

**EVENTUAL PRACTICAL STABILITY AND CONE VALUED
LYAPUNOV FUNCTIONS FOR DIFFERENTIAL EQUATIONS
WITH “MAXIMA”**

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ABSTRACT. A special type of practical stability for differential equations with “maxima” is introduced. The definitions incorporate two different measures and a scalar product on a cone. The application of a dot product allows us to use scalar comparison of ordinary differential equations for investigation of stability properties of the solutions. At the same time, the fixed vector, involved in the definition, plays a role of a weight of the components of the solution. Some sufficient conditions for eventual d-practical stability in terms of two measures of nonlinear differential equations with “maxima” are obtained. The proofs are based on the Razumikhin method and cone valued Lyapunov functions. An example illustrates the practical application of the proven results and the advantage of the introduced type of stability.

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1. INTRODUCTION

It is well-known ([5]) that stability and even asymptotic stability themselves are neither necessary nor sufficient to ensure practical stability. The desired state of a system may be mathematically unstable; however, the system may oscillate sufficiently close to the desired state, so that its performance is deemed acceptable. The practical stability is neither weaker nor stronger than the usual stability; an equilibrium can be stable in the usual sense, but not practically stable, and vice versa. For example, an aircraft may oscillate around a mathematically unstable path, yet its performance may be acceptable. Practical stability is, in a sense, a uniform boundedness of the solution relative to the initial conditions, but the bound must be sufficiently small ([1], [3], [10], [11], [12], [13]).

In the last few decades, great attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many

problems in control theory correspond to the maximal deviation of the regulated quantity ([7]). Such kind of problems could be adequately modeled by differential equations that contain the maxima operator. A. D. Mishkis also points out the necessity to study differential equations with “maxima” in his survey [6]. Note that various conditions for stability of differential equations with “maxima” are obtained by D. D. Bainov et al. ([8], [9]).

In this paper, *eventual d-practical stability* of differential equations with “maxima” is defined. The definition combines the ideas of two different measures and a dot product. The dot product, introduced into the definition, plays a role of weights of components of the solution. The Razhumikhin method and cone valued Lypunov functions are used to obtain sufficient conditions for the introduced stability of solutions of differential equations with “maxima.” Comparison results for scalar differential equations are applied. An appropriate example illustrates the application of the obtained sufficient conditions and the main advantages of the considered type of stability.

2. MAIN RESULTS

Let $r > 0$ be a given number, $t_0 \in \mathbb{R}_+$, and $\phi \in C([-r, 0], \mathbb{R}^n)$.

Consider the initial value problem for nonlinear differential equations with “maxima” (DEM),

$$\begin{aligned} x' &= F(t, x(t), \max_{s \in [t-r, t]} x(s)), \quad \text{for } t \geq t_0, \\ x(t) &= \phi(t - t_0), \quad \text{for } t \in [t_0 - r, t_0], \end{aligned} \quad (2.1)$$

where $x \in \mathbb{R}^n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $F = (F_1, F_2, \dots, F_n)$.

Note that for $x : [t - r, t] \rightarrow \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$, we denote

$$\max_{s \in I(t)} x(s) = \left(\max_{s \in [t-r, t]} x_1(s), \max_{s \in [t-r, t]} x_2(s), \dots, \max_{s \in [t-r, t]} x_n(s) \right).$$

We denote by $x(t; t_0, \phi)$ the solution of DEM (2.1). In our further investigations, we will assume that solution $x(t; t_0, \phi)$ is defined on $[t_0 - r, \infty)$ for any initial function $\phi \in C([-r, 0], \mathbb{R}^n)$.

Let $x, y \in \mathbb{R}^n$. Denote by $(x \bullet y)$ the dot product of both vectors x and y . Let $\mathcal{K} \subset \mathbb{R}^n$ be a cone. Consider the set

$$\mathcal{K}^* = \{\varphi \in \mathbb{R}^n : (\varphi \bullet x) \geq 0 \text{ for any } x \in \mathcal{K}\}.$$

We assume that \mathcal{K}^* is a cone. Introduce the following sets

$$\begin{aligned} K &= \{a \in C(\mathbb{R}_+, \mathbb{R}_+) : a(s) \text{ is strictly increasing and } a(0) = 0\}, \\ \mathcal{G} &= \{h \in C([-r, \infty) \times \mathbb{R}^n, \mathcal{K}) : \inf_{x \in \mathbb{R}^n} h(t, x) = 0 \text{ for each } t \geq -r\}. \end{aligned} \quad (2.2)$$

Let ρ be positive constant, $\varphi_0 \in \mathcal{K}^*$, $h \in \mathcal{G}$. Define

$$\tilde{\mathcal{S}}(h, \rho, \varphi_0) = \{(t, x) \in [0, \infty) \times \mathbb{R}^n : (\varphi_0 \bullet h(t, x)) < \rho\}.$$

We introduce the following set (H) of conditions:

H1. The function $F \in C[\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $F(t, 0, 0) \equiv 0$.

