Communications in Applied Analysis **14** (2010), no. 4, 525–540

OSCILLATION OF SECOND ORDER DELAY DYNAMIC EQUATIONS WITH OSCILLATORY COEFFICIENTS

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ABSTRACT. In this paper, we extend the oscillation criteria established by Fite, Kamenev, Hille and Nehari for second order linear differential equations to second order linear delay dynamic equations with oscillatory coefficients on time scales. Our results are essentially new even for second order differential equations and difference equations. Finally we consider several examples to illustrate our main theorems.

Keywords. Oscillation; Delay dynamic equation; Time scales; Oscillatory coefficients; Second order

AMS (MOS) Subject Classification. 34C10, 34K11, 39A11, 39A13.

1. INTRODUCTION

In recent years, there has been an increasing interest in studying the oscillation and nonoscillation of solutions of second order delay dynamic equations on times scales which seeks to harmonize the oscillation of the continuous and the discrete, to include them in one comprehensive theory, and to eliminate obscurity from both. For the convenience we refer the reader to the papers [2-9,11,12,18,19] and the references cited therein. However, to the best of our knowledge, few researchers [3] consider the oscillation of the equation with oscillatory coefficients on time scales. In this paper, based on the classic results of oscillation theory, we consider the second order linear delay dynamic equation with oscillatory coefficients

$$x^{\Delta\Delta}(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i) = 0 \tag{1.1}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, where $\mathbb{T}$ is an arbitrary time scale which is unbounded above; $t_0 \in \mathbb{T}$ with $t_0 \geq 0$.

Throughout this paper, we always assume that

(A1) $\tau_i \in \mathbb{R}^+$ with $\tau_1 > \tau_2 > \cdots > \tau_m > 0$ such that $t - \tau_i \in \mathbb{T}$ for $t \in \mathbb{T}$;

Research partially supported by the NSF of Guangdong Province under Grant 8451063101000730.
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Received November 13, 2009

(A2) the functions p_i, $i = 1, \dots, m$, are real-valued $rd-$continuous functions defined on $[t_0, \infty)_{\mathbb{T}}$, and satisfy the following conditions:

(i) there exists a $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that for all $t \in [T_0, \infty)_{\mathbb{T}}$,

$$\begin{aligned}
&p_1(t) \geq 0,\\
&\theta_1 p_1(t) + p_2(t) \geq 0,\\
&\theta_2\theta_1 p_1(t) + \theta_2 p_2(t) + p_3(t) \geq 0,\\
&\qquad\qquad \cdots\cdots,\\
&\prod_{i=1}^{m-1} \theta_i p_1(t) + \prod_{i=2}^{m-1} \theta_i p_2(t) + \cdots + \theta_{m-1} p_{m-1}(t) + p_m(t) \geq 0,
\end{aligned}$$

where

$$\theta_1 = \frac{\tau_m}{\tau_1 - \tau_2 + \tau_m}, \quad \theta_i = \frac{\tau_{i-1} - \tau_i}{\tau_{i-1} - \tau_{i+1}}, \quad i = 2, 3, \dots, m-1;$$

(ii) for any $T \in [t_0, \infty)_{\mathbb{T}}$, there exists a $t^* \in [T, \infty)_{\mathbb{T}}$ such that $p_i(t) \geq 0$ on $[t^*, t^* + \tau_1 + \tau_m]_{\mathbb{T}}$, $i = 1, 2, \dots, m$.

Remark 2.1. Obviously, if $p_i(t) \geq 0$ for $t \in [T_0, \infty)_{\mathbb{T}}$, $i = 1, 2, \dots, m$, then $p_i(t)$ satisfy the conditions (i)–(ii). Conversely, if $p_i(t)$ satisfy the conditions (i)–(ii), then the functions $p_i(t)$ may be oscillatory functions except $p_1(t)$. For example, let $p_1(t) = \tau_1/\tau_2$ and $p_2(t) = \sin t$, then $p_1(t)$ and $p_2(t)$ satisfy the conditions (i)–(ii). Clearly, $p_2(t)$ is oscillatory on $[1, \infty)_{\mathbb{T}}$.

