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STOCHASTIC PROBLEMS IN SPACES OF ABSTRACT DISTRIBUTIONS

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1. INTRODUCTION

The Cauchy problem for stochastic equations with additive white noise in a Hilbert space H often arise as mathematical models of certain phenomena in different applications. Irregularity of the white noise leads to a problem of the choice of appropriate framework that allows to overcome, or "correct" this irregularity.

In this paper we discuss two ways of stating and solving the Cauchy problem for a linear operator-differential equation with the nonhomogenuity involving H-valued Gaussian white noise. These two ways differ by the choice of a space of generalized functions as a framework for setting and solving the problem. The first way employs the spaces of abstract distributions, or generalized (with respect to the time parameter t) functions with values in the space $L^2(\Omega; H)$ of square integrable H-valued random variables. It uses the techniques developed in (Fattorini, 1983; Melnikova & Alshansky, 1995). The second one employs spaces of generalized random variables or generalized functions with respect to the random parameter ω. It is based on the ideas of (Kuo, 1996; Holden, et. al., 1996; Melnikova et. al., 2003). The advantage of the first approach is that it allows to treat the stochastic Cauchy problem with additive noise even in the case when the operator coefficient in the equation is not the generator of a C_0-semigroup with no conditions of smoothness imposed on the initial value. Within the second approach the space $\Omega = \mathcal{S}'$ of tempered distributions with the normalized Gaussian measure (the white noise measure) defined on its Borel σ-field is taken as the probability space and the space of generalized H-valued random variables is introduced on it in such a way that the white noise becomes in a sense a smooth function. The advantage of this approach is that it allows to reduce the stochastic problem to a series of deterministic ones and solve the later by semigroup methods, although it yields solutions under conditions of certain smoothness of the initial value.

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2. STOCHASTIC CAUCHY PROBLEM IN SPACES OF ABSTRACT DISTRIBUTIONS WITH RESPECT TO t

Let H be a separable Hilbert space and Q be a symmetric positive-definite operator with eigenvalues $\{\sigma_j^2\}$, $\mathrm{Tr}Q = \sum_j \sigma_j^2 < \infty$. Suppose the corresponding eigenvectors $\{e_j\}$ form an orthonormal basis in H.

For any Banach space X by $\mathcal{D}'(X)$ and $\mathcal{S}'(X)$ denote the spaces of X-valued distributions (generalized functions) over spaces $\mathcal{D}$ and $\mathcal{S}$ of test functions respectively. Denote by $\mathcal{D}'_+(X)$ and $\mathcal{S}'_+(X)$ the subspaces of X-valued distributions and tempered distributions with bounded from below supports.

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and $\{W(t), t \geq 0\}$ be a Q-Wiener process.

Definition 1. Generalized random process $\{\mathbb{W}(\theta), \theta \in \mathcal{D}\}$, defined by

$$\langle \mathbb{W}, \theta \rangle := -\int_0^\infty W(t)\theta'(t)\,dt = \int_0^\infty \theta(t)\,dW(t)\,, \quad \theta \in \mathcal{D}\,, \tag{1}$$

is called the Q-white noise (associated with W).

The first integral in (1) is understood in Bochner sense and defines an element of $L_2(\Omega; H)$, thus $\mathbb{W} \in \mathcal{D}'\big(L_2(\Omega; H)\big)$. The second integral in (1) is an abstract Wiener integral. The equality follows from the generalization of the Ito formula for abstract functions.

Definition 2. Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be Banach spaces such that there is a bilinear operation $(u, v) \mapsto uv$ defined on $\mathcal{X} \times \mathcal{Y}$ with $uv \in \mathcal{Z}$, which is continuous with respect to second operand. For any $F \in \mathcal{D}'_+(\mathcal{X})$ and $G \in \mathcal{D}'_+(\mathcal{Y})$ define $F * G \in \mathcal{D}'_+(\mathcal{Z})$ by

$$\langle F * G\,, \theta \rangle := \big\langle (f * g)^{(n+m)}, \theta \big\rangle = (-1)^{n+m} \int_{\mathbb{R}} (f * g)(t)\theta^{n+m}(t)\,dt\,, \quad \theta \in \mathcal{D}\,,$$