H2. The vector $\varphi_0 \in \mathcal{K}^*$ and the functions $h_0, h \in \mathcal{G}$.

In our further investigations, we will use the following comparison scalar ordinary differential equation,

$$u' = g(t, u), \tag{2.3}$$

where $u \in \mathbb{R}$, $g(t, 0) \equiv 0$.

Definition 1. Let $h, h_0 \in \mathcal{G}$. The function h is *eventually φ_0 -stronger* than h_0 if, for a couple (λ, A) such that $0 < \lambda < A$, the inequality $(\varphi_0 \bullet h_0(t, x)) < \lambda$, for some $(t, x) \in [-r, \infty) \times \mathbb{R}^n$, implies $(\varphi_0 \bullet h(t, x)) < A$.

Remark 1. Let $x = (x_1, x_2, \dots, x_n)$, φ_0 be a vector with components greater than or equal to 1, and $h_0(t, x) = \sum_{k=1}^n |x_k|$, and $h(t, x) = \sqrt{\sum_{k=1}^n x_k^2}$. The function h is eventually φ_0 -stronger than h_0 .

We add the following condition to the set (H) of conditions above:

H3. The functions $h_0, h \in \mathcal{G}$ are such that h is eventually φ_0 -stronger than h_0 .

H4. The functions $h_0, h \in \mathcal{G}$ are such that, for $(t, x) \in \tilde{\mathcal{S}}(h_0, \rho_0, \varphi_0)$, the inequality $(\varphi_0 \bullet h(t, x)) \leq Q((\varphi_0 \bullet h_0(t, x)))$ holds, where $Q \in K$ is such that $Q(s) \leq s$ and $\rho_0 > 0$ is a constant.

Definition 2. We will say that the function $V(t, x) : \Omega \times \mathbb{R}^n \rightarrow \mathcal{K}$, $\Omega \subset \mathbb{R}_+$, $V = (V_1, V_2, \dots, V_n)$, belongs to the class \mathcal{L} if:

1. $V(t, x) \in C^1(\Omega \times \mathbb{R}^n, \mathcal{K})$;
2. There exist constants $M_i > 0$, $i = 1, 2, \dots, n$, such that $|V_i(t, x) - V_i(t, y)| \leq M_i \|x - y\|$ for any $t \in \Omega$ and $x, y \in \mathbb{R}^n$.

Let the function $V \in \mathcal{L}$, $V = (V_1, V_2, \dots, V_n)$ and $\phi \in C([-r, 0], \mathbb{R}^n)$. We define a derivative $\mathcal{D}V(t, x)$ of the function V along DEM (2.1) by the equalities

$$\mathcal{D}V_i(t, \phi(0)) = \frac{\partial V_i(t, \phi(0))}{\partial t} + \sum_{j=1}^n \frac{\partial V_i(t, \phi(0))}{\partial x_j} F_j(t, \phi(0), \sup_{s \in [-r, 0]} \phi(s)), \tag{2.4}$$

$$i = 1, 2, \dots, n,$$

where $\mathcal{D}V(t, x) = (\mathcal{D}V_1(t, x), \mathcal{D}V_2(t, x), \dots, \mathcal{D}V_n(t, x))$.

We will introduce the definition of a new type of eventual practical stability for differential equations with “maxima” based on the ideas of stability in terms of two measures ([4]) and the dot product.

