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WELL-POSEDNESS FOR AN ABSTRACT SEMILINEAR VOLTERRA INTEGRO-FRACTIONAL-DIFFERENTIAL PROBLEM

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ABSTRACT. We consider an abstract second order semilinear integrodifferential equation involving fractional time derivatives of order between 0 and 2. Well-posedness is established under appropriate conditions on the initial data and the nonlinearities. These conditions which depend on the order of the fractional derivatives determine the exact underlying space.

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1. INTRODUCTION

Of concern is the following problem

$$\begin{cases} u''(t) = Au(t) + f(t) \ + \int_0^t g\left(t, s, u(s), D^{\beta_1}u(s), \dots, D^{\beta_n}u(s)\right) ds, t > 0 \\ u(0) = u_0 \in X, \ u'(0) = u_1 \in X \end{cases}$$

where $0 < \beta_i \leq 2$, $i = 1, \dots, n$. Here the prime denotes time differentiation and $D^{\beta_i}, i = 1, \dots, n$ denotes fractional time differentiation (in the sense of Riemann-Liouville or Caputo). The operator A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \geq 0$ of bounded linear operators in the Banach space X, f and g are nonlinear functions from $\mathbf{R}^+$ to X and $\mathbf{R}^+ \times \mathbf{R}^+ \times X \times \cdots \times X$ to X, respectively, u_0 and u_1 are given initial data in X.

This problem has been extensively studied in case $\beta_1 = \cdots = \beta_n = 0$ or 1 (see [1-5,7,8] and references therein, to cite but a few). Well-posedness has been proved using fixed point theorems and the theory of strongly continuous cosine families in Banach spaces developed in [15,16]. Several results on classical solutions and mild solutions have been proved under different conditions on the nonlinearities and the initial data. In case $\beta_1 = \beta_2 = \cdots = \beta_n = 1$, the natural underlying space where to look for mild solutions is the space of continuously differentiable functions. In [10] the present author (with Kirane and Medved) discussed the case where $0 < \beta_1, \beta_2, \dots, \beta_n < 1$. A suitable space for the initial data and a suitable underlying space for the existence of mild solutions have been found.

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Here we consider the case where $0 < \beta_i < 2$, $i = 1, \ldots, n$. This situation is more delicate as derivatives of order more that 1 will require functions (in the underlying space) to be more regular than C^1. On the other hand, when trying to prove existence and uniqueness of classical solutions one is tempted to require more regularity on the cosine family. It seems that the former difficulty cannot be avoided whereas we will see that we can "throw" part of the requirement of smoothness of the cosine family on the considered nonlinearities. This observation is valid for mild solutions as well.

For simplicity we will discuss the problem

$$\begin{cases} u''(t) = Au(t) + f(t) + \int_0^t g(t, s, u(s), D^\gamma u(s))\, ds, t > 0 \\ u(0) = u_0 \in X, \ u'(0) = u_1 \in X \end{cases} \tag{1.1}$$

with $1 < \gamma < 2$.

The next section of this paper contains some notation and preliminary results needed in our proofs. Section 3 treats existence and uniqueness of classical solutions in appropriate spaces when the fractional derivatives are of Caputo type. Section 4 is devoted to the case of Riemann-Liouville fractional derivatives.

2. PRELIMINARIES

As announced earlier, in this section we present some notation, assumptions and results needed in our proofs later. We start by the definitions of the different types of fractional derivatives used in this paper.

Definition 2.1. The integral

$$(I_{a+}^\alpha h)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{h(t)dt}{(x-t)^{1-\alpha}}, \ x > a$$

is called the Riemann-Liouville fractional integral of h of order $\alpha > 0$ when the right side exists.

Here Γ is the usual Gamma function

$$\Gamma(z) := \int_0^\infty e^{-s} s^{z-1} ds, \ z > 0.$$

In particular, for $\nu > -1$, we find

$$I^\alpha x^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)} x^{\nu+\alpha}. \tag{2.1}$$

Definition 2.2. The (left hand) Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by

$$(D_a^\alpha h)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{h(t)dt}{(x-t)^{\alpha-n+1}}, \ x > a, \ n = [\alpha] + 1$$

whenever the right side is pointwise defined.

