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GENERALIZED HARDY-HILBERT'S INEQUALITY

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ABSTRACT. In this paper, we have studied some extensions of Hardy-Hilbert's inequality with an improved weight coefficient. We have also established reverse inequalities of Hardy-Hilbert type inequalities.

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1. INTRODUCTION

If a_n, $b_n \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $0 < \sum_{n=1}^{\infty} a_n{}^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n{}^q < \infty$, then the famous Hardy-Hilbert inequality (see Hardy et al. [2] texolowa) is given by

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \Big\{ \sum_{n=1}^{\infty} a_n^p \Big\}^{1/p} \Big\{ \sum_{n=1}^{\infty} b_n^q \Big\}^{1/q}, \tag{1.1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. This inequality plays important role in analysis. Considerable attention has been given to develop some types of strengthened inequality by estimating the weight-coefficient. Gau [3] considered the general case and proved a new inequality for the weight coefficient $w(q,n)$ as

$$w(q,n) < \frac{\pi}{\sin(\pi/p)} - \frac{\theta_p}{n^{1/p}} \Big(q > 1, \frac{1}{p} + \frac{1}{q} = 1, n \in N \Big), \tag{1.2}$$

where $\theta_p = p - 1$. Yang and Gau [8] found the best possible value for $\theta_p = \theta = 1 - C = 0.42278433^{+}$, where C is Euler's constant. They also proved the following new Hardy-Hilbert's inequality

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)} < \Big\{ \sum_{n=1}^{\infty} \Big[\frac{\pi}{\sin(\pi/p)} - \frac{1-C}{n^{1/p}} \Big] \Big\}^{\frac{1}{p}} \Big\{ \sum_{n=1}^{\infty} \Big[\frac{\pi}{\sin(\pi/p)} - \frac{1-C}{n^{1/q}} \Big] \Big\}^{\frac{1}{q}}_{.} \tag{1.3}$$

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Yang [5] proved a strengthened version of Hardy-Hilbert's inequality as follows

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \Big\{\sum_{n=1}^{\infty}\Big[\frac{\pi}{\sin(\pi/p)}-\frac{1}{2n^{1/p}+n^{-1/q}}\Big]a_n^p\Big\}^{1/p} \times\Big\{\sum_{n=1}^{\infty}\Big[\frac{\pi}{\sin\pi/p}-\frac{1}{2n^{1/q}+n^{-1/p}}\Big]b_n^q\Big\}^{1/q}. \tag{1.4}$$

Yang [6] has given reverse of the Hardy-Hilbert type inequality as If $0 < p < 1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_n, b_n \geq 0$, such that $0<\sum_{n=1}^{\infty}\frac{a_n^p}{2n+1}<\infty$ and $0<\sum_{n=1}^{\infty}\frac{b_n^q}{2n+1}<\infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n+1)^2} > 2\Big\{\sum_{n=0}^{\infty}\Big[1-\frac{1}{4(n+1)^2}\Big]\frac{a_n^p}{2n+1}\Big\}^{\frac{1}{p}} \times\Big\{\sum_{n=0}^{\infty}\Big[1-\frac{1}{6(n+1)(2n+1)}\Big]\frac{b_n^q}{2n+1}\Big\}^{\frac{1}{q}}, \tag{1.5}$$

where the constant factor is the best possible.

In this paper, we have generalized the results of [5] and [6], which is related to the double series of the form

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}.$$

For this series, we have estimated the weight-coefficient of the following form

$$w(q_k,n)=\sum_{m=1}^{\infty}\frac{1}{m+n}\Big(\frac{n}{m}\Big)^{1/q_k}(q_k>1,\ p_k+q_k=1,\ n\in N). \tag{1.6}$$

2. PRELIMINARIES

The sequence space l_p has been generalized to $l(p)$, in the following manner (see Simmons [4]).

