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ON THE ASYMPTOTICS OF THE DIFFERENCE EQUATION $x_n = \dfrac{x_{n-3} - (x_n + x_{n-1})^3}{1 + x_n x_{n-1} + x_n x_{n-2} + x_{n-1} x_{n-2}}$

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ABSTRACT. The aim of this paper is to show the existence of a solution of the difference equation in the title converging to zero as $n \to \infty$, and to determine its asymptotic behaviour.

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1. INTRODUCTION

In recent investigations of dynamical systems rational difference equations of higher order are of main importance, c.f. Kulenovíc and G. Ladas [4] and the references therein. A special example is the equation:

$$x_n = \frac{x_{n-3}}{1 + x_{n-1} x_{n-2}} \tag{1}$$

$n \in \mathbb{N}$, about which in Ref. [5] it was shown that every solution converges as $n \to \infty$ to a 3-periodic solution $(\dots, p, q, r, p, q, r, \dots)$ with $pqr = 0$. In Ref. [3] L. Berg investigated the difference equation (1). This paper showed that there is a existence of a solution of equation (1) converging to zero as $n \to \infty$ and the author determined its asymptotic behaviour.

2. ASYMTOTIC OF THE DIFFERENCE EQUATION

By the similar method of Ref. [3] we shall consider the difference equation:

$$x_n = \frac{x_{n-3} - (x_n + x_{n-1})^3}{1 + x_n x_{n-1} + x_n x_{n-2} + x_{n-1} x_{n-2}}, \quad n = 0, 1, 2, 3 \dots \tag{2}$$

In order to find the asymptotic behaviour of a solution of equation (2) tending to zero as $n \to \infty$ we proceed as recomended in Ref. [2], i.e. we assume first that

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such a solution exists for a continuous argument $n = t$, and that it is continuously differentiable.

Writing $x_n = x(t)$, approximating $x_{n-1} = x(t-1)$, $x_{n-2} = x(t-2)$, $x_{n-3} = x(t-3)$, according to Taylor we have up to smaller terms:

$$x(t-1) = x - x',\ x(t-2) = x - 2x',\ x(t-3) = x - 3x'$$

or more exactly

$$x(t-3) = x - 3x' + \frac{9}{2}x''$$

we approximate equation (2) by the differential equation:

$$x + x^2(x-x') + x(x-x')(x-2x') + x^2(x-2x') + x^3 + (x-x')^3 + 3x^2(x-x') \\ + 3x(x-x')^2 = x - 3x' + \frac{9}{2}x''$$

After neglecting $8xx'^2 - x'^3$ from this equation, it turns into the equation

$$11x^3 - 18x^2x' = -3x' + \frac{9}{2}x'' \tag{3}$$

from which we first find approximating $x' = -\frac{11}{3}x^3$, and from this $x" = -11x^2x'$ as well as $x = \sqrt{\frac{3}{22t}}$. From (3) we obtain $11x^3 - 18x^2x' = -3x' - \frac{99}{2}x^2x'$ which can be integrated by

$$x = \sqrt{\frac{3}{22t + 63\ln x}}$$

disregarding the constant of integration. Obviously, a solution x tending to zero satisfies $x \sim \sqrt{\frac{3}{22t}}$ as before, so that by iteration we have

$$x = \sqrt{\frac{3}{22t}}\left(1 + \frac{63}{88}\frac{\ln t}{t}\right) \tag{4}$$

up to smaller terms as $n \to \infty$. This result encourages us to expect a solution of equation (2) of the form

$$x = \frac{1}{\sqrt{n}}\left(a + \frac{b\ln n}{n} + \frac{c\ln^2 n + d\ln n + e}{n^2}\right) \tag{5}$$

up to smaller terms as $n \to \infty$. Replacing this as an ansatz into equation (2), we find by means of the DERIVE system in accordance with equation (4)

$$a = \frac{\sqrt{66}}{22},\ b = \frac{63\sqrt{66}}{1936},\ c = \frac{11907\sqrt{66}}{340736},\ d = -\frac{3969\sqrt{66}}{85184},\ e = 0 \tag{6}$$

