation of**

Parameters. The solution representation of (3.2) in (3.25) is called the **Stochastic Version of Lagrange-type Variation of Constants Formula** [15, 16]. \square

In the following, we present a relationship between the solution processes of linear non-homogeneous systems differential equations (3.2) and

$$dz = [z(-A(t) + B^2(t)) - r(t)] dt - [zB(t) + s(t)] dw(t), \quad z(t_0), \quad (3.29)$$

where A, B, p, q and w are as defined in (3.2), and r and s are continuous column vector functions. This is a non-homogeneous stochastic system of differential equations corresponding to an adjoint system of differential equations (3.17).

Lemma 3.7 (Stochastic Version of Green's Formula [4, 8]). *Let $y(t)$ and $z(t)$ be solution processes of (3.2) and (3.29). Then, for $t, t_0 \in J$,*

$$\begin{aligned} & z(t)y(t) - z(t_0)y(t_0) \\ &= \int_{t_0}^t [z(u)[p(u) - B(u)q(u)] - [r(u) + s(u)B(u)]y(u) - s(u)q(u) du \\ & \quad + \int_{t_0}^t [z(u)q(u) - s(u)y(u)] dw(u). \end{aligned} \quad (3.30)$$

for all $t \geq t_0$ and $t \in I, t_0 \in J$.

Proof. We compute the Itô-Doob differential of $z(t)y(t)$ as:

$$\begin{aligned} d(z(t)y(t)) &= dz(t)y(t) + z(t)dy(t) + dzdy \\ &= [z(t)(-A(t) + B^2(t)) - r(t)]y(t) dt - [z(t)B(t) + s(t)]y(t) dw(t) \\ & \quad + z(t)[A(t)y(t) + p(t)] dt + z(t)[B(t)y(t) + q(t)] dw(t) \\ & \quad - [z(t)B(t) + s(t)] dw(t)[B(t)y(t) + q(t)] dw(t) \\ &= [-z(t)(A(t)y(t) + z(t)B^2(t))y(t) - r(t)y(t)] dt \\ & \quad + [z(t)A(t)y(t) + z(t)p(t)] dt + [z(t)B(t)y(t) + z(t)q(t)] dw(t) \\ & \quad - [z(t)B(t)y(t) + s(t)y(t)] dw(t) \\ & \quad - [z(t)B^2(t)y(t) + s(t)B(t)y(t)] - [z(t)B(t)q(t) + s(t)q(t)] dt \\ &= [z(t)p(t) - r(t)y(t) - s(t)B(t)y(t) - z(t)B(t)q(t) - s(t)q(t)] dt \\ & \quad + [z(t)q(t) - s(t)y(t)] dw(t) \\ &= [z(t)(p(t) - B(t)q(t)) - (r(t) + s(t)B(t))y(t) - s(t)q(t)] dt \\ & \quad + [z(t)q(t) - s(t)y(t)] dw(t). \end{aligned} \quad (3.31)$$

This implies that

$$\begin{aligned}
 & z(t)y(t) - z(t_0)y(t_0) \\
 = & \int_{t_0}^t [z(u)[p(u) - B(u)q(u)] - [r(u) + s(u)B(u)]y(u) - s(u)q(u)] du \\
 & + \int_{t_0}^t [z(u)q(u) - s(u)y(u)] dw(u).
 \end{aligned} \tag{3.32}$$

□

ACKNOWLEDGMENT

This research was supported by Mathematical Sciences Division, US Army Research Office, Grant No. W911NF-07-01-0283.

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