Definition 2.1. Let a bounded sequence, $p=(p_k)$ of strictly positive numbers, with $0<p_k\leq \sup p_k = H<\infty$. Then

$$l(p)=\{x=(x_k):\sum|x_k|^{p_k}<\infty\}.$$

A natural metric on $l(p)$ is

$$d(x,y)=\Big(\sum_{k=0}^{\infty}|\ x_k-y_k\ |^{p_k}\Big)^{1/M},$$

where $M=\max(1,H)$.

Das and Nanda [1] have generalized Holder's inequality in $l(p)$ space, which is given below.

Lemma 2.1 (Das and Nanda [1]). *Let $(p_n)_{n=1}^{\infty}$ is a real sequence be defined by $\frac{1}{p_n}+\frac{1}{q_n}=1$, for all n. Let $a_n, b_n \geq 0$. We write*

$$A_m=\sum_{n=1}^{m} {a_n}^{p_n}, B_m=\sum_{n=1}^{m} {b_n}^{q_n},$$
$$A=\sum_{n=1}^{\infty} {a_n}^{p_n}, B=\sum_{n=1}^{\infty} {b_n}^{q_n}$$

and whenever the series on the right converge.
(a) Let $p_n>1$ for all n. Then
(i)

$$\sum_{k=1}^{m} a_k b_k \leq \alpha_m \beta_m, \tag{2.1}$$

$$\text{where } \alpha_m=\sup_{1\leq n\leq m}\frac{1}{p_n}+\sup_{1\leq n\leq m}\frac{1}{q_n},\ \ \beta_m=\sup_{1\leq n\leq m} A_m^{1/p_n}B_m^{1/q_n}.$$

(ii) If $a\in l(p)$, $b\in l(q)$, then $ab\in l$ and

$$\sum_{k=1}^{\infty} a_k b_k \leq \alpha\beta, \tag{2.2}$$

$$\text{where } \alpha=\sup_{n\geq 1}\frac{1}{p_n}+\sup_{n\geq 1}\frac{1}{q_n},\ \ \beta=\sup_{n\geq 1}\ (A^{1/p_n}B^{1/q_n}).$$

(b) Let $0<p_n<1$ for all n. Then
(i)

$$\sum_{k=1}^{m} a_k^{p_k} \leq \gamma_m\Big[\sup_{1\leq n\leq m} p_n+\sup_{1\leq n\leq m}(1-p_n)\Big], \tag{2.3}$$

$$\text{where } \gamma_m=\sup_{1\leq n\leq m}\Big[\Big(\sum_{k=1}^{m} b_k^{p'_k}\Big)^{1-p_n}\Big(\sum_{k=1}^{m} a_k b_k\Big)^{p_n}\Big];$$

(ii) if $a\in l(p)$ and $b\in l(p')$, then

$$\sum_{k=1}^{\infty} a_k^{p_k} \leq \gamma\Big[\sup p_n+\sup(1-p_n)\Big], \tag{2.4}$$

$$\text{where } \gamma=\sup_{n}\Big[\Big(\sum_{k=1}^{\infty} b_k^{p'_k}\Big)^{1-p_n}\Big(\sum_{k=1}^{\infty} a_k b_k\Big)^{p_n}\Big].$$

It may be observed that by taking $p_n=$ constant, we get the usual Holder's inequality for l^p space.

Lemma 2.2 (Yang [7]). *If for $r=0,1,2,3,4$, $f^{(r)}(\infty)=0$, $f^{(2r-1)}(x)<0$, $f^{(2r)}(x)\geq 0$, $x\in[1,\infty)$, and $\int_1^{\infty} f(x)dx<\infty$, then*

$$\sum_{m=1}^{\infty} f(m) \leq \int_1^{\infty} f(x)dx+\frac{1}{2}f(1)-\frac{1}{12}f'(1). \tag{2.5}$$

Lemma 2.3 (Yang [5])**.** *If $x > 1$, $n \in N$, then*

$$f_n(x) + g_n(x) > \frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}. \tag{2.6}$$