Definition 2.3. The fractional derivative of order $\alpha > 0$ in the sense of Caputo is given by

$$\left({}^C D_a^\alpha h\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{h^{(n)}(t)dt}{(x-t)^{\alpha-n+1}},\ x > a,\ n = [\alpha] + 1.$$

In particular

$$(D_a^\beta h)(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x \frac{h(t)dt}{(x-t)^\beta}, \quad x > a,\ 0 < \beta < 1$$

$$\left({}^C D_a^\beta h\right)(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{h'(t)dt}{(x-t)^\beta}, \quad x > a,\ 0 < \beta < 1 \tag{2.2}$$

and

$$(D_a^\gamma h)(x) = \frac{1}{\Gamma(2-\gamma)} \left(\frac{d}{dx}\right)^2 \int_a^x \frac{h(t)dt}{(x-t)^{\gamma-1}}, \quad x > a,\ 1 < \gamma < 2$$

$$\left({}^C D_a^\gamma h\right)(x) = \frac{1}{\Gamma(2-\gamma)} \int_a^x \frac{h''(t)dt}{(x-t)^{\gamma-1}}, \quad x > a,\ 1 < \gamma < 2. \tag{2.3}$$

Remark 2.4. The fractional integral of order α is well defined on L^p, $p \geq 1$ (see [14]). Further, from Definition 2.2, it is clear that the Riemann-Liouville fractional derivative is defined for any function $h \in L^p$, $p \geq 1$ for which $k_{n-\alpha} * h$ is n times differentiable (where $k_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$ and $*$ is the incomplete convolution). In fact, as domain of $D_0^\alpha = D^\alpha$ we can take

$$D(D^\alpha) = \{h \in L^p(0,T) : k_{n-\alpha} * h \in W^{n,p}(0,T)\}$$

where

$$W^{n,p}(0,T) := \left\{ u : \exists \varphi \in L^p(0,T) : u(t) = \sum_{k=0}^{n-1} c_k \frac{t^k}{k!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \varphi(s) ds \right\}.$$

In particular, we know that the absolutely continuous functions ($p = n = 1$) are differentiable almost everywhere and therefore the Riemann-Liouville fractional derivative of order $0 < \beta < 1$ exists a.e. In this case (for an absolutely continuous function) the derivative is summable (see Lemma 2.2 [13]) and the fractional derivative in the sense of Caputo of order $0 < \beta < 1$ exists as well. Moreover, we have the following general relationship between the two types of fractional derivatives

$$\begin{aligned}(D_a^\alpha h)(x) &= \sum_{k=0}^{n-1} \frac{h^{(k)}(a)}{\Gamma(1+k-\alpha)} (x-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{h^{(n)}(t)dt}{(x-t)^{\alpha-n+1}} \\ &= \sum_{k=0}^{n-1} \frac{h^{(k)}(a)}{\Gamma(1+k-\alpha)} (x-a)^{k-\alpha} + \left({}^C D_a^\alpha h\right)(x),\ x > a\end{aligned} \tag{2.4}$$

in case $h(x) \in AC^n[a,b] := \{f : [a,b] \to \mathbf{R} \text{ and } (D^{n-1} f)(x) \in AC[a,b]\}$.

See [6,9,11-14] for more on fractional derivatives.

We will assume that

(H1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbf{R}$, of bounded linear operators in the Banach space X.

The associated sine family $S(t)$, $t \in \mathbf{R}$ is defined by

$$S(t)x := \int_0^t C(s)x ds,\ t \in \mathbf{R},\ x \in X.$$

It is known (see [16,17]) that there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$|C(t)| \leq Me^{\omega|t|},\ t \in \mathbf{R} \text{ and } |S(t) - S(t_0)| \leq M \left| \int_{t_0}^t e^{\omega|s|} ds \right|,\ t, t_0 \in \mathbf{R}. \tag{2.5}$$

If we define

$$E := \{x \in X : C(t)x \text{ is once continuously differentiable on } \mathbf{R}\}$$

then we have

Lemma 2.5 (see [16,17]). *Assume that (H1) is satisfied. Then*

(i) $S(t)X \subset E$, $t \in \mathbf{R}$,

(ii) $S(t)E \subset D(A)$, $t \in \mathbf{R}$,

(iii) $\frac{d}{dt}C(t)x = AS(t)x$, $x \in E$, $t \in \mathbf{R}$,

(iv) $\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax$, $x \in D(A)$, $t \in \mathbf{R}$.