Definition 3. Let $\varphi_0 \in \mathcal{K}^*$, $h, h_0 \in \mathcal{G}$, and λ and A be constants such that $0 < \lambda < A$. The system of differential equations with “maxima” (2.1) is said to be

(S1) *d-eventually practically stable* in terms of measures h_0 and h with a vector φ_0 if, for any given couple (λ, A) such that $0 < \lambda < A$, there exists $\tau(\lambda, A) > 0$ such that, for some $t_0 \geq \tau(\lambda, A)$ and $\phi \in C([-r, 0], \mathbb{R}^n)$ such that $\sup_{s \in [-r, 0]} (\varphi_0 \bullet h(t_0 + s, \phi(s))) < \lambda$, the inequality $(\varphi_0 \bullet h(t, x(t; t_0, \phi))) < A$ holds for $t \geq t_0$, where $x(t; t_0, \phi)$ is a solution of DEM (2.1);

(S2) *uniformly d-eventually practically stable* in terms of measures h_0 and h with a vector φ_0 if, for given couple (λ, A) such that $0 < \lambda < A$, there exists $\tau(\lambda, A) > 0$, such that for any $t_0 \geq \tau(\lambda, A)$ and $\phi \in C([-r, 0], \mathbb{R}^n)$ such that $\sup_{s \in [-r, 0]} (\varphi_0 \bullet h(t_0 + s, \phi(s))) < \lambda$, the inequality $(\varphi_0 \bullet h(t, x(t; t_0, \phi))) < A$ holds for $t \geq t_0$.

Note that the vector φ_0 , introduced in Definition 3, plays the role of a weight of components of the solution. In the case $\varphi_0 = (1, 1, \dots, 1)$, $h(t, x) = h_0(t, x) \equiv (|x_1|, |x_2|, \dots, |x_n|)$, where $x = (x_1, x_2, \dots, x_n)$, the *d-eventually practical stability* defined above in terms of two measures reduces to eventually practical stability.

Note that in the case $r = 0$, the above definitions reduce to definitions for practical stability of ordinary differential equations given in the book [3].

In what follows, we will use following comparison result:

Lemma 1. *Let the following conditions be fulfilled:*

1. *The conditions H1, H2 are satisfied.*
2. *The function $V(t, x) : [t_0, T] \times \mathbb{R}^n \rightarrow \mathcal{K}$, $V \in \mathcal{L}$ is such that for any function $\psi \in C([-r, 0], \mathbb{R}^n)$ and any number $t \in [t_0, T]$ such that $(\varphi_0 \bullet V(t, \psi(0))) \geq (\varphi_0 \bullet V(t + s, \psi(s)))$ for $s \in [-r, 0)$ the inequality*

$$\left(\varphi_0 \bullet \mathcal{D}_{(2.1)} V(t, \psi(0)) \right) \leq g(t, (\varphi_0 \bullet V(t, \psi(0))))$$

holds, where $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ and $g(t, 0) \equiv 0$.

3. *The function $x(t) = x(t; t_0, \varphi)$ is a solution of DEM (2.1), that is defined for $t \in [t_0 - r, T]$.*
4. *The function $u^*(t) = u^*(t; t_0, u_0)$ is the maximal solution of (2.3) with initial condition $u^*(t_0) = u_0$, that is defined for $t \in [t_0, T]$.*

Then the inequality $\max_{s \in [-r, 0]} (\varphi_0 \bullet V(t_0 + s, \varphi(s))) \leq u_0$ implies the validity of the inequality $(\varphi_0 \bullet V(t, x(t))) \leq u^(t)$ for $t \in [t_0, T]$.*

Proof. Let $u_n(t)$ be the maximal solution of the initial value problem

$$u' = g(t, u) + \frac{1}{n}, \quad t \in [t_0, T], \quad u(t_0) = u_0 + \frac{1}{n}, \quad (2.5)$$

where $\max_{s \in [-r, 0]} (\varphi_0 \bullet V(t_0 + s, \varphi(s))) \leq u_0$ and n is a natural number. Assume that $u_n(t)$ is defined for $t \in [t_0, T]$. Define a function $m(t) \in C([t_0, T], \mathbb{R}_+)$ by the equality

$m(t) = (\varphi_0 \bullet V(t, x(t)))$. Because of the fact that $u^*(t; t_0, u_0) = \lim_{n \rightarrow \infty} u_n(t)$, it is enough to prove that for any natural number n , the inequality

$$m(t) \leq u_n(t) \quad \text{for } t \in [t_0, T] \tag{2.6}$$

holds.