3. MAIN RESULTS

Lemma 3.1. *If (p_k) and (q_k) are real bounded sequences defined by $p_k^{-1} + q_k^{-1} = 1$ where $q_k > 1$ for all $k \in N, n \in N$ then*

$$w(q_k, n) < \sup_{k\geq 1}\Big\{\frac{\pi}{\sin(\pi/p_k)} - \frac{1}{n^{1/p_k}}[f_n(p_k) + g_n(p_k)]\Big\}, \quad \text{for all } k \geq 1. \tag{3.1}$$

Where $w(q_k, n)$ is defined by (1.6), and for $x > 1$

$$f_n(x) = x + \frac{1}{12x} + \frac{1}{(1+x)n} + \frac{1}{12xn^2} + \frac{1}{3(1+3x)n^3}$$

$$g_n(x) = \frac{-1}{12xn} - \frac{1}{2(1+2x)n^2} - \frac{7}{12} - \frac{1}{2n} + \frac{1}{12n^2} - \frac{7}{12n^3}$$

Proof. Let for all $k \geq 1$

$$g_k(x) = \frac{1}{(x+n)x^{1/q_k}}, \quad \text{where } x \in [1, \infty)(\ q_k \geq 1, n \in N). \tag{3.2}$$

We define $f(x) = \sup_{k\geq 1} g_k(x)$. By (2.5), we obtain that

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{1}{(m+n)m^{1/q_k}} \quad &\leq \quad \sup_{k\geq 1}\Big\{\int_1^{\infty} \frac{1}{(x+n)x^{1/q_k}}dx \\ &\qquad + \Big(\frac{1}{12} - \frac{1}{12p_k}\Big)\frac{1}{1+n} + \frac{1}{12(1+n)^2}\Big\}.\end{aligned} \tag{3.3}$$

Since for all $k \geq 1$

$$\begin{aligned}\int_0^{\frac{1}{n}} \frac{1}{(1+y)y^{1/q_k}}dy &= \int_0^{\frac{1}{n}} \sum_{\upsilon=0}^{\infty}(-1)^{\upsilon} y^{\upsilon-1/q_k}dy \\ &= \sum_{\upsilon=0}^{\infty}(-1)^{\upsilon}\int_0^{\frac{1}{n}} y^{\upsilon-1/q_k}dy = \frac{p_k}{n^{p_k}}\sum_{\upsilon=0}^{\infty}\frac{(-1)^{\upsilon}}{(1+\upsilon p_k)n^{\upsilon}} \\ &> \frac{p_k}{n^{p_k}}\sum_{\upsilon=0}^{3}\frac{(-1)^{\upsilon}}{(1+\upsilon p_k)n^{\upsilon}} \\ &= \frac{1}{n^{p_k}}\Big[p_k + \sum_{\upsilon=1}^{3}\frac{(-1)^{\upsilon}}{\upsilon n^{\upsilon}} - \sum_{\upsilon=1}^{3}\frac{(-1)^{\upsilon}}{\upsilon(1+\upsilon p_k)n^{\upsilon}}\Big]\end{aligned}$$

putting $x = ny$, we find that

$$\begin{aligned}\int_1^\infty \frac{1}{(x+n)x^{1/q_k}}\,dx &= \frac{1}{n^{1/q_k}}\int_0^{\frac{1}{n}} \frac{1}{(1+y)y^{1/q_k}}\,dy \\ &= \frac{1}{n^{1/q_k}}\Big[\int_0^\infty \frac{1}{(1+y)y^{1/q_k}}\,dy - \int_0^{1/n} \frac{1}{(1+y)y^{1/q_k}}\,dy\Big] \\ &= \frac{1}{n^{1/q_k}}\Big[\frac{\pi}{\sin(\frac{\pi}{p_k})} - \frac{p_k}{n^{p_k}}\sum_{\upsilon=0}^{\infty}\frac{(-1)^\upsilon}{(1+\upsilon p_k)n^\upsilon}\Big] \\ &< \frac{1}{n^{1/q_k}}\frac{\pi}{\sin(\frac{\pi}{p_k})} - \frac{1}{n}\Big[p_k + \sum_{\upsilon=1}^{3}\frac{(-1)^\upsilon}{\upsilon n^\upsilon} - \sum_{\upsilon=1}^{3}\frac{(-1)^\upsilon}{\upsilon(1+\upsilon p_k)n^\upsilon}\Big].\end{aligned}$$