Lemma 2.6 (see [16,17]). *Suppose that (H1) holds, $v : \mathbf{R} \to X$ a continuously differentiable function and $q(t) = \int_0^t S(t-s)v(s)ds$. Then, $q(t) \in D(A)$, $q'(t) = \int_0^t C(t-s)v(s)ds$ and $q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(0) = Aq(t) + v(t)$.*

Definition 2.7. A function $u \in C^2(I, X)$ $(I = [0, T])$ is called a classical solution of (1.1) with Caputo fractional derivative if $u(\cdot) \in D(A)$, satisfies the equation in (1.1) and the initial conditions are verified.

Definition 2.8. A continuous solution u, such that ${}^C D^\gamma u(t)$ ($D^\gamma u(t)$ in case of Riemann-Liouville fractional derivative) exists and is continuous, of the integro-differential equation

$$u(t) = C(t)u_0 + S(t)u_1 + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s) \int_0^s g(s, \tau, u(\tau), {}^C D^\gamma u(\tau)) d\tau\, ds \tag{2.6}$$

is called mild solution of problem (1.1).

It follows from [15] that, in case of continuity of the nonlinearities, solutions of (1.1) are solutions of the more general problem in Definition 2.8.

3. WELL POSEDNESS IN CASE OF CAPUTO DERIVATIVE

In this section we consider fractional derivatives in the sense of Caputo and prove existence and uniqueness of a classical solution. We will use the space

$$C_\gamma^C([0,T]) := \left\{ v \in C([0,T]) : {}^C D^\gamma v \in C([0,T]) \right\}$$

equipped with the norm $\|v\|_\gamma := \|v\|_C + \left\| {}^C D^\gamma v \right\|_C$ where $\|.\|_C$ is the uniform norm in $C([0,T])$.

Let X_A be the space $D(A)$ endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$. We need the following assumptions on f and g

(H2) $f : R^+ \to X$ is continuously differentiable.

(H3) $g : R^+ \times R^+ \times X_A \times X \to X$ is continuous and continuously differentiable with respect to its first variable.

(H4) g and g_1 (the derivative of g with respect to its first variable) are Lipschitz continuous, that is

$$\|g(t,s,x_1,y_1) - g(t,s,x_2,y_2)\| \le A_g \left(\|x_1 - x_2\| + \|y_1 - y_2\| \right),$$

$$\|g_1(t,s,x_1,y_1) - g_1(t,s,x_2,y_2)\| \le A_{g_1} \left(\|x_1 - x_2\| + \|y_1 - y_2\| \right),$$

for some positive constants A_g and A_{g_1}.

Lemma 3.1. *For $1 < \gamma < 2$, $u_0 \in D(A)$ and $u_1 \in E$ we have*

$${}^C D^\gamma C(t) u_0 = I^{2-\gamma} C(t) A u_0$$

and

$${}^C D^\gamma S(t) u_1 = I^{2-\gamma} A S(t) u_1.$$

Proof. These relations follow immediately from Definition 2.3 (or relation (2.3)) and Lemma 2.5. Indeed, as $u_0 \in D(A)$ and $u_1 \in E$, it is clear that

$$\begin{aligned} {}^C D^\gamma C(t) u_0 &= \frac{1}{\Gamma(2-\gamma)} \int_0^t (t-s)^{1-\gamma} \frac{d^2}{dt^2} C(s) u_0 ds \\ &= \frac{1}{\Gamma(2-\gamma)} \int_0^t (t-s)^{1-\gamma} C(s) A u_0 ds = I^{2-\gamma} C(t) A u_0 \end{aligned}$$

and

$$\begin{aligned} {}^C D^\gamma S(t) u_1 &= \frac{1}{\Gamma(2-\gamma)} \int_0^t (t-s)^{1-\gamma} \frac{d}{dt} C(s) u_1 ds \\ &= \frac{1}{\Gamma(2-\gamma)} \int_0^t (t-s)^{1-\gamma} A S(s) u_1 ds = I^{2-\gamma} A S(t) u_1. \end{aligned}$$

□

Lemma 3.2. *Suppose that (H1) holds, $S(t)$ is the sine family associated with the cosine family $C(t)$ and $w : R \to X$ a continuously differentiable function. Then, for $1 < \gamma < 2$ and $t \in [0,T]$ we have*

$$\left({}^C D^\gamma\right)\left(\int_0^\tau S(\tau - s)w(s)ds\right)(t) = \int_0^t I^{2-\gamma}C(t-s)w'(s)ds + I^{2-\gamma}C(t)w(0).$$