Throughout this paper, the knowledge and understanding of time scales and time-scale notation is assumed; for an excellent introduction to the calculus on time scales, see [1,4,5,14]. By a solution of (1.1) we mean a nontrivial real-valued function $x(t)$ satisfying (1.1) for $t \in [t_0, \infty)_{\mathbb{T}}$. Our attention is restricted to those solutions of (1.1) which exist on some half-line $[t_x, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| : t \in [T, \infty)_{\mathbb{T}}\} > 0$ for any $T \in [t_x, \infty)_{\mathbb{T}}$. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise, it is nonoscillatory. (1.1) is said to be oscillatory if all its solutions are oscillatory.

Here we are concerned with extending oscillation criteria for the second order linear differential equation

$$x''(t) + p(t)x(t) = 0, \quad p \in \mathbf{C}([t_0, \infty), \mathbb{R}^+). \tag{1.2}$$

In 1918, Fite [10] proved that if

$$\lim_{t\to\infty} \int_{t_0}^{t} p(s)ds = \infty, \tag{1.3}$$

then (1.2) is oscillatory.

In 1978, Kamenev [16] gave another condition for the oscillation of (1.2), i.e.,

$$\lim_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n p(s)ds = \infty, \quad n > 1. \tag{1.4}$$

In 1948, Hille [15] by a different approach proved that if

$$\liminf_{t\to\infty} t \int_t^\infty p(s)ds > \frac{1}{4}, \tag{1.5}$$

then (1.2) is oscillatory.

In 1957, Nehari [17] proved that if

$$\liminf_{t\to\infty} \frac{1}{t} \int_t^\infty s^2 p(s)ds > \frac{1}{4}, \tag{1.6}$$

then (1.2) is oscillatory.

Recently, under the restriction that the coefficients are positive, the oscillation criteria (1.3)–(1.6) have been obtained for dynamic equations, for example, see [6,7,8]. The purpose of this paper is establish oscillation criteria (1.3)–(1.6) for (1.1). Our results are essentially new even for second order differential equations and difference equations, i.e., $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$. Finally, we consider several examples to illustrate our main theorems.

2. MAIN RESULTS

For simplicity, let

$$\alpha(t,u) := \frac{t - \tau_m - u}{\sigma(t) - u},$$

$$\Theta(t) := \prod_{i=1}^{m-1} \theta_i p_1(t) + \prod_{i=2}^{m-1} \theta_i p_2(t) + \cdots + \theta_{m-1} p_{m-1}(t) + p_m(t).$$

We begin with the following lemmas.

Lemma 2.1. *Let $x(t)$ be an eventually positive solution of* (1.1)*. Then there exists a $T^* \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, such that*

$$x^{\Delta\Delta}(t) \le 0, \quad x^{\Delta}(t) > 0, \quad t \in [T^*, \infty)_{\mathbb{T}},$$

and

$$\sum_{i=1}^{m} p_i(t) x(t - \tau_i) \ge x(t - \tau_m)\Theta(t), \quad t \in [T^*, \infty)_{\mathbb{T}}.$$

Proof. Let $x(t)$ be an eventually positive solution of (1.1). Then there exists a $t_1 \in [t_0 + \tau_1, \infty)_{\mathbb{T}}$ such that

$$x(t - \tau_i) > 0, \quad i = 1, 2, \ldots, m, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

From the assumption (ii), there exists a $T^* \in [t_1, \infty)_{\mathbb{T}}$ such that $p_i(t) \geq 0$ on $[T^*, T^* + \tau_1 + \tau_m]_{\mathbb{T}}$, $i = 1, 2, \ldots, m$. Hence, for all $t \in [T^*, T^* + \tau_1 + \tau_m]_{\mathbb{T}}$, it follows from (1.1) that

$$x^{\Delta\Delta}(t) = -\sum_{i=1}^{m} p_i(t)x(t - \tau_i) \leq 0.$$

Now we claim that $x^{\Delta\Delta}(t) \leq 0$ on $[T^* + \tau_1 + \tau_m, T^* + \tau_1 + 2\tau_m]_{\mathbb{T}}$. Indeed, for any $t \in [T^* + \tau_1 + \tau_m, T^* + \tau_1 + 2\tau_m]_{\mathbb{T}}$, we have $t - \tau_1 - \tau_m, t - \tau_i \in [T^*, T^* + \tau_1 + \tau_m]_{\mathbb{T}}$, $i = 1, 2, \ldots, m$. Noting that $x(t) > 0$ and $x^{\Delta}(t)$ is nonincreasing on $[T^*, T^*+\tau_1+\tau_m]_{\mathbb{T}}$, we get