We then find that

$$\frac{1}{1+n} = \frac{1}{n}\Big(1+\frac{1}{n}\Big)^{-1} < \frac{1}{n}\Big(1-\frac{1}{n}+\frac{1}{n^2}\Big),$$

and

$$\frac{1}{(1+n)^2} = \frac{1}{n^2}\Big(1+\frac{1}{n}\Big)^{-2} < \frac{1}{n^2}\Big(1-\frac{2}{n}+\frac{3}{n^2}\Big).$$

Substituting the above results in(3.3), by (1.6) we have (3.1). □

Lemma 3.2. *If (p_k) and (q_k) are real bounded sequences defined by $p_k^{-1} + q_k^{-1} = 1$ where $q_k > 1$ for all $k \in N, n \in N$ then*

$$w(p_k, n) < \sup_{k\geq 1}\Big\{\frac{\pi}{\sin(\pi/p_k)} - \frac{1}{2n^{1/q_k} + n^{-1/p_k}}\Big\} \quad \textit{for all } k \in N. \tag{3.4}$$

Proof: Since for $n \geq 3$,

$$\Big(\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}\Big)\Big(1+\frac{1}{2n}\Big) = \frac{1}{2} + \frac{1}{n}\Big(\frac{1}{6} - \frac{1}{24n} - \frac{1}{2n^2} - \frac{1}{4n^3}\Big) > \frac{1}{2},$$

$$\text{then } \frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3} > \frac{1}{2+n^{-1}} (n \geq 3).$$

By (3.1) and (2.6), we have for all $k \geq 1$

$$\begin{aligned} w(q_k, n) &< \sup_{k\geq 1}\Big\{\frac{\pi}{\sin(\pi/p_k)} - \frac{1}{n^{1/p_k}}\Big(\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}\Big)\Big\} \\ &< \sup_{k\geq 1}\Big\{\frac{\pi}{\sin(\pi/p_k)} - \frac{1}{2n^{1/p_k} + n^{-1/q_k}}\Big\} \quad (n \geq 3). \end{aligned} \tag{3.5}$$

Taking $\theta_p = 1 - C$, by (1.2)(see Yang and Gau [8]), we find that for all $k \geq 1$

$$w(q_k, 1) < \sup_{k\geq 1}\Big\{\frac{\pi}{\sin(\pi/p_k)} - \frac{1-C}{1}\Big\} < \sup_{k\geq 1}\Big\{\frac{\pi}{\sin(\pi/p_k)} - \frac{1}{2\times 1 + 1}\Big\}. \tag{3.6}$$

Since $C < 3/5 = 0.6$, then we have

$$\frac{1}{2\times 2^{1/p_k} + 2^{-1/q_k}} < \frac{1-C}{2^{1/p_k}}, \quad \text{for all } k \geq 1$$

and

$$w(q_k,1) < \sup_{k\geq 1}\Big\{\frac{\pi}{\sin(\pi/p_k)} - \frac{1-C}{2^{1/p_k}}\Big\} < \sup_{k\geq 1}\Big\{\frac{\pi}{\sin(\pi/p_k)} - \frac{1}{2\times 2^{1/p_k}+2^{-1/q_k}}\Big\}.$$

It follows that for $n = 1, 2$, (3.4) also holds. Then (3.4) is valid for any $n \in N$.

Theorem 3.1. *If $a_n, b_n \geq 0, (p_k)$ and (q_k) are real bounded sequences defined by $\frac{1}{p_k} + \frac{1}{q_k} = 1$, where $p_k > 1$ for all $k \in N$, and $0 < \sum_{n=1}^{\infty} a_n^{p_k} < \infty, 0 < \sum_{n=1}^{\infty} b_n^{q_k} < \infty$. Then*