where $f : \mathbb{R} \to \mathcal{X}$, $g : \mathbb{R} \to \mathcal{Y}$ are continuous functions with bounded from below supports such that for any $\phi \in \mathcal{D}$ with $\operatorname{supp} \phi \subseteq \operatorname{supp} \theta$ the following equalities hold:

$$\langle F, \phi \rangle = (-1)^n \int_{\mathbb{R}} f(t)\phi^{(n)}(t)\,dt\,, \quad \langle G, \phi \rangle = (-1)^m \int_{\mathbb{R}} g(t)\phi^{(m)}(t)\,dt\,,$$

and $f * g$ is defined by $(f * g)(t) = \int_{\mathbb{R}} f(t-s)g(s)ds\,.$

It is proved in (Fattorini, 1983) that thus defined $F * G$ is independent of the choice of f and g.

Definition 3. Let A be a closed linear operator acting in H, $[\mathrm{dom}(A)]$ be the domain of A endowed with the graph norm, $B \in \mathcal{L}(\mathbb{H}; H)$, $\zeta \in H$ and $\mathbb{W}$ — be an $\mathbb{H}$-valued Q-white noise, defined by (1). We will call $X \in \mathcal{D}'_+([\mathrm{dom}\,A])$ the *generalized solution* of the Cauchy problem

$$X'(t) = AX(t) + B\mathbb{W}(t), \quad t \geq 0, \qquad X(0) = \zeta. \tag{2}$$

if it satisfies the equation

$$P * X = \delta \otimes \zeta + B\mathbb{W}\,, \quad (3)$$

[1]

where $P \in \mathcal{D}'_+\big(\mathcal{L}([\mathrm{dom}(A)]; H)\big)$ is defined by

$$P = \delta' \otimes I - \delta \otimes A\,. \quad (4)$$

Definition 4. Distribution $G \in \mathcal{D}'_+\big(\mathcal{L}(H;\mathcal{X})\big)$ is called *the convolution inverse* for $P \in \mathcal{D}'_+\big(\mathcal{L}(\mathcal{X};H)\big)$ if

$$G * P = \delta \otimes I_{\mathcal{X}}\,, \quad P * G = \delta \otimes I_H\,,$$

where $I_{\mathcal{X}}$ and I_H are identity operators in $\mathcal{X}$ and H correspondingly.

The properties of convolution inverse imply the next hteorem.

Theorem 1. *Let $P \in \mathcal{D}'_+\big(\mathcal{L}(\mathcal{X};H)\big)$ has convolution inverse $G \in \mathcal{D}'_+\big(\mathcal{L}(H;\mathcal{X})\big)$. Then the generalized stochastic process $\{X(\omega,\theta):\ \theta \in \mathcal{D}\}$, $\omega \in \Omega$, defined by*

$$X(\theta) = X(\cdot,\theta) := \langle G * F, \theta\rangle + \langle G * B\mathbb{W}, \theta\rangle\,, \quad (5)$$

is the unique solution of (3) *in* $\mathcal{D}'_+\big(L_2(\Omega,\mathcal{X})\big)$.

From this theorem follow the next results for the equation (3), corresponding to the Cauchy problem (2) with operator A generating different types of semigroups.

Corollary 1. *Let A be the generator of C_0-semigroup $\{S(t) \mid t \geq 0\}$. Then the generalized random process defined by*

$$X(\theta) = \int_0^\infty \theta(t)S(t)\zeta dt + \int_0^\infty \theta(t)dt \int_0^t S(t-s)f(s)ds$$
$$- \int_0^\infty \theta'(t)dt \int_0^t S(t-s)BW(s)ds, \quad \theta \in \mathcal{D}\,.$$

is the unique generalized solution of (2).

Proof. The assertion follows from the fact that the convolution inverse for P is the operator-valued distribution $\langle G,\theta\rangle = \int_0^\infty \theta(t)S(t)dt$ (the integral is understood in strong sense). It belongs to the class of exponential distributions

$$\mathcal{S}'_a\big(\mathcal{L}(H,[\mathrm{dom}\,A])\big) := \big\{G \in \mathcal{D}'_+\left(\mathcal{L}(H,[\mathrm{dom}\,A])\right) : e^{-at}G \in \mathcal{S}'\big(\mathcal{L}(H,[\mathrm{dom}\,A])\big)\big\}\,,$$

where a is such that $\|S(t)\| \leq Me^{at}$ for some $M > 0$ and all $t \geq 0$. □

[1]For $u \in \mathcal{D}'$, $h \in H$ by $u \otimes h$ we denote the element of $\mathcal{D}'(H)$ defined by $\langle u \otimes h, \theta\rangle := \langle u,\theta\rangle h$.

