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UPPER AND LOWER SOLUTION METHOD FOR SINGULAR $\phi-$LAPLACIAN BVPS WITH DERIVATIVE DEPENDING NONLINEARITY ON $[0, +\infty)$

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ABSTRACT. This article is concerned with the existence of positive solutions for a class of singular $\phi-$Laplacian boundary value problems posed on the half-line. The nonlinearity depends on the solution and its derivative, and may exhibit a space-singularity at the origin. Some existence results are proved using the upper and lower solution technique in a special Banach space. The nonlinearity obeys sign and growth conditions with respect to the unknown. The singularity is treated by approximation and truncation.

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1. INTRODUCTION

1.1 The mathematical problem. This paper is devoted to the existence of positive solutions to the following boundary value problem set on the positive half-line:

$$\begin{cases} (\phi(x'))'(t) + q(t)f(t, x(t), x'(t)) = 0, \ t > 0 \\ x(0) = 0, \quad \lim\limits_{t \to +\infty} x'(t) = 0. \end{cases} \tag{1.1}$$

The function $q : I \longrightarrow I$ is continuous and the nonlinearity $f : \mathbb{R}^+ \times I \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. Here $I := (0, +\infty)$ denotes the set of positive real numbers and $\mathbb{R}^+ = [0, +\infty)$. The nonlinear operator $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous, increasing homeomorphism such that $\phi(0) = 0$. It extends the usual p-Laplacian operator $\varphi_p(s) = |s|^{p-1}s$ for $p > 1$.

Boundary value problems on infinite intervals appear in many phenomena in applied mathematics and physics (see e.g. [1]). Second-order boundary value problems (bvps in short) on positive infinity have been widely investigated in the literature (see [3, 4, 5, 13] and the references therein).

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In [12], Yan *et al.* have obtained some existence results of unbounded positive solution to the bvp

$$\begin{cases} x''(t) + \phi(t)f(t, x(t), x'(t)) = 0, \ t > 0 \\ ax(0) - bx'(0) = x_0 \geq 0, \quad \lim\limits_{t\to+\infty} x'(t) = k > 0. \end{cases} \tag{1.2}$$

The upper and lower solution method is employed to prove existence of solutions and the multiplicity is discussed using the fixed point index theory.

In [6], the authors have studied the question of the existence of multiple solutions to the bvp

$$\begin{cases} x''(t) - k^2 x(t) = q(t)f(t, x(t), x'(t)) = 0, \quad t > 0 \\ x(0) = x(+\infty) = 0 \end{cases}$$

when the nonlinearity is sign-changing; they combined the fixed point index and the upper and lower solution technique to prove some existence results.

Lian *et al.* in [8] have considered the following boundary value problem with a p-Laplacian operator

$$\begin{cases} (\varphi_p(x'))'(t) + q(t)f(t, x(t), x'(t)) = 0, \quad t > 0 \\ \alpha x(0) - \beta x'(0) = 0, \quad \lim\limits_{t\to+\infty} x'(t) = 0. \end{cases} \tag{1.3}$$

Using a fixed point theorem in a cone due to Avery and Peterson, the existence of at least three positive solutions is proved. With a multi-point condition at 0, this bvp is investigated in [7] with the same method.

In [9], Liang and Zhang have considered the case of a nonlinearity which does not depend on the first derivative, namely the equation $(\varphi_p(x'))'(t) + a(t)f(t, x(t)) = 0$ associated with a multi-point condition at $t = 0$ and a Neumann condition at positive infinity. The p-Laplacian operator of derivation is extended to an increasing homeomorphism φ in Refs. [10, 11].

In this work, we aim to consider the solvability of Problem (1.1) when the nonlinearity depends on the first derivative, is sign-changing, and may be singular at $x = 0$. Our investigation mainly rely on the method of upper and lower solutions and approximation methods; in this respect, the regular and singular cases are studied separately. The paper is divided into three sections. In this section, we also present some preliminaries. In Section 2, we prove two existence results to Problem (1.1) corresponding to the regular case and the singular case respectively. The nonlinearity is assumed to satisfy a sign condition. The singular problems are approximated by a family of regular problems together with sequential arguments. Similar results are obtained in Section 3 under a Nagumo-type growth condition for a positive nonlinearity. Each existence theorem is illustrated by means of an example of application.

1.2 Auxiliary results.

Definition 1.1. Let E be a Banach space. A mapping $A : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Let

$$C_l([0,\infty),\mathbb{R}) = \{x \in C([0,\infty),\mathbb{R}) : \lim_{t\to\infty} x(t) \text{ exists}\}.$$

For $x \in C_l([0,\infty),\mathbb{R})$, define $\|x\| = \sup_{t\in\mathbb{R}^+} |x(t)|$. Then C_l is a Banach space. However, the basic space to study Problem (1.1) is denoted by

$$E = \{x \in C^1([0,\infty),\mathbb{R}),\ \lim_{t\to+\infty} \frac{|x(t)|}{1+t} \text{ exists and } \lim_{t\to+\infty} x'(t) = 0\}.$$

Then E is a Banach space normed by $\|x\| = \max\{\|x\|_1, \|x\|_2\}$ with $\|x\|_1 = \sup_{t\in\mathbb{R}^+} \frac{|x(t)|}{1+t}$ and $\|x\|_2 = \sup_{t\in\mathbb{R}^+} |x'(t)|$. The following compactness criterion is a useful tool to deal with Problem (1.1).

Lemma 1.2 ([2, p. 62])**.** *Let $M \subseteq C_l(\mathbb{R}^+,\mathbb{R})$. Then M is relatively compact in $C_l(\mathbb{R}^+,\mathbb{R})$ if the following conditions hold:*

(a) *M is uniformly bounded in $C_l(\mathbb{R}^+,\mathbb{R})$.*

(b) *The functions belonging to M are equicontinuous on $\mathbb{R}^+$, i.e. equicontinuous on every compact interval of $\mathbb{R}^+$.*

(c) *The functions from M are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(t) - x(+\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $x \in M$.*

As a consequence, we state without proof the following criterion:

Lemma 1.3. *Let $M \subseteq E$. Then M is relatively compact in E if the following conditions hold:*

(a) *M is bounded in E.*

(b) *The functions belonging to $\{u :\ u(t) = \frac{x(t)}{1+t},\ x \in M\}$ and to $\{z :\ z(t) = x'(t),\ x \in M\}$ are equicontinuous on $[0,+\infty)$.*

(c) *The functions belonging to $\{u :\ u(t) = \frac{x(t)}{1+t},\ x \in M\}$ and to $\{z :\ z(t) = x'(t),\ x \in M\}$ are equiconvergent at $+\infty$.*

2. EXISTENCE RESULT UNDER A SIGN CONDITION

2.1 The regular problem. Since the nonlinearity f may have a space-singularity at $x = 0$, we first consider a family of regular boundary value problems

$$\begin{cases} (\phi(x'))'(t) + q(t)f(t,x(t),x'(t)) = 0, & t > 0 \\ x(0) = k, \quad \lim_{t\to+\infty} x'(t) = 0, \end{cases} \tag{2.1}$$

where k is a positive real number.