Proof. From Definition 2.3 (or relation (2.3)) and Lemma 2.6 applied to w (which is a continuously differentiable function), we easily derive that

$$\left({}^C D^\gamma\right)\left(\int_0^\tau S(\tau - s)w(s)ds\right)(t) = \frac{1}{\Gamma(2-\gamma)}\int_0^t (t-\tau)^{1-\gamma}\left(\int_0^\tau S(\tau - z)w(z)dz\right)'' d\tau$$
$$= \frac{1}{\Gamma(2-\gamma)}\int_0^t (t-\tau)^{1-\gamma}\left[\int_0^\tau C(\tau - z)w'(z)dz + C(\tau)w(0)\right]d\tau$$
$$= \frac{1}{\Gamma(2-\gamma)}\int_0^t (t-\tau)^{1-\gamma}\int_0^\tau C(\tau - z)w'(z)dzd\tau + \frac{1}{\Gamma(2-\gamma)}\int_0^t (t-\tau)^{1-\gamma}C(\tau)w(0)d\tau$$
$$= \frac{1}{\Gamma(2-\gamma)}\int_0^t dz\int_z^t (t-\tau)^{1-\gamma}C(\tau - z)w'(z)d\tau + I^{2-\gamma}C(t)w(0)$$
$$= \frac{1}{\Gamma(2-\gamma)}\int_0^t dz\int_0^{t-z} (t-z-\sigma)^{1-\gamma}C(\sigma)w'(z)d\sigma + I^{2-\gamma}C(t)w(0)$$
$$= \int_0^t I^{2-\gamma}C(t-s)w'(s)ds + I^{2-\gamma}C(t)w(0),\ t \in [0,T].$$

□

Now we state and prove the main result of this section.

Theorem 3.3. *Assume that (H1)-(H4) hold. If $u_0 \in D(A)$ and $u_1 \in E$ then there exists $T > 0$ and a unique function $u : [0,T] \to X$, $u \in C([0,T]; X_A) \cap C^2([0,T]; X)$ which satisfies* (1.1) *with Caputo fractional derivative ${}^C D^\gamma u$.*

Proof. Let us define, for $t \in [0,T]$ and $u \in C_\gamma^C([0,T])$,

$$(Ku)(t) := C(t)u_0 + S(t)u_1 + \int_0^t S(t-s)f(s)ds$$
$$+ \int_0^t S(t-s)\int_0^s g\left(s,\tau,u(\tau),{}^C D^\gamma u(\tau)\right)d\tau\, ds. \tag{3.1}$$

Notice that $C(t)u_0 \in D(A)$ because $u_0 \in D(A)$ and we have $AC(t)u_0 = C(t)Au_0$. From the facts that $u_1 \in E$ and $S(t)E \subset D(A)$ ((ii) Lemma 2.5) it is clear that $S(t)u_1 \in D(A)$. Moreover, it follows from Lemma 2.5, (H2) and (H3) that both integral terms in (3.1) are in $D(A)$. Therefore, $Ku \in C([0,T]; D(A))$. In addition to that we have from Lemma 2.6,

$$
\begin{aligned}
(AKu)(t) &= C(t)Au_0 + AS(t)u_1 + \int_0^t C(t-s)f'(s)ds + C(t)f(0) - f(t) \\
&+ \int_0^t C(t-s)\left[g\left(s,s,u(s),{}^C D^\gamma u(s)\right) + \int_0^s g_1\left(s,\tau,u(\tau),{}^C D^\gamma u(\tau)\right)\right] d\tau\, ds \\
&- \int_0^t g\left(t,\tau,u(\tau),{}^C D^\gamma u(\tau)\right) d\tau, \quad t\in[0,T]. \qquad (3.2)
\end{aligned}
$$

Next, applying the operator ${}^C D^\gamma$ to both sides of (3.1), we get in virtue of Lemma 3.1

$$
\begin{aligned}
{}^C D^\gamma (Ku)(t) &= I^{2-\gamma}C(t)Au_0 + I^{2-\gamma}AS(t)u_1 + {}^C D^\gamma \int_0^t S(t-s)f(s)ds \\
&+ {}^C D^\gamma \int_0^t S(t-s)\int_0^s g\left(s,\tau,u(\tau),{}^C D^\gamma u(\tau)\right) d\tau\, ds, \quad t\in[0,T]. \qquad (3.3)
\end{aligned}
$$