$$x(t - \tau_1) > x(t - \tau_1) - x(t - \tau_1 - \tau_m) = \int_{t-\tau_1-\tau_m}^{t-\tau_1} x^{\Delta}(s)\Delta s \geq x^{\Delta}(t - \tau_1)\tau_m,$$

and

$$x(t - \tau_2) - x(t - \tau_1) = \int_{t-\tau_1}^{t-\tau_2} x^{\Delta}(s)\Delta s \leq x^{\Delta}(t - \tau_1)(\tau_1 - \tau_2).$$

Thus,

$$x(t - \tau_2) - x(t - \tau_1) < \frac{\tau_1 - \tau_2}{\tau_m} x(t - \tau_1),$$

i.e.,

$$x(t - \tau_1) > \frac{\tau_m}{\tau_1 - \tau_2 + \tau_m} x(t - \tau_2) = \theta_1 x(t - \tau_2).$$

As the same argument of the above, we can get

$$x(t - \tau_2) > \theta_2 x(t - \tau_3),\ x(t - \tau_3) > \theta_3 x(t - \tau_4),\ \ldots,\ x(t - \tau_{m-1}) > \theta_{m-1} x(t - \tau_m).$$

So we have

$$\begin{aligned} x^{\Delta\Delta}(t) &= -p_1(t)x(t - \tau_1) - p_2(t)x(t - \tau_2) - \cdots - p_m(t)x(t - \tau_m) \\ &< -\theta_1 p_1(t)x(t - \tau_2) - p_2(t)x(t - \tau_2) - \cdots - p_m(t)x(t - \tau_m) \\ &< -\theta_2[\theta_1 p_1(t) + p_2(t)]x(t - \tau_3) - \cdots - p_m(t)x(t - \tau_m) \\ &< \cdots \\ &< -\Theta(t)x(t - \tau_m) \leq 0. \end{aligned}$$

Therefore, $x^{\Delta\Delta}(t) \leq 0$ for $t \in [T^* + \tau_1 + \tau_m, T^* + \tau_1 + 2\tau_m]_{\mathbb{T}}$, which completes the proof of our claim.

On the other hand, we can easily get that $x^{\Delta\Delta}(t) \leq 0$ on $[T^* + \tau_1 + 2\tau_m, T^* + \tau_1 + 3\tau_m]_{\mathbb{T}}$, $\cdots$, $[T^* + \tau_1 + k\tau_m, T^* + \tau_1 + (k + 1)\tau_m]_{\mathbb{T}}$. Then $x^{\Delta\Delta}(t) \leq 0$, and

$$\sum_{i=1}^{m} p_i(t)x(t - \tau_i) \geq x(t - \tau_m)\Theta(t),\ t \in [T^*, \infty)_{\mathbb{T}}.$$

From the fact that $x^{\Delta\Delta}(t) \leq 0$ for $t \geq T^*$, we see that $x^{\Delta}(t)$ is nonincreasing on $[T^*, \infty)_{\mathbb{T}}$, which implies that $x^{\Delta}(t)$ is eventually negative or eventually positive on $[T^*, \infty)_{\mathbb{T}}$. If there exists a $t_2 \in [T^*, \infty)_{\mathbb{T}}$ such that $x^{\Delta}(t_2) =: d < 0$, then

$$x^{\Delta}(t) \leq x^{\Delta}(t_2) = d, \quad t \in [t_2, \infty)_{\mathbb{T}}.$$

Integrating both sides of the above from t_2 to t, we get

$$x(t) \le x(t_2) + d(t - t_2) \to -\infty \quad \text{as } t \to \infty,$$

which contradicts the fact that $x(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$, and hence $x^{\Delta}(t) > 0$ on $[T^*, \infty)_{\mathbb{T}}$. This completes the proof. □