Corollary 2. *Let A be the generator of an exponentially bounded integrated semigroup $\{S(t), t \geq 0\}$. Then the generalized random process defined by*

$$X(\theta) = (-1)^n \Big[\int_0^\infty \theta^{(n)}(t) S(t) \zeta dt + \int_0^\infty \theta^{(n)}(t) dt \int_0^t S(t-s) f(s) ds - \int_0^\infty \theta^{(n+1)}(t) dt \int_0^t S(t-s) BW(s)\, ds \Big], \quad \theta \in \mathcal{D}. \tag{6}$$

is the unique generalized solution of (2).

Proof. The assertion follows from the fact that in this case the convolution inverse for P is the operator-valued distribution

$$\langle G\,,\,\theta\rangle = (-1)^n \int_0^\infty \theta^{(n)}(t) S(t) dt, \quad \theta \in \mathcal{D}. \tag{7}$$

It also belongs to $\mathcal{S}'_a\big(\mathcal{L}(H, [\text{dom } A]\big)$. □

3. STOCHASTIC CAUCHY PROBLEM IN SPACES OF ABSTRACT DISTRIBUTIONS WITH RESPECT TO ω

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be the white noise probability space $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$, where μ is the normalized Gaussian measure on the Borel σ-algebra $\mathcal{B}$ of subsets of the space of tempered distributions $\mathcal{S}'$, i.e. the unique measure with

$$\int_{\mathcal{S}'} e^{i\langle\omega,\theta\rangle} d\mu(\omega) = e^{-\frac{1}{2}|\theta|_0^2}\,, \tag{8}$$

where $|\cdot|_0 = \|\cdot\|_{L^2(\mathbb{R})}$. Existence of μ is stated by the Bochner-Minlos theorem.

Consider $(L_2) := L_2(\mathcal{S}', \mu; \mathbb{R})$. We will denote its norm by $\|\cdot\|_0$. From (8) it follows that for any $\theta \in \mathcal{S}$ the function $\langle\cdot\,,\,\theta\rangle$ belongs to (L_2) and

$$\|\langle\cdot,\theta\rangle\|_0^2 = E\langle\cdot,\theta\rangle^2 = |\theta|_0^2\,.$$

Using this equality the mappping $\theta \mapsto \langle\cdot,\theta\rangle$ can be by continuity extended from $\mathcal{S}$ onto $L_2(\mathbb{R})$. Thus the random process

$$\beta(t) = \begin{cases} \langle\cdot, \mathbf{1}_{[0,t]}\rangle\,, & t \geq 0\,, \\ -\langle\cdot, \mathbf{1}_{[t,0)}\rangle\,, & t < 0\,, \end{cases}$$

is well defined. It is well known that it is a Brownian motion. It is easy to show that for the $\mathbb{R}$-valued generalized random process defined by

$$\langle\beta\,,\,\theta\rangle := \int_{\mathbb{R}} \beta(t)\theta(t)\, dt\,, \quad \theta \in \mathcal{S}\,,$$

it holds

$$\langle\dot{\beta}\,,\,\theta\rangle = -\int_{\mathbb{R}} \beta(t)\theta'(t)\, dt = \langle\cdot,\theta\rangle\,, \tag{9}$$

Thus the elements of $\mathcal{S}'$ in this context can be thought of as trajectories of the white noise associated with the Brownian motion $\beta(t)$. Therefore $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$ is referred to as the white noise probability space.