Regarding the third term in the right hand side of (3.3), we have by Lemma 3.2, for $t\in[0,T]$

$$
\left({}^C D^\gamma\right)\left(\int_0^\tau S(\tau-s)f(s)ds\right)(t) = \int_0^t I^{2-\gamma}C(t-s)f'(s)ds + I^{2-\gamma}C(t)f(0)
$$

and

$$
\begin{aligned}
&{}^C D^\gamma \int_0^t S(t-s)\int_0^s g\left(s,\tau,u(\tau),{}^C D^\gamma u(\tau)\right) d\tau\, ds \\
&\quad = \int_0^t I^{2-\gamma}C(t-s)\Big[g\left(s,s,u(s),{}^C D^\gamma u(s)\right) \\
&\quad\quad + \int_0^s g_1\left(s,\tau,u(\tau),{}^C D^\gamma u(\tau)\right) d\tau\Big] ds
\end{aligned}
$$

because of the continuous differentiability of $f(s)$ and

$$
\int_0^s g\left(s,\tau,u(\tau),{}^C D^\gamma u(\tau)\right) d\tau.
$$

Therefore $Ku\in C_\gamma^C([0,T])$ and K maps $C_\gamma^C([0,T])$ into $C_\gamma^C([0,T])$.

Now we want to prove that K is a contraction on C_γ^C endowed with the metric

$$
\rho(u,v) := \sup_{0\le t\le T}\left(\|u(t)-v(t)\| + \|A(u(t)-v(t))\| + \left\|{}^C D^\gamma u(t) - {}^C D^\gamma v(t)\right\|\right).
$$

For u,v in C_γ^C, we have from (3.1),

$$
\begin{aligned}
\|(Ku)(t)-(Kv)(t)\| \le \int_0^t \left(\int_0^{t-s} Me^{\omega\tau}d\tau\right) A_g \int_0^s &[\|u(\tau)-v(\tau)\|_A \\
&+ \left\|{}^C D^\gamma u(\tau) - {}^C D^\gamma v(\tau)\right\|] d\tau\, ds.
\end{aligned}
$$

Thus,

$$
\|(Ku)(t)-(Kv)(t)\| \le \frac{A_g T^2 M}{2}\left(\int_0^T e^{\omega\tau}d\tau\right)\rho(u,v). \qquad (3.4)
$$

From (3.2) we infer

$$\begin{aligned}
&\|(AKu)(t)-(AKv)(t)\| \\
&\leq MA_g \int_0^t e^{\omega(t-s)} \left(\|u(s)-v(s)\|_A + \left\|{}^C D^\gamma u(s) - {}^C D^\gamma v(s)\right\|\right) ds \\
&\quad + MA_{g_1} \int_0^t e^{\omega(t-s)} \int_0^s \left(\|u(\tau)-v(\tau)\|_A + \left\|{}^C D^\gamma u(s) - {}^C D^\gamma v(s)\right\|\right) d\tau\, ds \\
&\quad + A_g \int_0^t \left(\|u(s)-v(s)\|_A + \left\|{}^C D^\gamma u(s) - {}^C D^\gamma v(s)\right\|\right) ds.
\end{aligned}$$

Hence

$$\begin{aligned}
\|(AKu)(t)-(AKv)(t)\| &\leq MA_g \left(\int_0^T e^{\omega(T-s)} ds\right) \rho(u,v) \\
&\quad + MA_{g_1} \frac{T^2}{2} \left(\int_0^T e^{\omega(T-s)} ds\right) \rho(u,v) + A_g T \rho(u,v) \\
&\leq \left[(A_g + A_{g_1} T^2) M \left(\int_0^T e^{\omega(T-s)} ds\right) + A_g T\right] \rho(u,v). \qquad (3.5)
\end{aligned}$$