Note that for any natural number n , inequality $m(t_0) < u_n(t_0)$ holds. Assume inequality (2.6) is not true. Let n be a natural number such that there exists a point $\eta \in (t_0, T)$ such that $m(\eta) > u_n(\eta)$. Let $t_n^* = \max\{t \in [t_0, T] : m(s) < u_n(s) \text{ for } s \in [t_0, t]\}$, $t_n^* < T$. Therefore,

$$m(t_n^*) = u_n(t_n^*), \quad m(t) < u_n(t) \text{ for } t \in [t_0, t_n^*), \quad m(t) \geq u_n(t) \text{ for } t \in (t_n^*, t_n^* + \delta), \tag{2.7}$$

where $\delta > 0$ is a sufficiently small number. From inequalities (2.7) it follows that

$$m'(t_n^*) \geq u'_n(t_n^*) = g(t, u_n(t_n^*)) + \frac{1}{n} = g(t, m(t_n^*)) + \frac{1}{n}. \tag{2.8}$$

Since $g(t, u) + \frac{1}{n} > 0$ on $[t_n^* - r, t_n^*] \cap [t_0, T]$, it follows that the function $u_n(t)$ is nondecreasing on $[t_n^* - r, t_n^*] \cap [t_0, T]$. There are cases to consider:

If $t_n^* - r \geq t_0$, then $m(t_n^*) = v_n(t_n^*) \geq v_n(s) > m(s)$ for $s \in [t_n^* - r, t_n^*)$.

If $t_n^* - r < t_0$, then as above $m(t_n^*) > m(s)$ for $s \in [t_0, t_n^*)$, and $m(t_n^*) = v_n(t_n^*) \geq v_n(t_0) = u_0 + \frac{1}{n} > u_0 \geq \sup_{s \in [-r, 0]} V(t_0 + s, \phi(s)) \geq m(s)$ for $s \in [t_n^* - r, t_0)$.

Therefore, $m(t_n^*) > m(s)$ for $s \in [t_n^* - r, t_n^*)$, and according to Condition 1 of Lemma 1, using standard arguments, we get $m'(t_n^*) \leq g(t, m(t_n^*)) < g(t, m(t_n^*)) + \frac{1}{n}$, which is a contradiction to (2.8). Therefore the inequality (2.6) holds, and hence the conclusion of Lemma 1 follows. □

Remark 2. Note that the condition $(\varphi_0 \bullet V(t_0, \varphi(0))) \leq u_0$ is not enough for the validity of the conclusion of Lemma 1.

We will obtain sufficient conditions for d -eventual practical stability in terms of two measures of systems of differential equations with “maxima.” We will employ Lyapunov functions from the class \mathcal{L} . The proof is based on the Razumikhin method combined with a comparison method employing scalar ordinary differential equations.

Theorem 1. *Let the following conditions be fulfilled:*

1. *The conditions H1–H3 are satisfied.*
2. *There exists a function $V(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathcal{K}$, with $V \in \mathcal{L}$ such that*
 - (i) $b((\varphi_0 \bullet h(t, x))) \leq (\varphi_0 \bullet V(t, x)) \leq a((\varphi_0 \bullet h_0(t, x)))$, $(t, x) \in \tilde{\mathcal{S}}(h, \rho, \varphi_0)$, where $a, b \in K$;
 - (ii) *for any function $\psi \in C[[-r, 0], \mathbb{R}^n]$ and any number $t \geq 0$ such that $(\varphi_0 \bullet V(t, \psi(0))) > (\varphi_0 \bullet V(t + s, \psi(s)))$ for $s \in [-r, 0)$ and $(t, \psi(0)) \in \tilde{\mathcal{S}}(h, \rho, \varphi_0)$*

the inequality

$$\left(\varphi_0 \bullet \mathcal{D}_{(2.1)} V(t, \psi(0)) \right) \leq g(t, (\varphi_0 \bullet V(t, \psi(0))))$$

holds, where $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $g(t, 0) \equiv 0$, and $\rho > 0$ is a constant.

3. For any initial point $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ the solution of the scalar equation (2.3) exists on $[t_0, \infty)$, $t_0 \geq 0$.
4. The scalar differential equation (2.3) is eventually practically stable.

Then the system of differential equations with “maxima” (2.1) is d -eventually practically stable in terms of the measures h_0 and h with a vector φ_0 .

Proof. Let the couple (λ, A) such that $0 < \lambda < A$ be given.