Definition 2.1. A function $\alpha \in C(\mathbb{R}^+, I) \cap C^1(I, \mathbb{R})$ is called a lower solution of (2.1) if $\phi \circ \alpha' \in C^1(I, \mathbb{R})$ and satisfies

$$\begin{cases} -(\phi(\alpha'(t)))' \leq q(t) f(t, \alpha(t), \alpha'(t)), \quad t > 0 \\ \alpha(0) \leq k, \quad \lim\limits_{t \to +\infty} \alpha'(t) \leq 0. \end{cases}$$

A function $\beta \in C(\mathbb{R}^+, I) \cap C^1(I, \mathbb{R})$ is called an upper solution of (2.1) if $\phi \circ \beta' \in C^1(I, \mathbb{R})$ and satisfies

$$\begin{cases} -(\phi(\beta'(t)))' \geq q(t) f(t, \beta(t), \beta'(t)), \quad t > 0 \\ \beta(0) \geq k, \quad \lim\limits_{t \to +\infty} \beta'(t) \geq 0. \end{cases}$$

If there exist an upper solution β and a lower solution α of (2.1) with $\alpha \leq \beta$, then for all $t \geq 0$ and for all $N > 0$ we can define the set

$$D^N_{\alpha,\beta}(t) = \{(x, y) \in \mathbb{R} \times \mathbb{R} : \alpha(t) \leq x \leq \beta(t), -N \leq y \leq N\}.$$

Finally, for $t \geq 0$, let

$$\delta_N(t) = 1 + \sup_{(x,y) \in D^N_{\alpha,\beta}(t)} |f(t, x, y)|. \tag{2.2}$$

For fixed $t \geq 0$, $D^N_{\alpha,\beta}(t)$ is compact and so $\delta_N(t)$ is well defined. Regarding the regular problem (2.1), we state our main existence result.

Theorem 2.2. *Assume that α, β are lower and upper solutions of Problem (2.1) respectively with $\alpha \leq \beta$, $\sup\limits_{t \in \mathbb{R}^+} |\alpha'(t)| < \infty$ and $\sup\limits_{t \in \mathbb{R}^+} |\beta'(t)| < \infty$. Suppose further that there exists some constant $M \geq \max\{\sup\limits_{t \in \mathbb{R}^+} |\alpha'(t)|, \sup\limits_{t \in \mathbb{R}^+} |\beta'(t)|\}$ such that*

$$\int_0^{+\infty} q(\tau)\delta_M(\tau) d\tau < +\infty, \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)\delta_M(\tau) d\tau\right) ds < +\infty \tag{2.3}$$

and the following sign condition is satisfied:

$$(|y| - M) f(t, x, y) < 0, \quad \forall\, (t, x, y) \in \mathbb{R}^+ \times I \times \mathbb{R} \setminus \{-M, M\}. \tag{2.4}$$

Then, Problem (2.1) has at least one solution $x \in E$ such that

$$\alpha(t) \leq x(t) \leq \beta(t) \text{ and } 0 \leq x'(t) \leq M, \quad t \in \mathbb{R}^+.$$

Proof. Consider the truncation function at level M

$$T_M(y) = \begin{cases} -M, & y < -M, \\ y, & -M \leq y \leq M, \\ M, & M < y, \end{cases}$$

and the unit-interval retraction

$$\mu(x) = \begin{cases} x, & |x| \leq 1, \\ \frac{x}{|x|}, & |x| \geq 1. \end{cases}$$

Define the modified nonlinearity

$$f^*(t,x,y)=\begin{cases} f(t,\alpha(t),T_M(y))-\mu(x-\alpha(t)), & x<\alpha(t),\\ f(t,x,T_M(y)), & \alpha(t)\le x\le\beta(t),\\ f(t,\beta(t),T_M(y))-\mu(x-\beta(t)), & x>\beta(t),\end{cases}\tag{2.5}$$

and the modified problem

$$\begin{cases} -(\phi(x'))'(t)=q(t)f^*(t,x(t),x'(t)), \quad t>0\\ x(0)=k, \quad \lim\limits_{t\to+\infty}x'(t)=0.\end{cases}\tag{2.6}$$

We shall prove that this problem has a solution x such that $\alpha\le x\le\beta$ and $0<x'<M$ in which case $f^*=f$.

Part 1. To show that Problem (2.6) has at least one solution x, let the operator A be defined on E by

$$Ax(t)=k+\int_0^t\phi^{-1}\left(\int_s^{+\infty}q(\tau)f^*(\tau,x(\tau),x'(\tau))d\tau\right)ds.$$

In three steps, we prove that A has a fixed point in E.

Step 1. $A(E)\subset E$. For $x\in E$, we have by (2.3)

$$\begin{aligned}0\le\lim_{t\to+\infty}\frac{|Ax(t)|}{1+t}&\le\lim_{t\to+\infty}\frac{k}{1+t}+\frac{\int_0^t\phi^{-1}(\int_s^{+\infty}q(\tau)|f^*(\tau,x(\tau),x'(\tau))|d\tau)ds}{1+t}\\ &\le\lim_{t\to+\infty}\frac{k}{1+t}+\frac{\int_0^{+\infty}\phi^{-1}(\int_s^{+\infty}q(\tau)\delta_M(\tau)d\tau)ds}{1+t}=0\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to+\infty}|(Ax)'(t)|&=\lim_{t\to+\infty}|\phi^{-1}\left(\int_t^{+\infty}q(\tau)f^*(\tau,x(\tau),x'(\tau))\right)d\tau|\\ &\le\lim_{t\to+\infty}\phi^{-1}\left(\int_t^{+\infty}q(\tau)\delta_M(\tau)d\tau\right)=0.\end{aligned}$$

Step 2. A is continuous. Let some sequence $\{x_n\}_{n\ge1}\subseteq E$ be such that $\lim\limits_{n\to+\infty}x_n=x_0\in E$. Since $|f^*(\tau,x_n(\tau),x_n'(\tau))-f^*(\tau,x_0(\tau),x_0'(\tau))|\le2\delta_M(\tau)$, then by the continuity of f^* and the Lebesgue dominated convergence theorem, we have for all $s\in\mathbb{R}^+$,

$$\int_s^{+\infty}q(\tau)f^*(\tau,x_n(\tau),x_n'(\tau))d\tau\to\int_s^{+\infty}q(\tau)f^*(\tau,x_0(\tau),x_0'(\tau))d\tau,\quad\text{as } n\to+\infty.$$

Moreover, the continuity of ϕ^{-1} implies that, as $n\to+\infty$

$$\phi^{-1}\left(\int_s^{+\infty}q(\tau)f^*(\tau,x_n(\tau),x_n'(\tau))d\tau\right)\to\phi^{-1}\left(\int_s^{+\infty}q(\tau)f^*(\tau,x_0(\tau),x_0'(\tau))d\tau\right).$$