Finally, the relations (3.3) and Lemma 3.2 imply that

$$\begin{aligned}
&\left\|{}^C D^\gamma (Ku)(t) - {}^C D^\gamma (Kv)(t)\right\| \\
&\leq A_g \int_0^t I^{2-\gamma} |C(t-s)| \left[\|u(s)-v(s)\|_A + \left\|{}^C D^\gamma u(s) - {}^C D^\gamma v(s)\right\|\right] ds \\
&\quad + A_{g_1} \int_0^t I^{2-\gamma} |C(t-s)| \int_0^s \left[\|u(\tau)-v(\tau)\|_A + \left\|{}^C D^\gamma u(\tau) - {}^C D^\gamma v(\tau)\right\|\right] d\tau\, ds
\end{aligned}$$

or

$$\begin{aligned}
&\left\|{}^C D^\gamma (Ku)(t) - {}^C D^\gamma (Kv)(t)\right\| \\
&\leq \left[\frac{MA_g T^{2-\gamma}}{\Gamma(3-\gamma)} \int_0^T e^{\omega\tau} d\tau + \frac{MA_{g_1} T^{3-\gamma}}{\Gamma(3-\gamma)} \int_0^T e^{\omega\tau} d\tau\right] \rho(u,v) \\
&\leq \frac{MT^{2-\gamma}}{\Gamma(3-\gamma)} (A_g + A_{g_1} T) \left(\int_0^T e^{\omega\tau} d\tau\right) \rho(u,v). \qquad (3.6)
\end{aligned}$$

Choosing T sufficiently small in (3.4)-(3.6) we infer that K is a contraction on $C_\gamma^C([0,T])$ and hence there exists a unique mild solution $u \in C_\gamma^C([0,T])$. Furthermore, it is clear that $u \in C^2([0,T];X)$ and satisfies problem (1.1) with Caputo fractional derivative because

$$\begin{aligned}
u(t) = C(t)u_0 + S(t)u_1 + \int_0^t S(t-s) f(s) ds \\
+ \int_0^t S(t-s) \int_0^s g\left(s, \tau, u(\tau), {}^C D^\gamma u(\tau)\right) d\tau\, ds
\end{aligned}$$

is twice continuously differentiable and by Lemma 2.6

$$\begin{aligned} u''(t) &= AC(t)u_0 + AS(t)u_1 + A\int_0^t S(t-s)f(s)ds + f(t) \\ &\quad + A\int_0^t S(t-s)\int_0^s g\left(s,\tau,u(\tau),{}^C D^\gamma u(\tau)\right) d\tau\, ds \\ &\quad + \int_0^t g\left(t,\tau,u(\tau),{}^C D^\gamma u(\tau)\right) d\tau \\ &= Au(t) + f(t) + \int_0^t g\left(t,\tau,u(\tau),{}^C D^\gamma u(\tau)\right) d\tau. \end{aligned}$$

□

4. THE CASE OF R-L DERIVATIVE

Here we are concerned about solutions of problem (2.6) with Riemann-Liouville fractional derivative. The admissible space for the initial data would be "larger" than the one for classical solutions ($(u_0, u_1) \in D(A) \times E$) as expected. However, initial data here must be "more regular" than the one existing in the literature with a first order derivative in the nonlinearity ($(u_0, u_1) \in E \times X$). For $0 < \lambda < 2$, we define

$$E_\lambda := \{x \in X : D^\lambda C(t)x \text{ is continuous on } \mathbf{R}^+\}$$

and

$$C_\gamma^{RL}([0,T]) := \{v \in C([0,T]) : D^\gamma v \in C([0,T])\}$$

equipped with the norm $\|v\|_\gamma := \|v\|_C + \|D^\gamma v\|_C$ where $\|.\|_C$ is the uniform norm in $C([0,T])$.

We will need the following assumptions

(H5) $f : \mathbf{R}^+ \to X$ is a continuous function such that $D^{\gamma-1}f \in C([0,T])$, that is $f \in C_{\gamma-1}^{RL}([0,T])$

(H6) $g : \mathbf{R}^+ \times \mathbf{R}^+ \times X \times X \to X$ is continuous and continuously differentiable with respect to its first variable.

We will need the result below which is the analogue to Lemma 3.2 for the case of Riemann-Liouville derivatives.