Lemma 2.2. *Assume that there exists $T^* \ge t_0$, sufficiently large, such that*

$$x(t) > 0, \quad x^{\Delta}(t) > 0, \quad x^{\Delta\Delta}(t) \le 0 \quad \text{on } [T^*, \infty)_{\mathbb{T}}. \tag{2.1}$$

Then

$$x(t - \tau_m) > \alpha(t, T^*) x^{\sigma}(t) \quad \text{for } t \ge T_1 \ge T^*. \tag{2.2}$$

Proof. Since $x^{\Delta}(t)$ is nonincreasing on $[T^*, \infty)_{\mathbb{T}}$. We can choose $T_1 \ge T^* + \tau_m$ so that for $t \ge T_1$,

$$x^{\sigma}(t) - x(t - \tau_m) = \int_{t-\tau_m}^{\sigma(t)} x^{\Delta}(s)\Delta s \le x^{\Delta}(t - \tau_m)(\sigma(t) - t + \tau_m),$$

and so

$$\frac{x^{\sigma}(t)}{x(t - \tau_m)} \le 1 + \frac{x^{\Delta}(t - \tau_m)}{x(t - \tau_m)}(\sigma(t) - t + \tau_m).$$

Also, we see

$$x(t - \tau_m) > x(t - \tau_m) - x(T^*) = \int_{T^*}^{t-\tau_m} x^{\Delta}(s)\Delta s \ge x^{\Delta}(t - \tau_m)(t - \tau_m - T^*),$$

and thus,

$$\frac{x^{\Delta}(t - \tau_m)}{x(t - \tau_m)} < \frac{1}{t - \tau_m - T^*}.$$

Hence,

$$\frac{x^{\sigma}(t)}{x(t - \tau_m)} < 1 + \frac{\sigma(t) - t + \tau_m}{t - \tau_m - T^*} = \frac{\sigma(t) - T^*}{t - \tau_m - T^*} = \frac{1}{\alpha(t, T^*)},$$

and then we get the desired inequality (2.2). This completes the proof. □

Let us start with a direct extension of Fite theorem [10] to (1.1).

Theorem 2.1. *If there exists a positive Δ−differentiable function $v(t)$ such that for all sufficiently large T^*,*

$$\limsup_{t\to\infty} \int_{T^*}^{t} \Big[v(s)\Theta(s)\alpha(s, T^*) - \frac{(v_+^{\Delta}(s))^2}{4v(s)} \Big] \Delta s = \infty, \tag{2.3}$$

where $v_+^{\Delta}(s) := \max\{v^{\Delta}(s), 0\}$, then (1.1) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality we may assume that $x(t)$ is an eventually positive solution of (1.1), since the proof in the other case is similar. From Lemma 2.1, we have, for some $T^* \in [t_0, \infty)_{\mathbb{T}}$,

$$x^{\Delta\Delta}(t) \le -x(t - \tau_m)\Theta(t), \quad t \in [T^*, \infty)_{\mathbb{T}}. \tag{2.4}$$

Let

$$w(t) := v(t)\frac{x^{\Delta}(t)}{x(t)}. \tag{2.5}$$

By the product rule, the quotient rule and from (2.4) and (2.5), we have

$$\begin{aligned} w^{\Delta}(t) &= v^{\Delta}(t)\Big(\frac{x^{\Delta}(t)}{x(t)}\Big)^{\sigma} + v(t)\Big(\frac{x^{\Delta}(t)}{x(t)}\Big)^{\Delta} \\ &= \frac{v^{\Delta}(t)}{v^{\sigma}(t)}w^{\sigma}(t) + \frac{v(t)x^{\Delta\Delta}(t)}{x^{\sigma}(t)} - \frac{v(t)(x^{\Delta}(t))^2}{x(t)x^{\sigma}(t)} \\ &\le \frac{v^{\Delta}(t)}{v^{\sigma}(t)}w^{\sigma}(t) - v(t)\Theta(t)\frac{x(t-\tau_m)}{x^{\sigma}(t)} - \frac{v(t)(x^{\Delta}(t))^2}{x(t)x^{\sigma}(t)}. \end{aligned} \tag{2.6}$$