As a consequence

$$\begin{aligned}&\sup_{t\in\mathbb{R}^+}\frac{|Ax_n(t)-Ax_0(t)|}{1+t}\\ &=\sup_{t\in\mathbb{R}^+}\frac{\left|\int_0^t(\phi^{-1}(\int_s^{+\infty}q(\tau)f^*(\tau,x_n(\tau),x_n'(\tau))d\tau))ds-\int_0^t\phi^{-1}(\int_s^{+\infty}q(\tau)f^*(\tau,x_0(\tau),x_0'(\tau))d\tau)ds\right|}{1+t}\end{aligned}$$

$$\leq \sup_{t\in\mathbb{R}^+} \frac{\int_0^t \left|\phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x_n(\tau),x_n'(\tau))) - \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x_0(\tau),x_0'(\tau))d\tau)\right| ds}{1+t}$$
$$\to 0, \quad \text{as } n \to +\infty$$

and

$$\begin{aligned}
&|\phi(Ax_n)'(t) - \phi(Ax_0)'(t)| \\
&\quad = |\int_t^{+\infty} q(\tau)[f^*(\tau,x_n(\tau),x_n'(\tau)) - f^*(\tau,x_0(\tau),x_0'(\tau))]d\tau| \\
&\quad \leq \int_0^{+\infty} q(\tau)|f^*(\tau,x_n(\tau),x_n'(\tau)) - f^*(\tau,x_0(\tau),x_0'(\tau))|d\tau \\
&\quad \to 0, \quad \text{as } n \to +\infty.
\end{aligned}$$

Hence $A : E \to E$ is continuous.

Step 3. $A(E)$ is relatively compact.

(a) $A(E)$ is uniformly bounded. For $x \in E$, we have

$$\begin{aligned}
\sup_{t\in\mathbb{R}^+} \frac{|Ax(t)|}{1+t} &\leq \sup_{t\in\mathbb{R}^+} \frac{k + \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)|f^*(\tau,x(\tau),x'(\tau))|d\tau\right) ds}{1+t} \\
&\leq \sup_{t\in\mathbb{R}^+} \frac{k + \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)\delta_M(\tau)d\tau\right) ds}{1+t} \\
&\leq k + \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)\delta_M(\tau)d\tau\right) ds < +\infty
\end{aligned}$$

and

$$\begin{aligned}
\sup_{t\in\mathbb{R}^+} |(Ax)'(t)| &= \sup_{t\in\mathbb{R}^+} \left|\phi^{-1}\left(\int_t^{+\infty} q(\tau)f^*(\tau,x(\tau),x'(\tau))d\tau\right)\right| \\
&\leq \phi^{-1}\left(\int_0^{+\infty} q(\tau)\delta_M(\tau)d\tau\right) < +\infty.
\end{aligned}$$

Then $A(E)$ is bounded.

(b) For a given $T > 0$, $x \in E$, and $t, t' \in [0, T]$ $(t > t')$, we have the estimations

$$\begin{aligned}
\left|\frac{Ax(t)}{1+t} - \frac{Ax(t')}{1+t'}\right| &\leq k\left|\frac{1}{1+t} - \frac{1}{1+t'}\right| \\
&\quad + \Big|\frac{\int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau),x'(\tau))d\tau)ds}{1+t} \\
&\quad - \frac{\int_0^{t'} \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau),x'(\tau))d\tau)ds}{1+t'}\Big| \\
&\leq k\left|\frac{1}{1+t} - \frac{1}{1+t'}\right| + \left|\frac{1}{1+t} - \frac{1}{1+t'}\right| \left|\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau),x'(\tau))d\tau\right) ds\right| \\
&\quad + \Big|\frac{\int_{t'}^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau),x'(\tau))d\tau)ds}{1+t'}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{\int_t^{+\infty}\phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau),x'(\tau))d\tau)ds}{1+t}\Big| \\
&\leq k\left|\frac{1}{1+t}-\frac{1}{1+t'}\right| + \left|\frac{1}{1+t}-\frac{1}{1+t'}\right|\left|\int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau),x'(\tau))d\tau\right)ds\right| \\
&\quad + \left|\frac{1}{1+t'}-\frac{1}{1+t}\right|\left|\int_t^{+\infty}\phi^{-1}\left(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau),x'(\tau))d\tau\right)ds\right| \\
&\quad + \frac{1}{1+t'}\left|\int_{t'}^{t}\phi^{-1}\left(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau),x'(\tau))d\tau\right)ds\right| \\
&\leq k\left|\frac{1}{1+t}-\frac{1}{1+t'}\right| + 2|\frac{1}{1+t}-\frac{1}{1+t'}|\int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty} q(\tau)\left|f^*(\tau,x(\tau),x'(\tau))\right|d\tau\right)ds \\
&\quad + \frac{1}{1+t'}\int_{t'}^{t}\phi^{-1}(\int_s^{+\infty} q(\tau)\left|f^*(\tau,x(\tau),x'(\tau))\right|d\tau)ds \\
&\leq k\left|\frac{1}{1+t}-\frac{1}{1+t'}\right| + 2|\frac{1}{1+t}-\frac{1}{1+t'}|\int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty} q(\tau)\delta_M(\tau)d\tau\right)ds \\
&\quad + \frac{1}{1+t'}\int_{t'}^{t}\phi^{-1}(\int_s^{+\infty} q(\tau)\delta_M(\tau)d\tau)ds
\end{aligned}
$$

and

$$
\begin{aligned}
|\phi(Ax)'(t)-\phi(Ax)'(t')| &= \Big|\int_t^{+\infty} q(\tau)f^*(\tau,x(\tau),x'(\tau))d\tau \\
&\qquad -\int_{t'}^{+\infty} q(\tau)f^*(\tau,x(\tau),x'(\tau)d\tau)\Big| \leq \int_{t'}^{t} q(\tau)\delta_M(\tau)d\tau.
\end{aligned}
$$

Then, for any $\varepsilon>0$ and $T>0$, there exists $\delta>0$ such that $|\frac{Ax(t)}{1+t}-\frac{Ax(t')}{1+t'}|<\varepsilon$ and $|(Ax)'(t)-(Ax)'(t')|<\varepsilon$ for all $t,t'\in[0,T]$ with $|t-t'|<\delta$.

(c) For any $x\in E$, we have $\lim\limits_{t\to+\infty}\frac{Ax(t)}{1+t}=\lim\limits_{t\to+\infty}(Ax)'(t)=0$. Therefore

$$
\begin{aligned}
\sup_{x\in E}\left|\frac{Ax(t)}{1+t}-\lim_{t\to+\infty}\frac{Ax(t)}{1+t}\right| &= \sup_{x\in E}\left|\frac{k}{1+t}+\frac{\int_0^t\phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau),x'(\tau))d\tau)ds}{1+t}\right| \\
&\leq \sup_{x\in E}\frac{k}{1+t}+\frac{\int_0^{+\infty}\phi^{-1}(\int_s^{+\infty} q(\tau)\delta_M(\tau)d\tau)ds}{1+t} \\
&\to 0, \quad \text{as } t\to+\infty
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{x\in E}|(Ax)'(t)-\lim_{t\to+\infty}(Ax)'(t)| &= \sup_{x\in E}|\phi^{-1}\left(\int_t^{+\infty} q(\tau)f^*(\tau,x(\tau),x'(\tau))d\tau\right)| \\
&\leq \phi^{-1}\left(\int_t^{+\infty} q(\tau)\delta_M(\tau)d\tau\right) \\
&\to 0, \quad \text{as } t\to+\infty.
\end{aligned}
$$

By Lemma 1.3, $A(E)$ is relatively compact. Finally the Schauder's fixed point theorem implies that A has at least one fixed point $x \in E$, which is a solution to Problem (2.6).