Case 1. Let $A < \rho$. From Condition 4, it follows that there exist $\tau(\lambda, A) > 0$ and a point $t_0 \geq \tau(\lambda, A)$ such that $|u_0| < a(\lambda)$ implies

$$|u(t; t_0, u_0)| < b(A) \quad \text{for } t \geq t_0, \quad (2.9)$$

where $u(t; t_0, u_0)$ is a solution of the scalar differential equation (2.3) with initial condition $u(t_0) = u_0$.

Choose a function $\phi \in C([-r, 0], \mathbb{R}^n)$ such that $\sup_{s \in [-r, 0]} (\varphi_0 \bullet h_0(t_0 + s, \phi(s))) < \lambda$, where t_0 is defined above. Let $x(t) = x(t; t_0, \phi)$ be a solution of DEM (2.1) with the initial function ϕ . From assumption H3, it follows that the inequality

$$(\varphi_0 \bullet h(t, \phi(t - t_0))) < A \quad \text{for } t \in [t_0 - r, t_0] \quad (2.10)$$

holds.

We claim that

$$(\varphi_0 \bullet h(t, x(t))) < A \quad \text{for } t \geq t_0 \quad (2.11)$$

holds.

Assume the claim is not true. From the choice of the initial function ϕ and inequality (2.10), it follows there exists a point $t^* > t_0$ such that

$$\begin{aligned} (\varphi_0 \bullet h(t, x(t))) &< A, \quad \text{for } t \in [t_0 - r, t^*), \\ (\varphi_0 \bullet h(t^*, x(t^*))) &= A. \end{aligned} \quad (2.12)$$

Since $A < \rho$, the inclusion $x(t; t_0, \phi) \in \tilde{\mathcal{S}}(h, \rho, \varphi_0)$ is valid for $t \in [t_0 - r, t^*]$.

Let $u_0^* = \max_{s \in [-r, 0]} (\varphi_0 \bullet V(t_0 + s, \phi(s)))$. From Lemma 1 and Condition (ii), it follows that

$$(\varphi_0 \bullet V(t, x(t))) \leq u^*(t; t_0, u_0^*) \quad \text{for } t \in [t_0, t^*], \quad (2.13)$$

where $u^*(t; t_0, u_0^*)$ is a solution of scalar differential equation (2.3) with initial condition $u(t_0) = u_0^*$.

From Condition (i) and the choice of the initial function ϕ , for $s \in [-r, 0]$, we obtain

$$(\varphi_0 \bullet V(t_0 + s, \phi(s))) \leq a((\varphi_0 \bullet h_0(t_0 + s, \phi(s)))) < a(\lambda). \quad (2.14)$$

Inequality (2.14) proves that $|u_0^*| < a(\lambda)$, and therefore, according to inequalities (2.9) and (2.13), we get

$$(\varphi_0 \bullet V(t_0 + s, \phi(s))) \leq u^*(t; t_0, u_0) < b(A) \quad \text{for } t \in [t_0, t^*]. \quad (2.15)$$

From inequality (2.15), the choice of t^* , and Condition (i), we get

$$b(A) = b((\varphi_0 \bullet h(t^*, x(t^*)))) \leq (\varphi_0 \bullet V(t^*, x(t^*))) \leq u^*(t^*; t_0, u_0) < b(A).$$

This is a contradiction, which proves (2.11).

Case 2. Let $A \geq \rho$. We repeat the proof of Case 1, but instead of the number a , everywhere we use the number ρ . \square

Note the Condition H3 could be replaced by the Condition H4 in the sufficient condition for d -eventual practical stability:

Theorem 2. *Let the Conditions H1, H2, H4 and Conditions 2, 3, 4 of Theorem 1 be fulfilled. Then the system of differential equations with “maxima” (2.1) is d -eventually practically stable in terms of the measures h_0 and h with a vector φ_0 .*

The proof of Theorem 2 is similar to the proof of Theorem 1. In this case we consider the constant $\rho_1 = \min\{\rho, \rho_0\}$, and from the choice of the initial function ϕ , it follows that $(t_0 + s, \phi(s)) \in \tilde{\mathcal{S}}(h_0, \rho_1, \varphi_0)$ for $s \in [-r, 0]$. Condition H4 immediately shows the validity of inequality (2.10).

Theorem 3. *Let the following conditions be fulfilled:*

1. *The Conditions 1, 2, 3 and 4 of Theorem 1 are satisfied.*
2. *The scalar differential equation (2.3) is uniformly eventually practically stable.*

Then the system of differential equations with “maxima” (2.1) is uniformly d -eventually practically stable in terms of the measures h_0 and h with the vector φ_0 .