From Lemma 2.2,

$$x(t-\tau_m) > \alpha(t, T^*)x^{\sigma}(t).$$

Noting that $x^{\Delta}(t) > 0$ and $x^{\Delta}(t)$ is nonincreasing on $[T^*, \infty)_{\mathbb{T}}$, from (2.6) and the above inequality, we can obtain

$$\begin{aligned} w^{\Delta}(t) &\le -v(t)\Theta(t)\alpha(t, T^*) + \frac{v^{\Delta}(t)}{v^{\sigma}(t)}w^{\sigma}(t) - \frac{v(t)}{(v^{\sigma}(t))^2}(w^{\sigma}(t))^2 \\ &\le -v(t)\Theta(t)\alpha(t, T^*) + \frac{v_+^{\Delta}(t)}{v^{\sigma}(t)}w^{\sigma}(t) - \frac{v(t)}{(v^{\sigma}(t))^2}(w^{\sigma}(t))^2. \end{aligned} \tag{2.7}$$

Completing the square of (2.7), we get

$$w^{\Delta}(t) \le -v(t)\Theta(t)\alpha(t, T^*) + \frac{(v_+^{\Delta}(t))^2}{4v(t)}.$$

Integrating both sides of the above from T^* to t, and rearranging the terms, we have

$$\int_{T^*}^{t}\Big[v(s)\Theta(s)\alpha(s, T^*) - \frac{(v_+^{\Delta}(s))^2}{4v(s)}\Big]\Delta s \le -w(t) + w(T^*) \le w(T^*),$$

which contradicts (2.3), and hence the proof is complete. $\square$

We are now ready to state and prove a Kamenev-type criterion [16] for (1.1).

Theorem 2.2. *If there exist a positive $\Delta-$differentiable function $v(t)$ and a number $n > 1$ such that for all sufficiently large T^*,*

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{T^*}^{t}\Big[(t-s)^n v(s)\Theta(s)\alpha(s, T^*) - \frac{\delta^2(t,s)}{4v(s)(t-s)^n}\Big]\Delta s = \infty, \tag{2.8}$$

where

$$\delta(t, s) := (t-s)^n v_+^{\Delta}(s) - nv^{\sigma}(s)(t-\sigma(s))^{n-1},$$

and $v_+^{\Delta}(s)$ is defined in Theorem 2.1, then (1.1) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1). As in the proof of Theorem 2.1, we may assume that $x(t)$ is an eventually positive solution of (1.1). Define $w(t)$ as (2.5). Proceeding as in the proof of Theorem 2.1, there exists a $T^* \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, so that (2.7) holds. Multiplying both sides of (2.7), with t replaced by s, by $(t-s)^n$, integrating with respect to s from T^* to t, we get

$$\begin{aligned}
\int_{T^*}^t (t-s)^n v(s)\Theta(s)\alpha(s,T^*)\Delta s &\le -\int_{T^*}^t (t-s)^n w^\Delta(s)\Delta s \\
&\quad + \int_{T^*}^t (t-s)^n \frac{v_+^\Delta(s)}{v^\sigma(s)} w^\sigma(s)\Delta s - \int_{T^*}^t (t-s)^n \frac{v(s)}{(v^\sigma(s))^2}(w^\sigma(s))^2\Delta s \\
&= (t-T^*)^n w(T^*) + \int_{T^*}^t ((t-s)^n)^{\Delta_s} w^\sigma(s)\Delta s + \int_{T^*}^t (t-s)^n \frac{v_+^\Delta(s)}{v^\sigma(s)} w^\sigma(s)\Delta s \\
&\quad - \int_{T^*}^t (t-s)^n \frac{v(s)}{(v^\sigma(s))^2}(w^\sigma(s))^2\Delta s. \qquad (2.9)
\end{aligned}$$

We now claim that if $t \ge \sigma(s)$ and $n \ge 1$ then

$$((t-s)^n)^{\Delta_s} \le -n(t-\sigma(s))^{n-1}. \qquad (2.10)$$

For this, we consider the following two cases:

Case 1. If $\mu(t) = 0$, then

$$((t-s)^n)^{\Delta_s} = -n(t-s)^{n-1}.$$

Case 2. If $\mu(t) \neq 0$, then

$$\begin{aligned}
((t-s)^n)^{\Delta_s} &= \frac{1}{\mu(s)}[(t-\sigma(s))^n - (t-s)^n] \\
&= \frac{-1}{\sigma(s)-s}[(t-s)^n - (t-\sigma(s))^n]. \qquad (2.11)
\end{aligned}$$