Part 2. To prove that $\alpha(t) \leq x(t) \leq \beta(t)$, $\forall\, t \in \mathbb{R}^+$, we argue by contradiction assuming that some point $t^* \in \mathbb{R}^+$ exists and satisfies $x(t^*) > \beta(t^*)$. Define

$$t_1 = \inf\{t < t^* : x(s) > \beta(s), \quad \forall s \in [t, t^*]\},$$

$$t_2 = \sup\{t > t^* : x(s) > \beta(s), \quad \forall s \in [t^*, t]\}.$$

Since $x(0) \leq \beta(0)$ then $t_1 \geq 0$, $x(t_1) = \beta(t_1)$ and $x'(t_1) > \beta'(t_1)$. In the other hand since $\lim\limits_{t\to+\infty}(x' - \beta')(t) \leq 0$, there are two cases to consider.

(a) If $t_2 < +\infty$, then there exists $t_0 \in (t_1, t_2)$ such that $(x - \beta)(t_0) = \max\limits_{t\in[t_1,t_2]}(x - \beta)(t)$. Since $x'(t_0) = \beta'(t_0)$ and $|x'(t_0)| = |\beta'(t_0)| \leq M$, then

$$\begin{aligned}
&(\phi(x'(t_0)))' - (\phi(\beta'(t_0)))' \\
&\quad \geq -q(t_0)f^*(t_0, x(t_0), x'(t_0)) + q(t)f(t_0, \beta(t_0), \beta'(t_0)) \\
&\quad = -q(t_0)[f(t_0, \beta(t_0), \beta'(t_0)) - \mu(x(t_0) - \beta(t_0)) - f(t_0, \beta(t_0), \beta'(t_0))] \\
&\quad = q(t_0)\mu(x(t_0) - \beta(t_0)) > 0.
\end{aligned}$$

The continuity of $(\phi(x'))' - (\phi(\beta'))'$ implies that there exists $\delta > 0$ such that $(\phi(x'))'(t) - (\phi(\beta'))'(t) > 0, \forall\, t \in [t_0, t_0 + \delta]$; hence $x' - \beta'$ is increasing on $[t_0, t_0 + \delta]$ which is a contradiction to $(x - \beta)(t_0) = \max\limits_{t\in[t_1,t_2]}(x - \beta)(t)$.

(b) If $t_2 = +\infty$, then $\lim\limits_{t\to+\infty}(x' - \beta')(t) = 0$ and there are again two cases to consider.

(b1) There exists $T > t^*$ such that $(x' - \beta')(t) = 0$ in $[T, +\infty)$. Then $x'(T) = \beta'(T)$ and $(x - \beta)(T) = \max\limits_{t\in[T,+\infty)}(x - \beta)(t)$. As in case (a), we reach a contradiction.

(b2) There exists $T > 0$ such that $x'(t) - \beta'(t) > 0$ on $[T, +\infty)$. Then there exists $\bar{l} > 0$ such that $x(t) - \beta(t) = |x(t) - \beta(t)| > \bar{l}$, for all $t > T$. In this case we have $\mu(x(t) - \beta(t)) > \min\{1, \bar{l}\} = \bar{l}$, for all $t > T$. Since $\lim\limits_{t\to+\infty} x'(t) = 0$ and $\lim\limits_{t\to+\infty} x'(t) - \beta'(t) = 0$, then $\lim\limits_{t\to+\infty} \beta'(t) = 0$. From the continuity of f, we can choose T large enough such that, for all $t > T$, we have $-M \leq x'(t) \leq M$ and

$$\begin{aligned}
|f(t, \beta(t), \beta'(t)) - f(t, \beta(t), x'(t))| &\leq |f(t, \beta(t), \beta'(t)) - f(t, \beta(t), 0)| \\
&\quad + |f(t, \beta(t), 0) - f(t, \beta(t), x'(t))| \leq \frac{\bar{l}}{2}.
\end{aligned}$$

Consequently, for all $t > T$, we have the estimates

$$\begin{aligned}
0 &> \phi(\beta'(t)) - \phi(x'(t)) \\
&= \int_t^{+\infty} (\phi(x'))'(s) - (\phi(\beta'))'(s)ds
\end{aligned}$$

$$\geq \int_t^{+\infty} q(s)[f(s,\beta(s),\beta'(s)) - f^*(s,x(s),x'(s))]ds$$
$$\geq \int_t^{+\infty} q(s)[f(s,\beta(s),\beta'(s)) - f(s,\beta(s),T_M(x'(s))) + \mu(x(s)-\beta(s))]ds$$
$$\geq \int_t^{+\infty} q(s)[f(s,\beta(s),\beta'(s)) - f(s,\beta(s),x'(s)) + \mu(x(s)-\beta(s))]ds.$$
$$\geq \frac{\bar{l}}{2}\int_t^{+\infty} q(s)ds > 0,$$

which is a contradiction. In the same way, we can prove that $\alpha \leq x$, which completes the proof.

Part 3. We prove that $0 \leq x'(t) \leq M$, $\forall\, t \in \mathbb{R}^+$. Since $\alpha \leq x \leq \beta$, then the sign condition (2.4) yields $f^*(t,x(t),x'(t)) = f(t,x(t),T_M(x(t))) \geq 0$. This implies that x' is nonincreasing and since $\lim\limits_{t\to+\infty} x'(t) = 0$, then $x'(t) \geq 0, \forall\, t \in \mathbb{R}^+$. Now suppose that there exists $t_0 \in \mathbb{R}^+$ such that $x'(t_0) > M$; hence $x'(t) > M, \forall\, t \in [0,t_0]$. Let

$$\bar{t} = \sup\{t > t_0 : x'(s) > M,\ \forall\, s \in [0,t]\}.$$

Then $x'(\bar{t}) = M$ and $x'(t) > M$ for any $t \in [0,\bar{t})$. This implies that for any $t \in [0,\bar{t}]$, we have

$$(\phi(x'(t)))' = -q(t)f^*(t,x(t),x'(t)) = -q(t)f(t,x(t),M) = 0.$$

Indeed, the continuity of f both with the sign condition (2.4) imply that $f(t,x,M) = f(t,x,-M) = 0$. Hence there exists a constant λ such that $x'(t) = \lambda, \forall\, t \in [0,\bar{t}]$. Since $x'(\bar{t}) = M$, then $x'(t) = M, \forall\, t \in [0,\bar{t}]$, which is a contradiction. Therefore x is a solution of Problem (1.1). $\square$

2.2 The singular problem. Define the function

$$\rho(t) = \begin{cases} t, & t \in [0,1] \\ \frac{1}{t}, & t \in (1,+\infty), \end{cases}$$

let $\widetilde{\rho}(t) = \frac{\rho(t)}{1+t}$, and $F(t,x,y) = f(t,(1+t)x,y)$. Assume that the following hypotheses hold.