The proof of Theorem 3 is similar to the proof of Theorem 1 where we consider an arbitrary point t_0 .

Note that Condition H3 could be replaced by Condition H4 in Theorem 3:

Theorem 4. *Let the Conditions H1, H2, H4, Conditions 2, 3, 4 of Theorem 1 and Condition 2 of Theorem 2 be fulfilled. Then the system of differential equations with “maxima” (2.1) is uniformly d -eventually practically stable in terms of the measures h_0 and h with a vector φ_0 .*

In the case $\varphi_0 = (1, 1, \dots, 1)$ and $h(t, x) = h_0(t, x) \equiv (|x_1|, |x_2|, \dots, |x_n|)$, where $x = (x_1, x_2, \dots, x_n)$, the above results reduce to the following.

Theorem 5. *Let the following conditions be fulfilled:*

1. *The condition H1 is satisfied.*
2. *There exists a function $V(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathcal{K}$, with $V \in \mathcal{L}$ such that*
 - (i) *$b(\|x\|) \leq \sum_{i=1}^n V_i(t, x) \leq a(\|x\|)$, for $t \in \mathbb{R}_+$, $\|x\| < A$, where $a, b \in K$ and $a(\lambda) < b(A)$;*
 - (ii) *for any $\psi \in C([-r, 0], \mathbb{R}^n)$ and $t \geq 0$ such that $\sum_{i=1}^n V_i(t, \psi(0)) > \sum_{i=1}^n V_i(t + s, \psi(s))$ for $s \in [-r, 0)$ and $\sup_{s \in [t_0 - r, t_0]} \|\psi(s)\| < A$, the inequality*

$$\sum_{i=1}^n \mathcal{D}_{(2.1)} V_i(t, \psi(0)) \leq g(t, \sum_{i=1}^n V_i(t, \psi(0)))$$

holds, where $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $g(t, 0) \equiv 0$, and $\rho > 0$ is a constant.

3. *For any initial point $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ the solution of scalar equation (2.3) exists on $[t_0, \infty)$, $t_0 \geq 0$.*

If the scalar differential equation (2.3) is eventually practically stable, then the system of differential equations with “maxima” (2.1) is eventually practically stable. If the scalar differential equation (2.3) is uniformly eventually practically stable, then the system of differential equations with “maxima” (2.1) is uniformly eventually practically stable.

3. APPLICATIONS

Now we will illustrate our results.

Consider the following system of differential equations with “maxima,”

$$\begin{aligned} x'(t) &= y(t) \left(x^2(t) + y^2(t) \right) \sin^2 t + e^{-t} \max_{s \in [t-r, t]} x(s), \\ y'(t) &= -\frac{1}{2} x(t) \left(x^2(t) + y^2(t) \right) \sin^2 t + e^{-t} \max_{s \in [t-r, t]} y(s), \quad t \geq t_0, \end{aligned} \tag{3.1}$$

with initial conditions

$$x(t) = \phi_1(t - t_0), \quad y(t) = \phi_2(t - t_0) \quad \text{for } t \in [t_0 - r, t_0], \tag{3.2}$$

where $x, y \in \mathbb{R}$, $r > 0$ is a sufficiently small constant, and $t_0 \geq 0$.

Let $h_0(t, x, y) = (|x|, |y|)$, $h(t, x, y) = (x^2, y^2)$. Consider $V : \mathbb{R}^2 \rightarrow \mathcal{K}$, $V = (V_1, V_2)$, $V_1(x, y) = \frac{1}{2}(x + 2y)^2$, $V_2(x, y) = \frac{1}{2}(x - y)^2$, where $\mathcal{K} = \{(x, y) : x \geq 0, y \geq 0\} \subset \mathbb{R}^2$ is a cone. For the vector $\varphi_0 = (1, 2)$, then $(\varphi_0 \bullet h(t, x, y)) = x^2 + 2y^2$, $(\varphi_0 \bullet V(x, y)) = \frac{1}{2}(x + 2y)^2 + (x - y)^2 = \frac{3}{2}(x^2 + 2y^2)$ and $(\varphi_0 \bullet h_0(t, x, y)) = |x| + 2|y|$.