In the terminology of Ref. [1] equation (5) with the coefficients (6) represents an asymptotic solution of equation (2). However, we shall show that it represents in fact the asymptotic behaviour of a real solution of equation (2). For this reason we use the following Theorem 2 of Stevíc, which is a generalization of Theorem 1 in Ref. [2] to equation of order $k \geq 1$ see also S. Stevíc [7]:

Theorem 2.1. *Let $f : \mathbb{R}_+^k \to \mathbb{R}_+$ be a continuous and nondecreasing function in each argument, and let $\{y_n\}$ and $\{z_n\}$ be sequences with $y_n < z_n$ for $n \geq n_0$ and such that*

$$y_{n-k} \leq f(y_n, y_{n-1}, \ldots, y_{n-k+1}),\ \ f(z_n, z_{n-1}, \ldots, z_{n-k+1}) \leq z_{n-k},\ \ \textit{for } n \geq n_0 + k - 1. \tag{7}$$

Then the difference equation

$$x_{n-k} = f(x_n, x_{n-1}, \ldots, x_{n-k+1}) \tag{8}$$

has a solution x_n such that

$$y_n \leq x_n \leq z_n \textit{ for } n \geq n_0 \tag{9}$$

Based on Theorem 1.1 we prove Theorem 1.2.

Theorem 2.2. *Equation (2) possesses a solution with the finite asymptotic expansion (5) as $n \to \infty$ and the coefficients (6).*

Proof. By means of abbreviation

$$F(x_n, x_{n-1}, x_{n-2}, x_{n-3}) = f(x_n, x_{n-1}, x_{n-2}) - x_{n-3}$$

with $f(x_n, x_{n-1}, x_{n-2}) = x_n(1+x_nx_{n-1}+x_{n-1}x_{n-2}+x_nx_{n-2})+(x_n+x_{n-1})^3$. Difference equation (8) turns into

$$F(x_n, x_{n-1}, x_{n-2}, x_{n-3}) = 0 \tag{10}$$

and the inequalities (7) turn into

$$F(z_n, z_{n-1}, z_{n-2}, z_{n-3}) \leq 0 \leq F(y_n, y_{n-1}, y_{n-2}, y_{n-3}) \tag{11}$$

These inequalities together with equation (10) can be interpreted as a certain intermediate value property of the function $F(x_n, x_{n-1}, x_{n-2}, x_{n-3})$. Then the premisses concerning the arguments of f are satisfied. Inserting the ansatz (5) into

$$F(x_n, x_{n-1}, x_{n-2}, x_{n-3}) = x_n(1 + x_nx_{n-1} + x_{n-1}x_{n-2} + x_nx_{n-2}) + (x_n + x_{n-1})^3 - x_{n-3}$$

we obtain again by means of the DERIVE systerm as $n \to \infty$

$$F \sim \frac{a}{2}\Big(22a^2 - 3\Big)\frac{1}{\sqrt{n^3}}$$

and taking into account successively the coefficients (7)

$$F \sim 12\Big(b - \frac{63\sqrt{66}}{1936}\Big)\frac{1}{\sqrt{n^5}}$$

$$F \sim -12\Big(c - \frac{11907\sqrt{66}}{340736}\Big)\frac{\ln^2 n}{\sqrt{n^7}} \tag{12}$$

$$F \sim -12\Big(d + \frac{3969\sqrt{66}}{85184}\Big)\frac{\ln n}{\sqrt{n^7}}$$

$$F \sim -12(e+0)\frac{1}{\sqrt{n^7}}$$

as well as

$$F \sim \frac{35721\sqrt{66}}{7496192}\frac{\ln^3 n}{\sqrt{n^9}}$$

Choosing

$$y_n = x_n - \frac{p}{n^{\frac{5}{2}}},\ z_n = x_n + \frac{p}{n^{\frac{5}{2}}}$$

with some constant $p > 0$, we see from (12) with $e - p$ respectively $e + p$ instead of e that the inequalities (11) are satisfied for sufficiently large n. Hence, using the coefficients (6) and considering that $p > 0$ can be chosen arbitrarily. The proof is complete. □

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