(H_1) There exists a constant $M > 0$ such that

$$(|y| - M)f(t,x,y) < 0, \quad \forall\,(t,x,y) \in \mathbb{R}^+ \times I \times \mathbb{R} \setminus \{-M,M\}.$$

(H_2) There exist $p \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $g \in C(I \times \mathbb{R},\mathbb{R}^+)$ such that $g(\cdot,y)$ is nonincreasing and $g(x,\cdot)$ is a nondecreasing with

$$|F(t,x,y)| \leq p(t)g(x,y), \quad \forall\,(t,x,y) \in \mathbb{R}^+ \times I \times \mathbb{R} \tag{2.7}$$

and for each $c > 0$,

$$\int_0^{+\infty} q(\tau)p(\tau)g(c\widetilde{\rho}(\tau),M)d\tau < +\infty,$$

$$\text{and} \quad \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)p(\tau)g(c\widetilde{\rho}(\tau), M)d\tau\right) ds < +\infty.$$

(H_3) For each $\delta \in (0, M)$, there exists a $\psi_\delta \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that $\psi_\delta(t) > 0$ in $[1, +\infty)$ and

$$|F(t, x, y)| \geq \psi_\delta(t), \quad \forall\, t \in \mathbb{R}^+,\ \forall\,(x, y) \in (0, M] \times [0, M - \delta]$$

with

$$\int_0^{+\infty} q(\tau)\psi_\delta(\tau)d\tau < +\infty. \tag{2.8}$$

Using Theorem 2.2, we will prove the following one.

Theorem 2.3. *Under Assumptions $(H_1) - (H_3)$, Problem* (1.1) *has at least one positive solution.*

Proof. We argue using an approximation method. First choose a decreasing sequence $(\varepsilon_n)_{n\geq 1}$ with $\lim_{n\to+\infty} \varepsilon_n = 0$ and $\varepsilon_1 < M$. Then, consider the sequence of regular boundary value problems

$$\begin{cases} (\phi(x'))'(t) + q(t)f(t, x(t), x'(t)) = 0, & t > 0 \\ x(0) = \varepsilon_n, \quad \lim_{t\to+\infty} x'(t) = 0. \end{cases} \tag{2.9}$$

Step 1. For each $n \geq 1$, we claim that Problem (2.9) has at least one solution x_n. To this end, let $\alpha_n(t) = \varepsilon_n$; then $|\alpha_n'(t)| = 0 < M$ and Assumption (H_1) implies that $f(t, \alpha_n(t), \alpha_n'(t)) = f(t, \alpha_n(t), 0) > 0$. Then

$$\begin{cases} -(\phi(\alpha_n'(t)))' = -\phi(0) = 0 \leq q(t)f(t, \alpha_n(t), \alpha_n'(t)), & t > 0 \\ \alpha(0) \leq \varepsilon_n, \quad \lim_{t\to+\infty} \alpha_n'(t) = 0. \end{cases}$$

Then, for any $n \geq 1$, α_n is a lower solution for Problem (2.9). Let $\beta(t) = Mt + M$; then $\beta'(t) = M$ and Assumption (H_1) implies that

$$\begin{cases} -(\phi(\beta'(t)))' = 0 = f(t, \beta(t), \beta'(t)),\ t > 0 \\ \quad \beta(0) = M \geq \varepsilon_n \quad \lim_{t\to+\infty} \beta'(t) = M \geq 0. \end{cases}$$

Then β is an upper solution for (2.9). Furthermore for any $t \in \mathbb{R}^+$ and $(x, y) \in [\alpha_n(t), \beta(t)] \times [-M, M]$, we have by (H_2)

$$\begin{aligned} |f(t, x, y)| &= |F(t, \frac{x}{1+t}, y)| \\ &\leq p(t)g(\frac{x}{1+t}, y) \\ &\leq p(t)g(\frac{\varepsilon_n}{1+t}, M) \\ &\leq p(t)g(\varepsilon_n\widetilde{\rho}(t), M) := \delta_M(t) \end{aligned}$$

with $\int_0^{+\infty} q(\tau)\delta_M(\tau)d\tau < +\infty$ and $\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)\delta_M(\tau)d\tau\right) ds < +\infty$. Then all conditions of Theorem 2.2 are met and so, for any $n \geq 1$, Problem (2.9) has at least one solution x_n such that

$$\alpha_n(t) \leq x_n(t) \leq \beta(t) \text{ and } 0 \leq x_n'(t) \leq M, \quad \forall\, t \in \mathbb{R}^+.$$

Step 2. The sequence $(x_n)_{n\geq 1}$ is relatively compact in E.

(a) The sequence $(x_n)_{n\geq 1}$ is bounded in E. By Step 1, we have, for all $n \geq 1$,

$$\|x_n\| = \max\{\|x_n\|_1, \|x_n\|_2\} \leq \max\{\|\beta\|_1, M\} = M.$$

Let $\delta \in (0, M)$. Since $\lim\limits_{t\to+\infty} x_n'(t) = 0$, then there exists $t^* > 1$ such that

$$0 \leq x_n'(t) \leq M - \delta, \ \forall\, t \geq t^*$$

and since $0 < \frac{x_n(t)}{1+t} \leq M$ for all $t \in \mathbb{R}^+$, then by (H_3) there exists $\psi_\delta \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$f(t, x_n(t), x_n'(t)) = F(t, \frac{x_n(t)}{1+t}, x_n'(t)) \geq \psi_\delta(t), \ \text{ for } \ t \in \mathbb{R}^+. \tag{2.10}$$

Note that $0 \leq x' \leq M$ yields the positivity of f. Let

$$c^* = \phi^{-1}\left(\int_{t^*}^{+\infty} q(\tau)\psi_\delta(\tau)d\tau\right) > 0.$$

Then we distinguish between two cases.

(a1) If $t \in [0, t^*]$, then

$$\begin{aligned}
x_n(t) &\geq \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau, x_n(\tau), x_n'(\tau))d\tau\right) ds \\
&\geq \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)F(\tau, \frac{x_n(\tau)}{1+\tau}, x_n'(\tau))d\tau\right) ds \\
&\geq \int_0^t \phi^{-1}\left(\int_{t^*}^{+\infty} q(\tau)\psi_\delta(\tau)d\tau\right) ds \\
&= t\phi^{-1}\left(\int_{t^*}^{+\infty} q(\tau)\psi_\delta(\tau)d\tau\right) \\
&\geq tc^* \geq \rho(t)c^*.
\end{aligned}$$

(a2) If $t \in (t^*, +\infty)$, then

$$\begin{aligned}
x_n(t) &\geq \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau, x_n(\tau), x_n'(\tau))d\tau\right) ds \\
&\geq \int_0^{t^*} \phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau, x_n(\tau), x_n'(\tau))d\tau\right) ds \\
&\geq \int_0^{t^*} \phi^{-1}\left(\int_{t^*}^{+\infty} q(\tau)f(\tau, x_n(\tau), x_n'(\tau))d\tau\right) ds \\
&\geq \int_0^{t^*} \phi^{-1}\left(\int_{t^*}^{+\infty} q(\tau)\psi_\delta(\tau)d\tau\right) ds
\end{aligned}$$

$$
\begin{aligned}
&= t^*\phi^{-1}\left(\int_{t^*}^{+\infty} q(\tau)\psi_\delta(\tau)d\tau\right)ds \\
&= t^*c^* \geq c^* \geq \rho(t)c^*.
\end{aligned}
$$