Using Theorem 41 [13],

$$x^n - y^n \ge ny^{n-1}(x-y), \quad x \ge y > 0, \quad n \ge 1,$$

we have

$$(t-s)^n - (t-\sigma(s))^n \ge n(t-\sigma(s))^{n-1}(\sigma(s)-s).$$

Then, from (2.11) and the above, we get

$$((t-s)^n)^{\Delta_s} \le -n(t-\sigma(s))^{n-1},$$

which proves (2.10) holds. Substituting (2.10) in (2.9), completing the square, we obtain

$$\begin{aligned}
\int_{T^*}^t (t-s)^n v(s)\Theta(s)\alpha(s,T^*)\Delta s &\le (t-T^*)^n w(T^*) - n\int_{T^*}^t (t-\sigma(s))^{n-1} w^\sigma(s)\Delta s \\
&\quad + \int_{T^*}^t (t-s)^n \frac{v_+^\Delta(s)}{v^\sigma(s)} w^\sigma(s)\Delta s - \int_{T^*}^t (t-s)^n \frac{v(s)}{(v^\sigma(s))^2}(w^\sigma(s))^2\Delta s \\
&= (t-T^*)^n w(T^*) + \int_{T^*}^t \frac{\delta(t,s)}{v^\sigma(s)} w^\sigma(s)\Delta s - \int_{T^*}^t (t-s)^n \frac{v(s)}{(v^\sigma(s))^2}(w^\sigma(s))^2\Delta s \\
&\le (t-T^*)^n w(T^*) + \int_{T^*}^t \frac{\delta^2(t,s)}{4v(s)(t-s)^n}\Delta s.
\end{aligned}$$

So we have

$$\frac{1}{t^n}\int_{T^*}^{t}\Big[(t-s)^n v(s)\Theta(s)\alpha(s,T^*)-\frac{\delta^2(t,s)}{4v(s)(t-s)^n}\Big]\Delta s\le w(T^*)<\infty,$$

which contradicts (2.8), and hence the proof is complete. □

In the following, we extend the theorems of Hille [15] and Nehari [17] to (1.1). For convenience, we introduce the following notations, for all sufficiently large T^*, set

$$p_* := \liminf_{t\to\infty} t\int_t^\infty \Theta(s)\alpha(s,T^*)\Delta s,\quad q_* := \liminf_{t\to\infty}\frac{1}{t}\int_{T^*}^{t}(\sigma(s))^2\Theta(s)\alpha(s,T^*)\Delta s,$$
$$l_* := \liminf_{t\to\infty}\frac{\sigma(t)}{t},\quad l^* := \limsup_{t\to\infty}\frac{\sigma(t)}{t}.$$

In order for the definition of p_* to make sense, we assume that

$$\int_{t_0}^{\infty}\Theta(s)\alpha(s,T^*)\Delta s<\infty.$$

Theorem 2.3. *If for all sufficiently large T^*,*

$$\liminf_{t\to\infty} t\int_t^\infty \Theta(s)\alpha(s,T^*)\Delta s>\frac{1}{4}, \tag{2.12}$$

then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1). As in the proof of Theorem 2.1, we may assume that $x(t)$ is an eventually positive solution of (1.1). Define $w(t)$ as (2.5) by putting $v(t)=1$. Proceeding as in the proof of Theorem 2.1, there is a $T^*\in[t_0,\infty)_{\mathbb{T}}$ sufficiently large, so that (2.6) holds with $v(t)=1$. From Lemma 2.2 and (2.6), we obtain

$$w^\Delta(t)\le -\Theta(t)\alpha(t,T^*)-\frac{(x^\Delta(t))^2}{x(t)x^\sigma(t)}. \tag{2.13}$$

Since $x^{\Delta\Delta}(t)\le 0$ on $[T^*,\infty)_{\mathbb{T}}$, we have

$$w^\Delta(t)\le -\Theta(t)\alpha(t,T^*)-w(t)w^\sigma(t). \tag{2.14}$$

From (2.14), we see

$$w^\Delta(t)\le -w(t)w^\sigma(t)\qquad \text{for } t\in[T^*,\infty)_{\mathbb{T}}.$$

and so,

$$\left(-\frac{1}{w(t)}\right)^\Delta=\frac{w^\Delta(t)}{w(t)w^\sigma(t)}\le -1\qquad \text{for } t\in[T^*,\infty)_{\mathbb{T}}.$$