Let $\mathcal{T}$ be the set of all finite multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots)$ ($\alpha_i \in \mathbb{N} \cup \{0\}$). It is well known (Kuo, 1996; Holden et. al., 1996) that the stochastic Hermite polynomials defined by $\mathbf{h}_\alpha(\omega) := \prod_{i=1}^\infty h_{\alpha_i}(\langle \omega, \xi_i \rangle), \quad \omega \in \mathcal{S}'$, form an orthogonal basis in (L_2) with $\big(\mathbf{h}_\alpha, \mathbf{h}_\beta\big)_{(L_2)} = \alpha! \delta_{\alpha\beta}$, where $\alpha! = \prod\limits_i \alpha!$. For any $t \in \mathbb{R}$ we have

$$\langle \cdot, \mathbf{1}_{[0,t]} \rangle = \Big\langle \cdot, \sum_{n=1}^\infty \int_0^t \xi_n(s) ds \, \xi_n \Big\rangle = \sum_{n=1}^\infty \int_0^t \xi_n(s) ds \, \langle \cdot, \xi_n \rangle = \sum_{n=1}^\infty \int_0^t \xi_n(s) ds \, \mathbf{h}_{\epsilon_n}, \quad (10)$$

where $\epsilon_n = (0, 0, \ldots, \underset{n}{1}, 0, \ldots)$ and the series is convergent in (L_2).

Let $n(\cdot, \cdot) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection with

$$n(i,j) \geq ij \,, \quad i\,, j \in \mathbb{N}\,. \quad (11)$$

Define

$$\beta_j(t) = \sum_{i=1}^\infty \int_0^t \xi_i(s) \, ds \, \mathbf{h}_{\epsilon_{n(i,j)}} \,. \quad (12)$$

It is easy to show that $\beta_j(t)$ are independent equally distributed Brownian motions as the sets $\{\epsilon_{n(i,j)}\}_{i \in \mathbb{N}}$ and $\{\epsilon_{n(i,k)}\}_{i \in \mathbb{N}}$ are disjoint for $j \neq k$. The equality

$$W(t) = \sum_{j=1}^\infty \sigma_j \beta_j(t) e_j \,, \quad (13)$$

where $\{e_j\}$ is as in section 1, defines an H-valued Q-Wiener process on $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$. Note that all the results of section 1 remain valid in this particular case. Moreover, the structure of the white noise probability space allows to define H-valued generalized with respect to $\omega \in \mathcal{S}'$ functions, i.e. generalized random variables. In spaces of these generalized random variables the Q-white noise (and even the cylindrical white noise defined by $W_c(t) = \sum_{j=1}^\infty \beta_j(t) e_j$), become smooth functions of t thus resolving the problem of irregularity stated in the introduction.

Let $(L_2)(H) = L^2(\mathcal{S}', \mu; H)$ be the space of H-valued Bochner square integrable with respect to μ functions on $\mathcal{S}'$. Denote its norm by $\|\cdot\|_{0,H}$ ($\|\cdot\|_{0,\mathbb{R}} = \|\cdot\|_0$). The set $\{\mathbf{h}_\alpha(\omega) e_j\}_{\alpha \in \mathcal{T}, j \in \mathbb{N}}$ form an orthogonal basis of $(L_2)(\mathbb{H})$. For example for $W(t)$ defined by (13) by (12) we have

$$W(t) = \sum_{i,j \in \mathbb{N}} \sigma_j \int_0^t \xi_i(s) \, ds \, \mathbf{h}_{\epsilon_{n(i,j)}} e_j \,. \quad (14)$$

Let $(\mathcal{S})_p(H)$ be the subspace of $(L_2)(H)$ consisting of all $\varphi = \sum_{\alpha,j} \varphi_{\alpha,j} \mathbf{h}_\alpha e_j = \sum_\alpha \varphi_\alpha \mathbf{h}_\alpha \in (L_2)(H)$ ($\varphi_\alpha = \sum_j \varphi_{\alpha,j} e_j \in H$), satisfying

$$\|\varphi\|_{p,H}^2 = \sum_{\alpha,j} \alpha! \, \varphi_{\alpha,j}^2 (2\mathbb{N})^{2p\alpha} = \sum_\alpha \alpha! \, \|\varphi_\alpha\|_{\mathbb{H}}^2 (2\mathbb{N})^{2p\alpha} < \infty \,,$$

Denote by $(\mathcal{S})_{-p}(H)$ its dual which can be identified with the set of all formal expansions

$$\Phi = \sum_{\alpha,j} \Phi_{\alpha,j} \mathbf{h}_\alpha e_j = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha\,, \quad \left(\text{where } \Phi_\alpha = \sum_j \Phi_{\alpha,j} e_j \in H\right),$$

satisfying the condition

$$\|\Phi\|^2_{-p,H} = \sum_{\alpha,j} \alpha!\, \Phi^2_{\alpha,j} (2\mathbb{N})^{-2p\alpha} = \sum_\alpha \alpha!\, \|\Phi_\alpha\|^2_H (2\mathbb{N})^{-2p\alpha} < \infty\,. \tag{15}$$