To sum up, for any $t \in \mathbb{R}^+$ and $n \geq 1$, we have proved that $x_n(t) \geq \rho(t)c^*$. Using (H_1) and the monotonicity of g, we infer the estimates

$$
\begin{aligned}
q(s)f(s, x_n(s), x_n'(s)) &= q(s)F\left(s, \frac{x_n(s)}{1+s}, x_n'(s)\right) \\
&\leq q(s)p(s)g\left(\frac{x_n(s)}{1+s}, x_n'(s)\right) \\
&\leq q(s)p(s)g(c^*\widetilde{\rho}(s), M).
\end{aligned}
$$

(b) For any $T > 0$ and $t, t' \in [0, T]$ $(t > t')$, the following estimates hold

$$
\begin{aligned}
&|\frac{x_n(t)}{1+t} - \frac{x_n(t')}{1+t'}| \leq \varepsilon_n|\frac{1}{1+t} - \frac{1}{1+t'}| \\
&\quad + \Big|\frac{\int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)f(\tau, x_n(\tau), x_n'(\tau))d\tau)ds}{1+t} \\
&\quad - \frac{\int_0^{t'} \phi^{-1}(\int_s^{+\infty} q(\tau)f(\tau, x_n(\tau), x_n'(\tau))d\tau)ds}{1+t'}\Big| \\
&\leq \varepsilon_n|\frac{1}{1+t} - \frac{1}{1+t'}| + 2|\frac{1}{1+t} - \frac{1}{1+t'}|\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau, x_n(\tau), x_n'(\tau))d\tau\right)ds \\
&\quad + \frac{1}{1+t'}\int_{t'}^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau, x_n(\tau), x_n'(\tau))d\tau\right)ds \\
&\leq M|\frac{1}{1+t} - \frac{1}{1+t'}| + 2|\frac{1}{1+t} - \frac{1}{1+t'}|\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)p(\tau)g(c^*\widetilde{\rho}(\tau), M)d\tau\right)ds \\
&\quad + \frac{1}{1+t'}\int_{t'}^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)p(\tau)g(c^*\widetilde{\rho}(\tau), M)d\tau\right)ds
\end{aligned}
$$

and

$$
\begin{aligned}
|\phi(x_n'(t)) - \phi(x_n'(t'))| &= |\int_{t'}^t q(\tau)f_n(\tau, x_n(\tau), x_n'(\tau))d\tau)| \\
&\leq \int_{t'}^t q(\tau)p(\tau)g(c^*\widetilde{\rho}(\tau), M)d\tau.
\end{aligned}
$$

Then, for any $\varepsilon > 0$ and $T > 0$, there exists $\delta > 0$ such that for all $t, t' \in [0, T]$ and $|t - t'| < \delta$, $|\frac{x_n(t)}{1+t} - \frac{x_n(t')}{1+t'}| < \varepsilon$ and $|x_n'(t) - x_n'(t')| < \varepsilon$.

(c) For any $n \geq 0$, (H_2) implies that $\lim\limits_{t\to+\infty} \frac{x_n(t)}{1+t} = \lim\limits_{t\to+\infty} x_n'(t) = 0$. Therefore

$$
\begin{aligned}
\sup_{n\geq 1}|\frac{x_n(t)}{1+t} - 0| &= \sup_{n\geq 1}\frac{\varepsilon_n + \int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)f(\tau, x_n(\tau), x_n'(\tau))d\tau)ds}{1+t} \\
&\leq \frac{M + \int_0^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau)p(\tau)g(c^*\widetilde{\rho}(\tau), M)d\tau)ds}{1+t}
\end{aligned}
$$

$$\to 0, \text{ as } t \to +\infty$$

and

$$\begin{aligned}\sup_{n\geq n_0} |x_n'(t) - 0| &= \sup_{n\geq n_0} \phi^{-1}\Big(\int_t^{+\infty} q(\tau) f_n(\tau, x_n(\tau), x_n'(\tau)) d\tau\Big) \\ &\leq \phi^{-1}\Big(\int_t^{+\infty} p(\tau) g(c^*\widetilde{\rho}(\tau), M) d\tau\Big) \\ &\to 0, \text{ as } t \to +\infty.\end{aligned}$$

As a consequence $(x_n)_{n\geq 1}$ is relatively compact in E by Lemma 1.3; hence there exists a subsequence $(x_{n_k})_{k\geq 1}$ with $\lim_{k\to+\infty} x_{n_k} = x_0$. Since $x_{n_k}(t) \geq \widetilde{\rho}(t)c^*, \forall\, k \geq 1$, we have $x_0(t) \geq \widetilde{\rho}(t)c^*, \forall\, t \in \mathbb{R}^+$. The continuity of f and ϕ^{-1} and the Lebesgue dominated convergence theorem, imply that, for all $t \in \mathbb{R}^+$,

$$\begin{aligned}x_0(t) &= \lim_{k\to+\infty} x_{n_k}(t) \\ &= \lim_{k\to+\infty} \left[\varepsilon_{n_k} + \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x_{n_k}(\tau), x_{n_k}'(\tau)) d\tau\right) ds\right] \\ &= \int_0^t \phi^{-1}\Big(\int_s^{+\infty} q(\tau) f(\tau, x_0(\tau), x_0'(\tau)) d\tau\Big) ds.\end{aligned}$$

Therefore x_0 is a positive solution of Problem (1.1), which completes the proof of Theorem 2.3. □

Example 2.4. Consider the singular boundary value problem

$$\begin{cases} (\phi((x'(t)))' + q(t) f(t, x(t), x'(t)) = 0, & t > 0 \\ x(0) = 0, \quad \lim_{t\to+\infty} x'(t) = 0, \end{cases} \tag{2.11}$$

where $\phi(t) = t^5$, $q(t) = e^{-t}$ and

$$f(t,x,y) = m(t) \begin{cases} \frac{e^y - e^{-3}}{x}, & y < -3 \\ \frac{3-|y|}{x}, & y \geq -3 \end{cases} \text{ where } m(t) = \begin{cases} t, & t \in [0,1] \\ \frac{1}{t}, & t \in (1,+\infty). \end{cases}$$

(H_1) It is clear that the function $f : \mathbb{R}^+ \times I \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies $(|y| - 3) f(t, x, y) < 0, \forall\, (t, x, y) \in \mathbb{R}^+ \times I \times \mathbb{R} \setminus \{-3, 3\}$.
(H_2) $F(t,x,y) = f(t, x(1+t), y) = \frac{m(t)}{1+t} \begin{cases} \frac{e^y - e^{-3}}{x}, & y < -3 \\ \frac{3-|y|}{x}, & y \geq -3. \end{cases}$ Let

$$p(t) = \frac{m(t)}{1+t} \text{ and } g(x,y) = \begin{cases} \frac{3}{x}, & y < 3 \\ \frac{y}{x}, & y \geq 3. \end{cases}$$

Then $p \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $g \in C(I \times \mathbb{R}, \mathbb{R}^+)$ where $g(\cdot, y)$ is nonincreasing, $g(x, \cdot)$ is a nondecreasing and

$$|F(t,x,y)| \leq p(t) g(x,y), \quad \forall\, (t,x,y) \in \mathbb{R}^+ \times I \times \mathbb{R}.$$

Therefore, for any $c > 0$

$$\int_0^{+\infty} q(\tau)p(\tau)g(c\widetilde{\rho}(\tau), M)d\tau = \frac{3}{c} < \infty,$$

$$\text{and} \quad \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)p(\tau)g(c\widetilde{\rho}(\tau), M)d\tau\right) ds = 5\left(\frac{3}{c}\right)^{\frac{1}{5}} < \infty.$$

(H_3) For any $\delta \in (0, 3)$ and $(x, y) \in (0, 3] \times [0, 3-\delta]$, we have that

$$|F(t, x, y)| \geq \frac{\delta}{3}p(t) := \psi_\delta(t)$$

and

$$\int_0^{+\infty} q(\tau)\psi_\delta(\tau)d\tau < \infty.$$

Therefore all conditions of Theorem 2.3 are satisfied and thus Problem (2.11) has at least one positive solution.