Therefore,

$$-\frac{1}{w(t)}+\frac{1}{w(T^*)}=\int_{T^*}^{t}\frac{w^\Delta(s)}{w(s)w^\sigma(s)}\Delta s\le -\int_{T^*}^{t}\Delta s=-(t-T^*),$$

which implies that $\lim_{t\to\infty} w(t) = 0$. Integrating both sides of (2.14) from t to ∞, using the fact that $\lim\limits_{t\to\infty} w(t) = 0$, we have

$$w(t) \geq \int_t^\infty \Theta(s)\alpha(s,T^*)\Delta s + \int_t^\infty w(s)w^\sigma(s)\Delta s.$$

Multiplying both sides of the above by t, we obtain

$$\begin{aligned} tw(t) &\geq t\int_t^\infty \Theta(s)\alpha(s,T^*)\Delta s + t\int_t^\infty w(s)w^\sigma(s)\Delta s \\ &= t\int_t^\infty \Theta(s)\alpha(s,T^*)\Delta s + t\int_t^\infty \frac{1}{s\sigma(s)}(sw(s)\sigma(s)w^\sigma(s))\Delta s \\ &= t\int_t^\infty \Theta(s)\alpha(s,T^*)\Delta s + t\int_t^\infty (sw(s)\sigma(s)w^\sigma(s))\big(\frac{-1}{s}\big)^\Delta \Delta s. \end{aligned} \tag{2.15}$$

Let

$$r_* := \liminf_{t\to\infty} tw(t) \quad \text{and} \quad r^* := \limsup_{t\to\infty} tw(t).$$

Then, for any $\varepsilon > 0$, by the definition of r_*, r^* l_* and l^*, we can pick $t \in [T^*,\infty)_{\mathbb{T}}$, sufficiently large, such that

$$r_* - \varepsilon \leq tw(t) \leq r^* + \varepsilon, \quad l_* - \varepsilon \leq \frac{\sigma(t)}{t} \leq l^* + \varepsilon. \tag{2.16}$$

Taking the lim inf of (2.15) as $t \to \infty$ and substituting (2.16) in it, we get

$$r_* \geq p_* + (r_* - \varepsilon)^2.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$r_* \geq p_* + r_*^2, \tag{2.17}$$

and then

$$p_* \leq r_* - r_*^2 \leq \frac{1}{4},$$

which contradicts (2.12), and then the proof is complete. □

Theorem 2.4. *If for all sufficiently large T^*,*

$$\liminf_{t\to\infty} \frac{1}{t}\int_{T^*}^t (\sigma(s))^2\Theta(s)\alpha(s,T^*)\Delta s > \frac{l^*}{l^*+1}, \tag{2.18}$$

then (1.1) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1). Proceeding as in the proof of Theorem 2.3, there is a $T^* \in [t_0,\infty)_{\mathbb{T}}$, sufficiently large, so that (2.14) holds. Multiplying both sides of (2.14) by $(\sigma(t))^2$, integrating from T^* to t ($t \geq T^*$), dividing by t, and rearranging the terms, we have

$$\begin{aligned} tw(t) \leq \frac{1}{t}T^{*2}w(T^*) &+ \frac{1}{t}\int_{T^*}^t (s^2)^\Delta w(s)\Delta s - \frac{1}{t}\int_{T^*}^t (\sigma(s))^2\Theta(s)\alpha(s,T^*)\Delta s \\ &- \frac{1}{t}\int_{T^*}^t (\sigma(s))^2 w(s)w^\sigma(s)\Delta s \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{t}T^{*2}w(T^*) - \frac{1}{t}\int_{T^*}^{t}(\sigma(s))^2\Theta(s)\alpha(s,T^*)\Delta s + \frac{1}{t}\int_{T^*}^{t}(s+\sigma(s))w(s)\Delta s \\
&\quad - \frac{1}{t}\int_{T^*}^{t}(\sigma(s))^2 w(s)w^{\sigma}(s)\Delta s \\
&= \frac{1}{t}T^{*2}w(T^*) - \frac{1}{t}\int_{T^*}^{t}(\sigma(s))^2\Theta(s)\alpha(s,T^*)\Delta s + \frac{1}{t}\int_{T^*}^{t}\Big(1+\frac{\sigma(s)}{s}\Big)(sw(s))\Delta s \\
&\quad - \frac{1}{t}\int_{T^*}^{t}\frac{\sigma(s)}{s}(sw(s)\sigma(s)w^{\sigma}(s))\Delta s.
\end{aligned}
$$