Lemma 4.1. *Suppose that (H1) holds, $S(t)$ is the sine family associated with the cosine family $C(t)$ and $w : \mathbf{R} \to X$ a function in $C_{\gamma-1}^{RL}([0,T])$. Then, for $1 < \gamma < 2$, we have*

$$(D^\gamma)\left(\int_0^\tau S(\tau-s)w(s)ds\right)(t) = \int_0^t C(\sigma)D^{\gamma-1}w(t-\sigma)d\sigma,\ t \in [0,T].$$

Proof. From Definition 2.2 and Lemma 2.6 applied to w which is a continuous function such that $D^{\gamma-1}w \in C([0,T])$, we easily derive that

$$(D^{\gamma})\left(\int_0^{\tau} S(\tau-s)w(s)ds\right)(t) = \frac{1}{\Gamma(2-\gamma)}\left(\frac{d}{dt}\right)^2 \int_0^t (t-\tau)^{1-\gamma}\int_0^{\tau} S(\tau-s)w(s)dsd\tau$$

$$= \frac{1}{\Gamma(2-\gamma)}\left(\frac{d}{dt}\right)^2 \int_0^t \int_s^t (t-\tau)^{1-\gamma}S(\tau-s)w(s)d\tau\, ds$$

$$= \frac{1}{\Gamma(2-\gamma)}\left(\frac{d}{dt}\right)^2 \int_0^t ds \int_0^{t-s} (t-s-\sigma)^{1-\gamma}S(\sigma)w(s)d\sigma$$

$$= \frac{1}{\Gamma(2-\gamma)}\frac{d}{dt}\int_0^t \frac{d}{dt}\left(\int_0^{t-s} \sigma^{1-\gamma}S(t-s-\sigma)w(s)d\sigma\right)ds$$

$$= \frac{1}{\Gamma(2-\gamma)}\frac{d}{dt}\int_0^t \left(\int_0^{t-s} \sigma^{1-\gamma}C(t-s-\sigma)w(s)d\sigma\right)ds$$

$$= \frac{1}{\Gamma(2-\gamma)}\frac{d}{dt}\int_0^t \left(\int_0^{s} (s-\sigma)^{1-\gamma}C(\sigma)w(t-s)d\sigma\right)ds$$

$$= \frac{1}{\Gamma(2-\gamma)}\frac{d}{dt}\int_0^t C(\sigma)\int_0^{t-\sigma} (t-\sigma-v)^{1-\gamma}w(v)dvd\sigma$$

$$= \int_0^t C(\sigma)D^{\gamma-1}w(t-\sigma)d\sigma + C(t)I^{2-\gamma}w(0)$$

$$= \int_0^t C(\sigma)D^{\gamma-1}w(t-\sigma)d\sigma$$

□

The following result can be found in [14] (Remark 2.4 (2.68)).

Lemma 4.2. *Let $\alpha > 0$, $\beta < 0$ and $\varphi \in L^1(a,b)$ be such that $I^{n+\beta}\varphi \in AC^n([a,b])$. Then*

$$I^{\alpha}_{a+}I^{\beta}_{a+}\varphi = I^{\alpha+\beta}_{a+}\varphi - \sum_{k=0}^{n-1}\frac{\varphi^{(n-k-1)}_{n+\beta}(a)}{\Gamma(\alpha-k)}(x-a)^{\alpha-k-1}$$

where $\varphi_{n+\beta}(x) = I^{n+\beta}_{a+}\varphi(x)$ and $n = [-\beta]+1$

Here is the main result of this section

Theorem 4.3. *Assume that (H1) and (H4)-(H6) hold. If $(u_0, u_1) \in E_{\gamma} \times E_{\gamma-1}$, then there exists $T > 0$ and a unique solution $u \in C^{RL}_{\gamma}([0,T])$ of problem (2.6) with Riemann-Liouville fractional derivative.*

Proof. For $t \in [0,T]$, consider the operator

$$(Ku)(t) := C(t)u_0 + S(t)u_1 + \int_0^t S(t-s)f(s)ds$$
$$+ \int_0^t S(t-s)\int_0^s g(s,\tau,u(\tau),D^{\gamma}u(\tau))d\tau\, ds. \quad (4.1)$$

If $u \in C_\gamma^{RL}([0,T])$ then clearly $Ku \in C([0,T];X)$. Moreover, applying the operator D^γ to both sides of (4.1) and using Lemma 4.2 to the third and fourth terms in the right hand side of (4.1), we find

$$D^\gamma(Ku)(t) = D^\gamma C(t)u_0 + D^{\gamma-1}C(t)u_1 + \int_0^t C(t-s)D^{\gamma-1}f(s)ds$$
$$+ \int_0^t C(t-s)D^{\gamma-1}\left(\int_0^s g(s,\tau,u(\tau),D^\gamma u(\tau))d\tau\right)ds. \tag{4.2}$$