Let $(\mathcal{S})(H)$ be $\cap_{p\in\mathbb{N}}(\mathcal{S}_p)(H)$ with the projective limit topology and $(\mathcal{S})^*(H)$ be $\cup_{p\in\mathbb{N}}(\mathcal{S}_{-p})(H)$ with the inductive limit topology. We have

$$\begin{aligned}(\mathcal{S})(H) \;&\subset\; \dots \subset (\mathcal{S}_{p+1})(H) \subset (\mathcal{S}_p)(H) \subset \dots \\ &\subset\; (L_2)(H) \subset \dots \subset (\mathcal{S}_{-p})(H) \subset (\mathcal{S}_{-p-1})(H) \subset \dots \subset (\mathcal{S})^*(H)\,.\end{aligned}$$

Define singular cylindrical white noise by

$$\mathbb{W}_c(t) = \sum_{i,j\in\mathbb{N}} \xi_i(t) \mathbf{h}_{\epsilon_{n(i,j)}} e_j = \sum_{n\in\mathbb{N}} \mathbb{W}_{\epsilon_n}(t) \mathbf{h}_{\epsilon_n}\,, \tag{16}$$

where $\mathbb{W}_{\epsilon_n}(t) = \xi_{i(n)}(t) e_{j(n)}$. for all $t\in\mathbb{R}$ we have $\mathbb{W}_c(t) \in (\mathcal{S})^*(H)$ since from (11) and properties of Hermite functions it follows $\xi_i^2(t)\big(2n(i,j)\big)^{-2} = O\left(j^{-2}\cdot i^{-\frac{5}{2}}\right)$ and consequently

$$\|\mathbb{W}_c(t)\|^2_{-1,H} = \sum_{i,j} \sigma_j^2 \xi_i^2(t)\big(2n(i,j)\big)^{-2} < \infty \implies \mathbb{W}(t) \in (\mathcal{S}_{-1})(H)\,.$$

Strong convergence in $(\mathcal{S})^*(H)$ is defined as the uniform one on bounded subsets of $(\mathcal{S})(H)$. It is characterized by the following

Proposition 1. *Let* $\Phi_n = \sum_\alpha \Phi_\alpha^{(n)} \mathbf{h}_\alpha\,, \Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})^*(H)$. *The following assertions are equivalent:*

(i) $\{\Phi_n\}$ *is strongly convergent to* Φ*;*

(ii) $\Phi_n \in (\mathcal{S}_{-p})(H)$ *for all* $n\in\mathbb{N}$ *and some* $p\in\mathbb{N}$, $\Phi \in (\mathcal{S}_{-p})(H)$ *and* $\lim_{n\to\infty} \|\Phi_n - \Phi\|_{-p} = 0$*;*

(iii) $\{\Phi_n\} \subset (\mathcal{S}_{-p})(H)$ *for some* $p\in\mathbb{N}$, *is bounded with respect to* $\|\cdot\|_{-p}$ *and* $\lim_{n\to\infty} \|\Phi_\alpha^{(n)} - \Phi_\alpha\|_{\mathbb{H}} = 0$ *for all* $\alpha \in \mathcal{T}$.

We have $\frac{dW_c(t)}{dt} = \mathbb{W}_c(t)$, where differentiation is defined via strong convergence.

We will call $\Phi(\cdot) : \mathbb{R} \to (\mathcal{S})^*(H)$ integrable on a measurable $C \subseteq \mathbb{R}$ if $\Phi(t) \in (\mathcal{S}_{-p})(H)$ for any $t \in C$ and some $p \in \mathbb{N}$ and Φ is Bochner integrable on C as a $(\mathcal{S}_{-p})(H)$-valued function.