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} &< \alpha \sup_{k\geq 1}\Big\{\sum_{n=1}^{\infty}\Big[\frac{\pi}{\sin(\pi/p_k)} - \frac{1}{2n^{1/p_k}+n^{-1/q_k}}\Big]a_n^{p_k}\Big\}^{\frac{1}{p_k}} \\ &\quad\times\Big\{\sum_{n=1}^{\infty}\Big[\frac{\pi}{\sin(\pi/p_k)} - \frac{1}{2n^{1/q_k}+n^{-1/p_k}}\Big]b_n^{q_k}\Big\}^{\frac{1}{q_k}} \qquad (3.7)\end{aligned}$$

and we also have

$$\begin{aligned}\sum_{m=1}^{\infty}\Big(\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}\Big)^{p_k} &< \alpha \sup_{k\geq 1}\Big[\frac{\pi}{\sin(\pi/p_k)}\Big]^{p_k-1} \\ &\quad\times\sum_{n=1}^{\infty}\Big[\frac{\pi}{\sin(\pi/p_k)} - \frac{1}{2n^{1/p_k}+n^{-1/q_k}}\Big]a_n^{p_k}. \qquad (3.8)\end{aligned}$$

Proof. By generalized Holder's inequality (2.2), we have

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[\frac{1}{(m+n)^{1/p_k}}\Big(\frac{m}{n}\Big)^{1/p_k q_k}a_m\Big]\Big[\frac{1}{(m+n)^{1/q_k}}\Big(\frac{n}{m}\Big)^{1/p_k q_k}b_n\Big] \\ &\leq \alpha\sup_{k\geq 1}\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m+n}\Big(\frac{m}{n}\Big)^{\frac{1}{q_k}}a_m^{p_k}\Big\}^{\frac{1}{p_k}}\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m+n}\Big(\frac{n}{m}\Big)^{\frac{1}{p_k}}b_n^{q_k}\Big\}^{\frac{1}{q_k}} \\ &= \alpha\sup_{k\geq 1}\Big\{\sum_{n=1}^{\infty}\Big[\sum_{m=1}^{\infty}\frac{1}{m+n}\Big(\frac{n}{m}\Big)^{1/q_k}\Big]a_n^{p_k}\Big\}^{1/p_k} \\ &\quad\times\Big\{\sum_{n=1}^{\infty}\Big[\sum_{m=1}^{\infty}\frac{1}{m+n}\Big(\frac{n}{m}\Big)^{1/p_k}\Big]b_n^{q_k}\Big\}^{1/q_k} \\ &= \alpha\sup_{k\geq 1}\Big\{\sum_{n=1}^{\infty}w(q_k,n)a_n^{p_k}\Big\}^{1/p_k}\Big\{\sum_{n=1}^{\infty}w(p_k,n)b_n^{q_k}\Big\}^{1/q_k}.\end{aligned}$$

Hence by (3.4), inequality (3.7) holds.
Since by (3.4), $w(p_k,n) < \frac{\pi}{\sin(\pi/p_k)}$, then by Holder's inequality (2.2), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} &= \sum_{n=1}^{\infty}\Big[\frac{a_n}{(m+n)^{1/p_k}}\Big(\frac{n}{m}\Big)^{1/p_kq_k}\Big]\Big[\frac{1}{(m+n)^{1/q_k}}\Big(\frac{m}{n}\Big)^{1/p_kq_k}\Big]\\ &\le \alpha\sup_{k\ge1}\Big\{\sum_{n=1}^{\infty}\Big[\frac{1}{(m+n)}\Big(\frac{n}{m}\Big)^{1/q_k}\Big]a_n^{p_k}\Big\}^{1/p_k}\Big\{\sum_{n=1}^{\infty}\Big[\frac{1}{(m+n)}\Big(\frac{m}{n}\Big)^{1/p_k}\Big\}^{1/q_k}\\ &= \alpha\sup_{k\ge1}\Big\{\sum_{n=1}^{\infty}\Big[\frac{1}{(m+n)}\Big(\frac{n}{m}\Big)^{1/q_k}\Big]a_n^{p_k}\Big\}^{1/p_k}\{w(p_k,n)\}^{1/q_k}\\ &< \alpha\sup_{k\ge1}\Big\{\sum_{n=1}^{\infty}\Big[\frac{1}{(m+n)}\Big(\frac{n}{m}\Big)^{1/q_k}\Big]a_n^{p_k}\Big\}^{1/p_k}\Big\{\frac{\pi}{\sin(\pi/p_k)}\Big\}^{1/q_k}\end{aligned}$$