Here we have used the relation

$$\begin{aligned} D^\gamma S(t)u_1 &= \frac{1}{\Gamma(2-\gamma)}\left(\frac{d}{dt}\right)^2 \int_0^t (t-\tau)^{1-\gamma}S(\tau)u_1 d\tau \\ &= \frac{1}{\Gamma(2-\gamma)}\left(\frac{d}{dt}\right)^2 \int_0^t \tau^{1-\gamma}S(t-\tau)u_1 d\tau \\ &= \frac{1}{\Gamma(2-\gamma)}\frac{d}{dt}\int_0^t \tau^{1-\gamma}C(t-\tau)u_1 d\tau + \frac{1}{\Gamma(2-\gamma)}\frac{d}{dt}t^{1-\gamma}S(0)u_1 \\ &= D^{\gamma-1}C(t)u_1. \end{aligned}$$

For the last term in (4.2), we can write with the help of (2.2)

$$\begin{aligned} &D^{\gamma-1}\left(\int_0^s g(s,\tau,u(\tau),D^\gamma u(\tau))d\tau\right) \\ &= \frac{1}{\Gamma(2-\gamma)}\int_0^s (s-\sigma)^{1-\gamma}\frac{d}{d\sigma}\left(\int_0^\sigma g(\sigma,\tau,u(\tau),D^\gamma u(\tau))d\tau\right)d\sigma \\ &= \frac{1}{\Gamma(2-\gamma)}\int_0^s (s-\sigma)^{1-\gamma}\left[\int_0^\sigma g_1(\sigma,\tau,u(\tau),D^\gamma u(\tau))d\tau + g(\sigma,\sigma,u(\sigma),D^\gamma u(\sigma))\right]d\sigma \\ &= I^{2-\gamma}\int_0^s g_1(s,\tau,u(\tau),D^\gamma u(\tau))d\tau + I^{2-\gamma}g(s,s,u(s),D^\gamma u(s)). \end{aligned}$$

Therefore

$$D^\gamma(Ku)(t) = D^\gamma C(t)u_0 + D^{\gamma-1}C(t)u_1 + \int_0^t C(t-s)D^{\gamma-1}f(s)ds$$
$$+ \int_0^t C(t-s)\left[I^{2-\gamma}\int_0^s g_1(s,\tau,u(\tau),D^\gamma u(\tau))d\tau + I^{2-\gamma}g(s,s,u(s),D^\gamma u(s))\right]ds. \tag{4.3}$$

This relation (4.3), together with our assumptions on the initial data, f, g, g_1 and the properties of the cosine family, shows that K maps $C_\gamma^{RL}([0,T])$ to $C_\gamma^{RL}([0,T])$.

Furthermore, for $u, v \in C_\gamma^{RL}([0,T])$, we see that

$$\begin{aligned} &\|(Ku)(t)-(Kv)(t)\| \\ &\le MA_g\int_0^t\left(\int_0^{t-s}e^{\omega\tau}d\tau\right)\int_0^s\left(\|u(\tau)-v(\tau)\| + \|D^\gamma u(\tau)-D^\gamma v(\tau)\|\right)d\tau\, ds \\ &\le MA_g\left(\int_0^T e^{\omega\tau}d\tau\right)\frac{T^2}{2}\|u(t)-v(t)\|_\gamma. \end{aligned}$$

Finally,

$$
\begin{aligned}
&\|(D^\gamma Ku)(t)-(D^\gamma Kv)(t)\| \\
&\leq A_{g_1}\int_0^t |C(t-s)|\, I^{2-\gamma}\int_0^s \left(\|u(\tau)-v(\tau)\|+\|D^\gamma u(\tau)-D^\gamma v(\tau)\|\right) d\tau\, ds \\
&\quad + A_g\int_0^t |C(t-s)|\, I^{2-\gamma}\left(\|u(s)-v(s)\|+\|D^\gamma u(s)-D^\gamma v(s)\|\right) ds \\
&\leq \frac{MT^{2-\gamma}}{\Gamma(3-\gamma)}\left(A_g+A_{g_1}T\right)\left(\int_0^T e^{\omega(T-s)}ds\right)\|u(t)-v(t)\|_\gamma .
\end{aligned}
$$

We conclude that, for T sufficiently small, K is a contraction on the complete metric space $C_\gamma^{RL}([0,T])$ and hence there exists a unique mild solution to (1.1). □

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