3. EXISTENCE RESULT UNDER A NAGUMO CONDITION

In this section, we consider Problem (1.1) with a positive nonlinearity $f : \mathbb{R}^+ \times I \times \mathbb{R} \to \mathbb{R}^+$.

Definition 3.1. A function $\alpha \in C(\mathbb{R}^+, \mathbb{R}^+) \cap C^1(I, \mathbb{R})$ is called a lower solution of (1.1) if $\phi \circ \alpha' \in C^1(I, \mathbb{R})$ with $\alpha(t) > 0$, $\forall\, t > 0$ and

$$\begin{cases} -(\phi(\alpha'(t)))' \leq q(t)f(t, \alpha(t), \alpha'(t)), & t > 0 \\ \alpha(0) = 0, \quad \lim\limits_{t\to+\infty} \alpha'(t) \leq 0. \end{cases}$$

A function $\beta \in C(\mathbb{R}^+, \mathbb{R}^+) \cap C^1(I, \mathbb{R})$ is called an upper solution of (1.1) if $\phi \circ \beta' \in C^1(I, \mathbb{R})$ with $\beta(t) > 0$, $\forall\, t > 0$ and

$$\begin{cases} -(\phi(\beta'(t)))' \geq q(t)f(t, \beta(t), \beta'(t)), & t > 0 \\ \beta(0) \geq 0, \quad \lim\limits_{t\to+\infty} \beta'(t) \geq 0. \end{cases}$$

Let α and β be a lower and upper solution of (1.1) respectively with $\alpha \leq \beta$. Then for all $t \in \mathbb{R}^+$ and for all $N > 0$, we may define $D^N_{\alpha,\beta}(t)$ and $\delta_N(t)$ as in (2.2). Our main existence result in this section is

Theorem 3.2. *Assume that α, β are lower and upper solution of* (1.1) *respectively with $\alpha \leq \beta$ and suppose that the following conditions are satisfied*
$(H_1)'$ *There exist a nondecreasing $\psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $l \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that*

$$f(t, x, y) \leq l(t)\psi(|y|), \ \forall\, t \in \mathbb{R}^+, \forall\, x \in [\alpha(t),\ \beta(t)],\ \forall\, y \in \mathbb{R}$$

with

$$\int_0^{+\infty} \frac{\phi^{-1}(s)}{\psi(\phi^{-1}(s))}ds > l_0(M-m) + M\int_0^{+\infty} q(\tau)l(\tau)d\tau, \tag{3.1}$$

where $l_0 = \sup_{t\in\mathbb{R}^+} q(t)(1+t)l(t)$, $M = \sup_{t\in\mathbb{R}^+} \frac{\beta(t)}{1+t}$ *and* $m = \inf_{t\in\mathbb{R}^+} \frac{\alpha(t)}{1+t}$.

$(H_2)'$ *For all* $N > 0$,

$$\int_0^{+\infty} q(\tau)\delta_N(\tau)d\tau < +\infty \ \text{ and } \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)\delta_N(\tau)d\tau\right) ds < +\infty.$$

Then Problem (1.1) *has at least one positive solution.*

Proof. Since $\lim_{t\to+\infty} \phi(t) = +\infty$, then by (3.1), we can choose some

$$N > \max\{\sup_{t\in\mathbb{R}^+} |\alpha'(t)|, \sup_{t\in\mathbb{R}^+} |\beta'(t)|\}$$

such that

$$\int_0^{\phi(N)} \frac{\phi^{-1}(s)}{\psi(\phi^{-1}(s))}ds > l_0(M-m) + M\int_0^{+\infty} q(\tau)l(\tau)d\tau. \tag{3.2}$$

Let μ, f^* and T_N be as in the proof of Theorem 2.2 and consider the auxiliary problem

$$\begin{cases} (\phi(x'))'(t) + q(t)f^*(t, x(t), x'(t)) = 0, \quad t > 0 \\ x(0) = 0, \quad \lim_{t\to+\infty} x'(t) = 0. \end{cases} \tag{3.3}$$

For $x \in E$, define the operator A by

$$Ax(t) = \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f^*(\tau, x(\tau), x'(\tau))d\tau\right) ds.$$

As in Parts 1, 2 of the proof of Theorem 2.2, we can show that A has a fixed point $x \in E$ such that $\alpha(t) \le x(t) \le \beta(t)$, $\forall\, t \in \mathbb{R}^+$. It remains to show that $|x'(t)| \le N$, $\forall\, t \in \mathbb{R}^+$. Since $\alpha \le x \le \beta$, then $f^*(t, x(t), x'(t)) = f(t, x(t), T_N(x(t))) \ge 0$, which implies that x' is nonincreasing. In addition, since $\lim_{t\to+\infty} x'(t) = 0$, then $x'(t) \ge 0, \forall\, t \in \mathbb{R}^+$ and there exists $T > 0$ such that $0 \le x'(t) < N$, $\forall\, t > T$. Suppose that there exists $t_0 \in \mathbb{R}^+$ such that $x'(t_0) > N$ and let

$$t' = \inf\{t < T, \quad x'(s) < N, \forall\, s \in [t, +\infty)\}.$$

Then $x'(t') = N$ and for all $t > t'$, $0 \le x'(t) < N$. From $(H_1)'$, we have

$$\begin{aligned}(\phi(x'(t)))' &= -q(t)f(t, x(t), x'(t)) \\ &\ge -q(t)l(t)\psi(|x'(t)|).\end{aligned}$$

Thus, for all $t > t'$, we have

$$\begin{aligned}\frac{x'(t)(\phi(x'(t)))'}{\psi(|x'(t)|)} &\ge -q(t)l(t)x'(t) \\ &\ge -q(t)l(t)\left[\left(\frac{x(t)}{1+t}\right)'(1+t) + \frac{x(t)}{1+t}\right] \\ &= -q(t)l(t)\left(\frac{x(t)}{1+t}\right)'(1+t) - q(t)l(t)\frac{x(t)}{1+t}.\end{aligned}$$