By (3.4), we find

$$\begin{aligned}\sum_{m=1}^{\infty}\Big(\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}\Big)^{p_k} &< \alpha\sup_{k\ge1}\Big[\frac{\pi}{\sin(\pi/p_k)}\Big]^{p_k/q_k}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+n)}\frac{n}{m}\Big)^{1/q_k}a_n^{p_k}\\ &= \alpha\sup_{k\ge1}\Big[\frac{\pi}{\sin(\pi/p_k)}\Big]^{p_k-1}\sum_{n=1}^{\infty}\Big[\sum_{m=1}^{\infty}\frac{1}{(m+n)}\Big(\frac{n}{m}\Big)^{1/q_k}\Big]a_n^{p_k}\\ &= \alpha\sup_{k\ge1}\Big[\frac{\pi}{\sin(\pi/p_k)}\Big]^{p_k-1}\sum_{n=1}^{\infty}w(q_k,n)a_n^{p_k}\\ &= \alpha\sup_{k\ge1}\Big[\frac{\pi}{\sin(\pi/p_k)}\Big]^{p_k-1}\sum_{n=1}^{\infty}\Big[\frac{\pi}{\sin(\pi/p_k)}-\frac{1}{2n^{1/p_k}+n^{-1/q_k}}\Big]a_n^{p_k}\end{aligned}$$

This proves (3.8). □

4. SOME REVERSE TYPE INEQUALITIES

In this section we have generalized the reverse inequalities of Hardy-Hilbert type inequalities in $l(p)$ space.

Lemma 4.1 (Yang [6]). *Define the weight function $w(n)$ as*

$$w(n) = \Big(n+\frac{1}{2}\Big)\sum_{k=0}^{\infty}\frac{1}{(k+n+1)^2}, \quad n\in N_0(=N\cup\{0\}). \tag{4.1}$$

then we have

$$1-\frac{1}{4(n+1)^2} < w(n) < 1-\frac{1}{6(n+1)(2n+1)} \quad (n\in N_0). \tag{4.2}$$

Theorem 4.2. *If $a_n, b_n \geq 0$, (p_k) and (q_k) are real sequences defined by $\frac{1}{p_k} + \frac{1}{q_k} = 1$, where $0 < p_k < 1$ for all $k \in N$, and $0 < \sum_{n=0}^{\infty} \frac{a_n^{p_k}}{2n+1} < \infty, 0 < \sum_{n=1}^{\infty} \frac{b_n^{q_k}}{2n+1} < \infty$. Then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^2} \geq 2\alpha \inf_{k\geq 1}\Big\{\sum_{n=0}^{\infty}\Big[1-\frac{1}{4(n+1)^2}\Big]\frac{a_n^{p_k}}{2n+1}\Big\}^{1/p_k} \times\Big\{\sum_{n=0}^{\infty}\Big[1-\frac{1}{4(n+1)^2}\Big]\frac{b_n^{q_k}}{2n+1}\Big\}^{1/q_k} \qquad (4.3)$$

where $\alpha = (\sup p_k + \sup(1-p_k))^{-1}$.

