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EXISTENCE OF SOLUTIONS FOR FRACTIONAL SEMILIEAR EVOLUTION BOUNDARY VALUE PROBLEM

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ABSTRACT. In this paper we prove the existence of solutions for fractional evolution equations with boundary conditions in Banach spaces. The results are obtained by using fractional calculus and the fixed point theorems.

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1. INTRODUCTION

Recently fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electro-dynamics of complex medium, polymer rheology, etc. (see [5,11,14,15,16]) involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. Theory of fractional differential equations has been extensively studied by Delbosco and Rodino [6] and Lakshmikantham et al [19-21]. In [3,8,17] the authors have proved the existence of solutions of abstract differential equations by using semigroup theory and fixed point theorem. Many partial fractional differential equations can be expressed as fractional differential equations in some Banach Spaces [10].

The following equation

$$\begin{cases} {}^cD^q_{0+}x(t) = f(t, x(t)), \quad 0 < t < 1, \\ x(0) + x'(0) = 0, x(1) + x'(1) = 0, \end{cases}$$

where ${}^cD^q_{0+}$ denotes the Caputo fractional derivative with $1 < q \leq 2$ was studied by S. Zhang [29] and the existence of positive solutions was obtained using classical fixed point theorems.

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Recently G. M. Mophou et al [23], were studied the Cauchy problem with nonlocal conditions

$$\begin{cases} D^q x(t) = Ax(t) + t^n f(t, x(t), Bx(t)), \quad t \in [0,T], n \in Z^+ \\ x(0) + g(x) = x_0, \end{cases}$$

in general Banach space X with $0 < q < 1$ and A is the infinitesimal generator of a C_0 semigroup of bounded linear operator. By means of the Krasnoselskii's theorem, existence of solutions was also obtained.

Subsequently several authors have investigated the problem for different types of nonlinear differential equations and integrodifferential equations including functional differential equations in Banach spaces.

In [28], the author studied both the local and global existence of solutions to the equation

$$\begin{cases} D_t^\alpha x(t) = f(t, x(t)), \quad t \in [0,T] \\ x^k(t_0) = x_0(k), \quad k = 0, 1, 2, \ldots, n-1 \end{cases}$$

in a finite dimensional space. The results are obtained via construction and the contraction mapping principle. Very recently N'Guerekata [12,13] discussed the existence of solutions of fractional abstract differential equations with nonlocal initial condition.

This paper is organized as follows. In Section 2 we introduce some preliminary results needed in the following sections. In Section 3 we present an existence result for Boundary value problem for fractional semilinear evolution equations in Banach spaces by using the fractional calculus and Sadovskii fixed point theorem.

2. PRELIMINARIES

We need some basic definitions[18,24,26] and properties of fractional calculus which are used in this paper.

Definition 2.1 ([8])**.** A real function f(t) is said to be in the space $C_\alpha, \alpha \in R$ if there exists a real number $p > \alpha$, such that $f(t) = t^p g(t)$, where $g \in C[0, \infty)$ and it is said to be in the space C_α^miff $f^{(m)} \in C_\alpha, m \in N$.

Definition 2.2. The fractional (arbitrary) order integral of the function $f \in L^1([a,b], R_+)$ of order $q \in R_+$ is defined by

$$I_a^q f(t) = \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^q f(t) = f(t) * \varphi_q(t)$, where $\varphi_q(t) = \frac{t^{q-1}}{\Gamma(q)}$ for $t > 0$, and $\varphi_q(t) = 0$ for $t \leq 0$, and $\varphi_q(t) \to \delta(t)$ as $q \to 0$, where δ is the delta function.

Definition 2.3. The Riemann-Liouville fractional integral operator of order $q > 0$, of a function $f \in C_\mu, \mu \geq -1$ is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \quad q > 0, \quad t > 0$$
$$I^0 f(x) = f(x),$$

Definition 2.4. The Caputo's derivative of fractional order q for a function f(t) is defined by

$$({}^c D^q f)(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q-n+1}} ds, \ n-1 < q < n, n = [q]+1,$$

where [q] denotes the integer part of real number q.

Definition 2.5. For a function f given on the interval $[a, b]$, the Riemann-Liouville fractional derivative of order q for a function f, is defined by

$$(D^q_{a+} f)(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(s)}{(t-s)^{q-n+1}} ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$, where Γ is the gamma function.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall use a modified fractional differential operator ${}^c D^q$ proposed by M. Caputo in his work on the theory of viscoelasticity.