An integration from t' to $+\infty$ yields

$$\begin{aligned}
\int_0^{\phi(N)} \frac{\phi^{-1}(s)}{\psi(\phi^{-1}(s))}ds &= \int_{\phi(x'(+\infty))}^{\phi(x'(t'))} \frac{\phi^{-1}(s)}{\psi(|\phi^{-1}(s)|)}ds = -\int_{t'}^{+\infty} \frac{x'(\tau)(\phi(x'(\tau)))'}{\psi(|x'(\tau)|)}d\tau \\
&\leq \int_{t'}^{+\infty} q(\tau)l(\tau)(\frac{x(\tau)}{1+\tau})'(1+\tau)d\tau + \int_{t'}^{+\infty} q(\tau)l(\tau)\frac{x(\tau)}{1+\tau}d\tau \\
&\leq \sup_{\tau\in\mathbb{R}^+} q(\tau)(1+\tau)l(\tau) \int_{t'}^{+\infty} \left(\frac{x(\tau)}{1+\tau}\right)' d\tau \\
&\qquad + \sup_{\tau\in\mathbb{R}^+} \frac{x(\tau)}{1+\tau} \int_0^{+\infty} q(\tau)l(\tau)d\tau \\
&\leq l_0(M-m) + M\int_0^{+\infty} q(\tau)l(\tau)d\tau
\end{aligned}$$

which is a contradiction to (3.2). Hence x is positive solution of (1.1). □

Example 3.3. Consider the singular boundary value problem

$$\begin{cases} ((x'(t))^3)' + \frac{7t}{(1+t)^5}\frac{|(x(t)-1)x'(t)|}{|x(t)|} = 0, \quad t > 0 \\ x(0) = 0, \quad \lim_{t\to+\infty} x'(t) = 0. \end{cases} \tag{3.4}$$

Here $\phi(t) = t^3$, $q(t) = \frac{7}{(1+t)^5}$ and $f(t, x, y) = \frac{t|y(x-1)|}{|x|}$.

Let $\alpha(t) = \frac{t}{1+t}$. Then $\alpha(t) > 0, \forall\, t > 0$, $\alpha(0) = 0$, $\alpha'(+\infty) = 0$ and $q(t)f(t, \alpha(t), \alpha'(t)) = \frac{7}{(1+t)^7}$ which imply that

$$-(\phi(\alpha'(t)))' = -\left(\phi\left(\frac{1}{(1+t)^2}\right)\right)' = \frac{6}{(1+t)^7} \leq q(t)f(t, \alpha(t), \alpha'(t)).$$

Hence α is a lower solution of (3.4). Let $\beta(t) = 1$. It is clear that β is an upper solution of (3.4) and $\alpha(t) \leq \beta(t)$, $\forall t \geq 0$.

$(H_1)'$ For $t \in \mathbb{R}^+$, $x \in [\alpha(t), \beta(t)]$ and $y \in \mathbb{R}$, we have

$$f(t, x, y) = \frac{t|y(x-1)|}{|x|} \leq \frac{t(1-\frac{t}{1+t})|y|}{\frac{t}{1+t}} = |y| = l(t)\psi(|y|),$$

where $l(t) = 1$ and $\psi(y) = y$. It is clear that $\psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ is nondecreasing and $l \in C(\mathbb{R}^+, \mathbb{R}^+)$. Moreover, we have

$$l_0 = \sup_{t\in\mathbb{R}^+} q(t)(1+t)l(t) = 7,\; M = \sup_{t\in\mathbb{R}^+} \frac{\beta(t)}{1+t} = 1,\; m = \inf_{t\in\mathbb{R}^+} \frac{\alpha(t)}{1+t} = 0,$$

and

$$\int_0^{+\infty} q(\tau)l(\tau)d\tau = \int_0^{+\infty} \frac{7}{(1+\tau)^5}d\tau < +\infty.$$

Then

$$\int_0^{+\infty} \frac{\phi^{-1}(s)}{\psi(\phi^{-1}(s))}ds = \int_0^{+\infty} ds = +\infty > l_0(M-m) + M\int_0^{+\infty} q(\tau)l(\tau)d\tau.$$

$(H_2)'$ For all $N > 0$ and $t \in \mathbb{R}^+$, we have $\delta_N(t) = \sup_{(x,y)\in D^N_{\alpha,\beta}(t)} f(t,x,y) + 1 = N + 1$ and

$$\int_0^{+\infty} q(\tau)\delta_N(\tau)d\tau = \int_0^{+\infty} \frac{7(N+1)}{(1+\tau)^5} < +\infty,$$

$$\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)\delta_N(\tau)d\tau\right) ds = \left(\frac{7(N+1)}{4}\right)^{\frac{1}{3}} \int_0^{+\infty} \frac{ds}{(1+s)^{\frac{4}{3}}} < +\infty.$$

Then all conditions of Theorem 3.2 are met. Hence Problem (3.4) has at least one solution x such that

$$\frac{t}{1+t} \leq x(t) \leq 1,\ \forall\, t \in \mathbb{R}^+.$$

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REFERENCES

[1] R. P. Agarwal, D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publisher, Dordrecht, 2001.

[2] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.

[3] S. Djebali, K. Mebarki, Multiple positive solutions for singular BVPs on the positive half-line, *Comput. Math. with Appli.* **55(12)** (2008), 2940–2952.

[4] S. Djebali, K. Mebarki, On the singular generalized Fisher-like equation with derivative depending nonlinearity, *Appl. Math. Comput.* **205** (2008), 336–351.

[5] S. Djebali, K. Mebarki, Multiple unbounded positive solutions for three-point bvps with sign-changing nonlinearities on the positive half-line, *Acta Appl. Math.* **109** (2010), 361–388.

[6] S. Djebali, O. Saifi, Positive solutions for singular BVPs on the positive half-line with sign changing and derivative depending nonlinearity, *Acta Appl. Math.* DOI 10.1007/s10440-009-9466-9 (27 pages).

[7] Y. Guo, C. Yu, and J. Wang, Existence of three positive solutions for $m-$point boundary value problems on infinite intervals, *Nonlinear Analysis* **71** (2009), 717–722.

[8] H. Lian, H. Pang, W. Ge, Triple positive solutions for boundary value problems on infinite intrvals, *Nonlinear Analysis* **67** (2007), 2199–2207.

[9] S. Liang, J. Zhang, The existence of countably many positive solutions for nonlinear singular $m-$point boundary value problems on the half line, *J. Comput. Appl. Math.* **222** (2008), 229–243.

[10] S. Liang, J. Zhang, The existence of countably many positive solutions for one-dimensional p-Laplacian with infinitely many singularities on the half-line, *Appl. Math. Comput.* **201** (2008), 210–220.

[11] S. Liang, J. Zhang, and Z. Wang, The existence of multiple positive solutions for multi-point boundary value problems on the half line, *J. Comput. Appl. Math.* **228(1)** (2009), 10–19.

[12] B. Yan, D. O'Regan, and R. P. Agarwal, Unbounded solutions for singular boundary value problems on the semi-infinite interval: Upper and lower solutions and multiplicity, *J. Comput. Appl. Math.* **197** (2006), 365–386.

[13] X. Zhang, Existence of positive solutions for multi-point boundary value problems on infinite intervals in Banach spaces, *Appl. Math. Comput.* **206** (2008), 932–941.