Proof. By the reverse Holder's inequality(2.4) and (4.1), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^2}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Big[\frac{a_m}{(m+n+1)^{2/p_k}}\Big]\Big[\frac{b_n}{(m+n+1)^{2/q_k}}\Big]\\
&\geq (\sup p_k + \sup(1-p_k))^{-1}\inf_{k\geq 1}\Big\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m^{p_k}}{(m+n+1)^2}\Big\}^{\frac{1}{p_k}}\Big\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{b_n^{q_k}}{(m+n+1)^2}\Big\}^{\frac{1}{q_k}}\\
&=\alpha\inf_{k\geq 1}\Big\{\sum_{n=0}^{\infty}\Big[\sum_{m=0}^{\infty}\frac{m+1/2}{(m+n+1)^2}\Big]\frac{2a_m^{p_k}}{2m+1}\Big\}^{\frac{1}{p_k}}\Big\{\sum_{n=0}^{\infty}\Big[\sum_{m=0}^{\infty}\frac{n+1/2}{(m+n+1)^2}\Big]\frac{2b_n^{q_k}}{2n+1}\Big\}^{\frac{1}{q_k}}\\
&=2\alpha\inf_{k\geq 1}\Big\{\sum_{m=0}^{\infty}w(m)\frac{a_m^{p_k}}{2m+1}\Big\}^{\frac{1}{p_k}}\Big\{\sum_{n=0}^{\infty}w(n)\frac{b_n^{q_k}}{2n+1}\Big\}^{\frac{1}{q_k}} \qquad (4.4)
\end{aligned}$$

where $\alpha = (\sup p_k + \sup(1-p_k))^{-1}$. Since $0 < p_k < 1$ and $q_k < 0$ for all $k \geq 1$, by (4.2), it follows that (4.3) is valid.

Theorem 4.3. *If $a_n \geq 0, (p_k)$ and (q_k) are real sequences defined by $\frac{1}{p_k} + \frac{1}{q_k} = 1$, where $0 < p_k < 1$ for all $k \in N$, and $0 < \sum_{n=0}^{\infty} \frac{a_n^{p_k}}{2n+1} < \infty$. Then*

$$\sum_{n=0}^{\infty}\Big(n+\frac{1}{2}\Big)^{p_k-1}\Big[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^2}\Big]^{p_k} > 2\alpha\inf_{k\geq 1}\sum_{n=0}^{\infty}\Big[1-\frac{1}{4(n+1)^2}\Big]\frac{a_n^{p_k}}{2n+1} \qquad (4.5)$$

where $\alpha = (\sup p_k + \sup(1-p_k))^{-1}$.

□

Proof. By the reverse Holder's inequality(2.4), (4.1) and (4.2), we have $w(n) < 1$ and for all $k \geq 1$

$$\begin{aligned}
\Big[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^2}\Big]^{p_k} &= \Big\{\sum_{m=0}^{\infty}\Big[\frac{a_m}{(m+n+1)^{2/p_k}}\Big]\Big[\frac{1}{(m+n+1)^{2/q_k}}\Big]\Big\}^{p_k} \\
&\geq \alpha \inf_{k\geq 1}\Big\{\sum_{m=0}^{\infty}\frac{a_m^{p_k}}{(m+n+1)^2}\Big\}\Big\{\sum_{m=0}^{\infty}\Big[\frac{1}{(m+n+1)^2}\Big\}^{p_k-1} \\
&= \alpha \inf_{k\geq 1}\Big\{\sum_{m=0}^{\infty}\frac{a_m^{p_k}}{(m+n+1)^2}\Big\}\Big\{w(n)\Big(n+\frac{1}{2}\Big)^{-1}\Big\}^{p_k-1} \\
&> \alpha \inf_{k\geq 1}\Big[\Big(n+\frac{1}{2}\Big)^{1-p_k}\sum_{m=0}^{\infty}\frac{a_m^{p_k}}{(m+n+1)^2}\Big]. \qquad (4.6)
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{n=0}^{\infty}\Big(n+\frac{1}{2}\Big)^{p_k-1}\Big[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^2}\Big]^{p_k} &> \alpha \inf_{k\geq 1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m^{p_k}}{(m+n+1)^2} \\
&= \alpha \inf_{k\geq 1}\sum_{m=0}^{\infty}\Big[\sum_{n=0}^{\infty}\frac{m+\frac{1}{2}}{(m+n+1)^2}\Big]\frac{2a_m^{p_k}}{2m+1} \\
&= 2\alpha \inf_{k\geq 1}\sum_{m=0}^{\infty} w(m)\frac{a_m^{p_k}}{2m+1} \qquad (4.7)
\end{aligned}$$

by (4.2), we have (4.5). □

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