Definition 2.6. Let X be a subset of Banach space. An operator $T : X \to X$ is called condensing if for any bounded subset $E \subset X$, with $\mu(E) \neq 0$, we get $\mu(T(E)) < \mu(E)$, when $\mu(E)$ denotes the measure of noncompactness of the set E.

Let $(X, \|\cdot\|)$ be a Banach space, and $I := [0, T], T > 0$, a compact interval of the real line R. Denote by $C = C([0, T], X)$ the Banach space of all continuous functions $[0, T] \to X$ endowed with the topology of uniform convergence (the norm in this space will be denoted by $\|\cdot\|_C$). For basic facts about fractional derivative and fractional calculus one can refer to the books [16,18,24].

3. MAIN RESULTS

Now consider the first order boundary value problem for semilinear fractional evolution equation

$$\begin{cases} {}^c D^q x(t) = A(t)x(t) + f(t, x(t), Bx(t)), & t \in I = [0, T], \\ ax(0) + bx(T) = c, \end{cases} \tag{1}$$

where ${}^cD^q$ is the Caputo fractional derivative and $A(t)$ is a bounded linear operator and $0 < q < 1$, $Bx(t) = \int_0^t K(t,s)x(s)ds$, K belongs to $C(D, R^+)$, the set of all positive continuous functions defined on D, with $D := \{(t,s) \in R^2 : 0 \le s \le t \le T\}$ and

$$B^* = \sup_{t\in[0,T]} \int_0^t K(t,s)ds < \infty,$$

$f : I \times X \times X \to X$, is continuous and a, b, c are real constants with $a + b \ne 0$. The fractional derivative ${}^cD^q$ is understood here in the Caputo sense, (i.e):

$${}^cD^q g(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} g'(s)ds,$$

for a continuous function $g : R^+ \to R$ provided that the right hand side is pointwise defined on R^+. The equation(1) is then equivalent to

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A(s)x(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), Bx(s)ds$$
$$-\frac{1}{a+b}\left[\frac{b}{\Gamma(q)} \int_0^T (T-s)^{q-1} f(s, x(s), Bx(s)ds - c\right], \quad \forall t \in [0,T]. \qquad (2)$$

See [18] for more details. We need the following assumptions to prove the existence of solutions of equation (1).

(**HA**). $A(t)$ is a bounded linear operator on X for each $t \in I$. The function $t \to A(t)$ is continuous in the uniform operator topology. We set

$$M = \max_{t\in[0,T]} \|A(t)\|.$$

(**HB**). $f : I \times X \times X \to X$ is continuous and there exist a constants $L_1 > 0, L_2 > 0$ such that

$$\|f(t,x,u) - f(t,y,v)\| \le L_1\|x-y\| + L_2\|u-v\| \quad \text{for all } x, y, u, v \in X.$$

For brevity let us take $\gamma = \frac{T^q}{\Gamma(q+1)}$ and $N = \max_{t\in I} \|f(t,0)\|$.
(**HC**). $f : I \times X \to X$ is continuous and there exists a function $\mu \in L^1(I, R^+)$ such that

$$\|f(t,x,y)\| \le \mu(t), \forall t \in I \quad x, y \in X.$$

Theorem 3.1. *Under assumptions* $(HA), (HB)$ *and if* $\gamma(M + L) < \frac{1}{2}$, *then Eq.(1) has a unique solution.*

Proof. Let $C = C([0,T]] : X)$. Define the mapping $F : C \to C$ by

$$(Fx)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A(s)x(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), Bx(s))ds$$
$$-\frac{1}{a+b}\left[\frac{b}{\Gamma(q)} \int_0^T (T-s)^{q-1} f(s, x(s), Bx(s))ds - c\right]$$

and we have to show that F has a fixed point. This fixed point is then a solution of the equation (1). Let $M = \max_{t\in I}\|A(t)\|$(see [25]). Then we can show that $FB_r \subset B_r$, where $B_r := \{x \in Z : \|x\| \le r\}$. From the assumptions, we have to choose $r \ge 2(N\gamma(1+\frac{|b|}{|a+b|}))$, then