It is easy to check the validity of Condition (i) of Theorem 1 for functions $a(s) = \frac{3}{2}s \in K$ and $b(s) = \frac{3}{2}s^2 \in K$. Let $\psi \in C([-r, 0], \mathbb{R}^2)$, $\psi = (\psi_1, \psi_2)$ be such that

$$\begin{aligned} (\varphi_0 \bullet V(\psi_1(0), \psi_2(0))) &= \frac{3}{2}(\psi_1^2(0) + 2\psi_2^2(0)) \\ &\geq \frac{3}{2}(\psi_1^2(s) + 2\psi_2^2(s)) = (\varphi_0 \bullet V(\psi_1(s), \psi_2(s))) \quad \text{for } s \in [-r, 0]. \end{aligned} \tag{3.3}$$

Then for $i = 1, 2$, we obtain

$$\begin{aligned} \psi_i(0) \max_{s \in [t-r, t]} \psi_i(s) &\leq |\psi_i(0)| \max_{s \in [t-r, t]} \psi_i(s) = \sqrt{(\psi_i(0))^2} \sqrt{\left(\max_{s \in [t-r, t]} \psi_i(s)\right)^2} \\ &\leq \sqrt{\frac{2}{3}(\varphi_0 \bullet V(\psi_1(0), \psi_2(0)))} \sqrt{\frac{2}{3}(\varphi_0 \bullet V(\psi_1(s), \psi_2(s)))} \\ &\leq \frac{2}{3}(\varphi_0 \bullet V(\psi_1(0), \psi_2(0))). \end{aligned}$$

Therefore, if inequality (3.3) is fulfilled, we have

$$\begin{aligned} &(\varphi_0 \bullet \mathcal{D}_{(3.1)}V(\psi_1(0), \psi_2(0))) \\ &= 3e^{-t} \left(\psi_1(0) \max_{s \in [t-r, t]} \psi_1(s) + 2\psi_2(0) \max_{s \in [t-r, t]} \psi_2(s) \right) \\ &\leq 6e^{-t}(\varphi_0 \bullet V(\psi_1(0), \psi_2(0))). \end{aligned}$$

Now, consider the scalar comparison equation $u' = 6e^{-t}u$ with initial condition $u(t_0) = u_0$, whose solution is $u(t) = u_0e^{6(e^{-t_0}-e^{-t})}$ and $|u(t)| \leq |u_0|e^{6e^{-t_0}}$ for $t \geq t_0$. For any numbers $0 < \lambda < A$, we choose a number $\tau > \max\{0, \ln 6 - \ln(\ln(\frac{A}{\lambda}))\} > 0$. Note $\tau = \tau(\lambda, A) > 0$. It is easy to check that for $t_0 > \tau$ and $|u_0| < \lambda$, the inequality $|u(t)| < A$ holds, i.e., the scalar comparison equation is uniformly eventually practically stable. Therefore, according to Theorem 2, the system of differential equations with “maxima” (3.1) is uniformly d -eventually practically stable in terms of two measures, i.e., for any numbers $0 < \lambda < A$, there exists a number $\tau = \tau(\lambda, A) > 0$ such that, if $t_0 > \tau$, then the inequality $\sup_{s \in [-r, 0]} (|\phi_1(s)| + 2|\phi_2(s)|) < \lambda$ implies $x^2(t; t_0, \phi) + 2y^2(t; t_0, \phi) < A$, for $t \geq t_0$.

Note that the choice of the vector φ_0 has a huge influence on the sufficient conditions. Let us, for example, consider the vector $\varphi_0 = (1, 1)$. In this case

$$(\varphi_0 \bullet V(x, y)) = \frac{1}{2}(x + 2y)^2 + \frac{1}{2}(x - y)^2 = x^2 + xy + \frac{5}{2}y^2$$

and Condition (i) of Theorem 1 is not satisfied for the for the above defined function $V(x, y)$.

We could change the Lyapunov function by $\tilde{V} : \mathbb{R}^2 \rightarrow \mathcal{K}$, $\tilde{V} = (\tilde{V}_1, \tilde{V}_2)$, $\tilde{V}_1(x, y) = \frac{1}{2}(x + y)^2$, $\tilde{V}_2(x, y) = \frac{1}{2}(x - y)^2$. In this case $(\varphi_0 \bullet \tilde{V}(x, y)) = x^2 + y^2$ and Condition (i) of Theorem 1 is satisfied. But in this case Condition (ii) is not satisfied.

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