Proposition 2. *If* $\Phi_\alpha(t)$ *is Bochner integrable on* C *as an* H*-valued function for any* $\alpha \in \mathcal{T}$ *and*

$$\sum_\alpha \left(\int_C \|\Phi_\alpha(t)\|_H dt\right)^2 (2\mathbb{N})^{-2p\alpha} < \infty$$

for some $p \in \mathbb{N}$, *then* $\Phi(\cdot) = \sum_{\alpha \in \mathcal{T}} \Phi_\alpha(\cdot)\mathbf{h}_\alpha : \mathbb{R} \to (\mathcal{S})^*(H)$ *is integrable on* C *and*

$$\int_C \Phi(t)\, dt = \sum_\alpha \int_C \Phi_\alpha(t)\, dt\, \mathbf{h}_\alpha .$$

We extend linear operators defined in H to linear operators in $(\mathcal{S})^*(H)$ in the next natural way. For $B \in \mathcal{L}(H)$ define $B\Phi := \sum_\alpha B\Phi_\alpha \mathbf{h}_\alpha$ for any $\Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})^*(H)$ which defines a linear continuous mapping in $(\mathcal{S})^*(H)$.

For a closed linear operator A in H define its domain in $(\mathcal{S})^*(H)$ as the set of all $\Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})^*(H)$ such that $\Phi_\alpha \in \operatorname{dom} A$ for all $\alpha \in \mathcal{T}$ and $\sum_\alpha \alpha! \|A\Phi_\alpha\|^2 \big(2\mathbb{N}\big)^{-2p\alpha} < \infty$ for some $p \in \mathbb{N}$. Denote it by $\big(\operatorname{dom} A\big)$. For $\Phi \in \big(\operatorname{dom} A\big)$ define $A\Phi := \sum_\alpha A\Phi_\alpha \mathbf{h}_\alpha$. It is easy to show the this defines a linear closed operator in $(\mathcal{S})^*(H)$.

Theorem 2. *Let* A *be the generator of a* C_0*-semigroup* $\{S(t), t \geq 0\}$ *in* H, $B \in \mathcal{L}(H)$. *Then for any* $\Phi \in \big(\operatorname{dom} A\big)$ *the Cauchy problem*

$$\frac{dX(t)}{dt} = AX(t) + B\mathbb{W}_c(t)\,, \quad t \geq 0\,, \quad X(0) = \Phi\,, \tag{17}$$

has a unique solution in $(\mathcal{S})^*(H)$, *which is given by*

$$X(t) = S(t)\Phi + \int_0^t S(t-s)B\mathbb{W}_c(s)ds\,.$$

Proof. The assertion follows from the fact that in $(\mathcal{S})^*(H)$ the problem (17) where $X(t) = \sum_\alpha X_\alpha(t)\mathbf{h}_\alpha$ with $X_\alpha \in H$ reduces to the next Cauchy problems for $X_\alpha(t)$:

$$\frac{dX_\alpha(t)}{dt} = AX_\alpha(t)\,, \quad X_\alpha(0) = \Phi_\alpha$$

for $\alpha \neq \epsilon_n$, $n \in \mathbb{N}$ and

$$\frac{dX_{\epsilon_n}(t)}{dt} = AX_{\epsilon_n}(t) + B\mathbb{W}_{\epsilon_n}(t)\,, \quad X_{\epsilon_n}(0) = \Phi_{\epsilon_n}$$

for $\alpha = \epsilon_n$. Each of them has the unique solution given by

$$X_\alpha(t) = \begin{cases} S(t)\Phi_{\epsilon_n} + \int_0^t S(t-s)B\mathbb{W}_{\epsilon_n}(s)\, ds\,, & \alpha = \epsilon_n\,, \\ S(t)\Phi_\alpha\,, & \alpha \neq \epsilon_n\,. \end{cases}$$

It follows from the estimate

$$\int_0^t \big\|S(t-s)B\mathbb{W}_{\epsilon_n}(s)\big\|_H\, ds \leq M\|B\| \int_0^t e^{a(t-s)}|\xi_{i(n)}(s)|\, ds \leq M_1\|B\|e^{at}\,,$$

where $M > 0$ and $a > 0$ are such that $\|S(t)\| \leq Me^{at}$ for $t \geq 0$, and $M_1 > 0$, that for $p \geq 1$ it holds

$$\sum_{n\in\mathbb{N}} \left(\int_0^t \big\|S(t-s)B\mathbb{W}_{\epsilon_n}(s)\big\|_H\, ds\right)^2 (2\mathbb{N})^{-2p\epsilon_n} < \infty\,.$$

By Proposition 2 it follows that the integral $\int_0^t S(t-s)B\mathbb{W}_c(s)ds$ exists in $(\mathcal{S})^*(H)$. □

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