$$\begin{aligned}
\|(Fx)(t)\| &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|A(s)\|\|x(s)\|ds \\
&+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,x(s),Bx(s))\|ds \\
&+ \frac{|b|}{|a+b|}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}\|f(s,x(s),Bx(s))\|ds\right] + \frac{|c|}{|a+b|} \\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|A(s)\|\|x(s)\|ds \\
&+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(\|f(s,x(s),Bx(s)) - f(s,0,0)\| + \|f(s,0,0)\|)ds \\
&+ \frac{|b|}{|a+b|}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}(\|f(s,x(s),Bx(s)) - f(s,0,0)\| + \|f(s,0,0)\|)ds\right] \\
&+ \frac{|c|}{|a+b|} \\
&\le Mr\frac{T^q}{\Gamma(q+1)} + ((L_1 + L_2B^*)r + N)\frac{T^q}{\Gamma(q+1)} \\
&+ \frac{|b|}{|a+b|}((L_1 + L_2B^*))r + N)\frac{T^q}{\Gamma(q+1)} + \frac{|c|}{|a+b|} \\
&\le Mr\gamma + ((L_1 + L_2B^*)r + N)\gamma\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|} \\
&\le r,
\end{aligned}$$

by the choice of L, a, b, c and r. Thus, F maps B_r into itself. Now, for $x, y \in Z$, we have

$$\begin{aligned}
\|(Fx)(t) - (Fy)(t)\| &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|A(s)(x(s) - y(s))\|ds \\
&+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,x(s),Bx(s)) - f(s,y(s),By(s))\|ds \\
&+ \frac{|b|}{|a+b|}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}\|f(s,x(s),Bx(s)) - f(s,y(s),By(s))\|ds\right] \\
&\le ((L_1 + L_2B^*) + M)\|x-y\|_C\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
&+ ((L_1 + L_2B^*) + M)\|x-y\|_C\left(\frac{|b|}{|a+b|}\right)\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}ds\right]
\end{aligned}$$

$$\leq \left[\frac{((L_1 + L_2 B^*) + M)T^q \left(1 + \frac{|b|}{|a+b|}\right)}{\Gamma(q+1)} \right] \|x - y\|_C.$$

Thus

$$\|Fx - Fy\|_C \leq \Omega_{a,b,c,L_1,L_2,M,T,q} \|x - y\|_C,$$

where $\Omega_{a,b,c,L_1,L_2,M,T,q} = \left[((L_1 + L_2 B^*) + M)\gamma \left(1 + \frac{|b|}{|a+b|}\right)\right]$. And since $\Omega_{a,b,c,L_1,L_2,M,T,q} < 1$, F is a contraction mapping and therefore there exists a unique fixed point $x \in B_r$ such that $Fx(t) = x(t)$. Any fixed point of F is the solution of the problem (1). □

Now recall this well known tool.

Theorem 3.2 (**Sadovskii**). . *Let B be a closed, convex and bounded subset of a Banach space X. If $F : B \to B$ is a condensing map, then F has a fixed point in B.*

Theorem 3.3. *Assume $(HA)-(HC)$ hold. If $M\gamma < 1$. Then the fractional evolution Eq.* (1) *with boundary condition has at least one solution on I provided that*

$$M\gamma + \mu(t)\gamma + \frac{|b|}{|a+b|} [\mu(t)\gamma] + \frac{|c|}{r|a+b|} < 1. \tag{3}$$

Proof. For each positive number r, let

$$B_r : \{x \in Z : \|x\| \leq r, 0 \leq t \leq T\},$$

then B_r, for each r, is a bounded, closed, convex set in Z. So F is well defined on B_r. We claim that there exists a positive number r such that $FB_r \subseteq B_r$. If it is not true, then for each positive number r, there is a function $x_r \in B_r$ but $Fx_r \notin B_r$, that is, $\|Fx_r(t)\| > r$ for some $t \in [0, T]$. However, on the other hand, we have

$$\begin{aligned} r &\leq \|(Fx_r)(t)\| \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|A(s)\| \|x_r(s)\| ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s, x_r(s), Bx_r)\| ds \\ &\quad + \frac{|b|}{|a+b|} \left[\frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} \|f(s, x_r(s), Bx_r)\| ds \right] + \frac{|c|}{|a+b|} \\ &\leq Mr\gamma + \mu(t) r\gamma + \frac{|b|}{|a+b|} [\mu(t) r\gamma] + \frac{|c|}{|a+b|}. \end{aligned}$$

Dividing both sides by r, we get

$$M\gamma + \mu(t)\gamma + \frac{|b|}{|a+b|} [\mu(t)\gamma] + \frac{|c|}{r|a+b|} \geq 1.$$

This contradicts expression (3). Hence $FB_r \subseteq B_r$, for some positive number r.
Now define the operators F_1 and F_2 on B_r as

$$F_1(x)(t) := \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A(s) x(s) ds$$

$$- \frac{1}{a+b}\left[\frac{b}{\Gamma(q)}\int_0^T (T-s)^{q-1} f(s,x(s),Bx(s))ds - c\right]$$

and

$$F_2(x)(t) := \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x(s),Bx(s))ds.$$

We will show that F_1 is a contraction mapping and F_2 is a compact operator. We have to prove that F_1 is a contraction, we take $x, y \in B_r$, then for each $t \in [0,T]$, we have

$$\begin{aligned}
\|(F_1(x)(t) - F_1(y)(t)\| &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|A(s)(x(s)-y(s))\|ds \\
&+ \frac{|b|}{|a+b|}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}\|f(s,x(s),Bx(s)) - f(s,y(s),By(s))\|ds\right] \\
&\le M\|x-y\|_C \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
&+ (L_1 + L_2 B^*)\|x-y\|_C \left(\frac{|b|}{|a+b|}\right)\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}ds\right] \\
&\le \left[\frac{(M + (L_1+L_2B^*)\left(\frac{|b|}{|a+b|}\right) T^q}{\Gamma(q+1)}\right]\|x-y\|_C.
\end{aligned}$$

Thus

$$\|(F_1x)(t) - (F_1y)(t)\| \le \Omega_{a,b,L,M,T,q}\|x-y\|_C,$$

where $\Omega_{a,b,L,M,T,q} = \left[M + (L_1 + L_2B^*)\left(\frac{|b|}{|a+b|}\right)\gamma\right]$. And since $\Omega_{a,b,L,M,T,q} < 1$, F_1 is a contraction mapping. We have to prove that F_2 is compact. Since x is continuous, then $(F_2x)(t)$ is continuous in view of (HB). Let us now note that F_2 is uniformly bounded on B_r. This follows from the inequality

$$\|(F_2x)(t)\| \le \frac{T^q\|\mu\|_{L^1}}{\Gamma(q+1)}.$$

Now let us prove that $(F_2x)(t)$ is equicontinuous. Let $t_1, t_2 \in I$, $t_1 < t_2$ and $x \in B_r$. Using the fact that f is bounded on the compact set $I \times B_r \times B(B_r)$ (thus $\sup_{(t,s)\in I\times B_r}\|f(s,x(s),Bx(s)\| := c_0 < \infty$), we will get

$$\begin{aligned}
\|F_2x(t_2) - F_2x(t_1)\| &= \|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1} - (t_1-s)^{q-1}]f(s,x(s),Bx(s))ds \\
&+ \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s))ds\| \\
&\le \frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1} - (t_1-s)^{q-1}]\|f(s,x(s),Bx(s))\|ds \\
&+ \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\|f(s,x(s),Bx(s))\|ds
\end{aligned}$$

$$\leq \frac{c_0}{\Gamma(q)} \| \int_0^{t_1} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] ds + \frac{c_0}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds$$
$$\leq \frac{c_0}{\Gamma(q+1)} [(t_2 - t_1)^q + t_1^q - t_2^q] + \frac{c_0}{\Gamma(q+1)} (t_2 - t_1)^q$$
$$\leq \frac{c_0}{\Gamma(q+1)} |2(t_2 - t_1)^q + t_1^q - t_2^q|,$$

which does not depend on x. So $F_2(B_r)$ is relatively compact. As $t_2 \to t_1$, the right hand side of the above inequality tends to zero. By the Arzela-Ascoli Theorem, F_2 is a compact operator. These arguments show that $F = F_1 + F_2$ is a condensing mapping on B_r, and by the Sadovskii fixed point theorem there exists a fixed point for F on B_r, which is a solution of the problem (1). The proof is complete. □

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