Taking the lim sup of both sides of the above as $t \to \infty$, we obtain

$$
r^* \le -q_* + (r^*+\varepsilon)(1+l^*+\varepsilon) - (r_*-\varepsilon)^2(l_*-\varepsilon),
$$

where r^* and r_* are as in the proof of Theorem 2.3. Since $\varepsilon > 0$ is arbitrary, we have

$$
r^* \le -q_* + r^*(1+l^*) - r_*^2 l_*,
$$

i.e.,

$$
q_* \le r^* l^* - r_*^2 l_*. \tag{2.19}
$$

On the other hand, from (2.13), we can get

$$
\begin{aligned}
w^{\Delta}(t) &\le -\Theta(t)\alpha(t,T^*) - \Big(\frac{x^{\Delta}(t)}{x(t)}\Big)^2\frac{x(t)}{x^{\sigma}(t)} \\
&= -\Theta(t)\alpha(t,T^*) - \frac{w^2(t)}{1+\mu(t)w(t)}.
\end{aligned}
$$

Multiplying the above by $(\sigma(t))^2$, integrating from T^* to t, dividing by t, and rearranging the terms, we get

$$
\begin{aligned}
tw(t) &\le \frac{1}{t}T^{*2}w(T^*) + \frac{1}{t}\int_{T^*}^{t}(s^2)^{\Delta}w(s)\Delta s - \frac{1}{t}\int_{T^*}^{t}(\sigma(s))^2\Theta(s)\alpha(s,T^*)\Delta s \\
&\quad - \frac{1}{t}\int_{T^*}^{t}\frac{(\sigma(s))^2w^2(s)}{1+\mu(s)w(s)}\Delta s \\
&= \frac{1}{t}T^{*2}w(T^*) - \frac{1}{t}\int_{T^*}^{t}(\sigma(s))^2\Theta(s)\alpha(s,T^*)\Delta s + \frac{1}{t}\int_{T^*}^{t}\psi(s,w(s))\Delta s,
\end{aligned} \tag{2.20}
$$

where

$$
\psi(s,w(s)) := (s+\sigma(s))w(s) - \frac{(\sigma(s))^2w^2(s)}{1+\mu(s)w(s)}.
$$

We claim that

$$
\psi(s,w(s)) \le 1 \quad \text{for } s \in [T^*,\infty)_{\mathbb{T}}.
$$

To see this observe that if we let

$$
g(s,u) := (s+\sigma(s))u - \frac{(\sigma(s))^2u^2}{1+\mu(s)u},
$$

then we have, after some simplification,

$$
g(s,u) = \frac{(2\sigma(s)-\mu(s))u(1+\mu(s)u) - (\sigma(s))^2u^2}{1+\mu(s)u} = \frac{(2\sigma(s)-\mu(s))u - s^2u^2}{1+\mu(s)u},
$$

since $s+\sigma(s) = 2\sigma(s) - \mu(s)$. We note that if $\mu(s) = 0$, then the maximum of $g(s,u)$ (with respect to u) occurs at $u_0 := 1/s$. Moreover in the case $\mu(s) > 0$, after some calculations, one finds that for fixed $s > 0$, the maximum of $g(s,u)$ for $u \geq 0$ occurs at $u_0 = 1/s$ also. Hence, we have

$$g(s,u) \leq g(s,u_0) = (s+\sigma(s))u_0 - \frac{(\sigma(s))^2 u_0^2}{1+\mu(s)u_0} = \frac{s+\sigma(s)}{s} - \frac{(\sigma(s))^2}{s(s+\mu(s))} = 1$$

for $u \geq 0$. Hence, we conclude that $\psi(s,w(s)) \leq 1$, and so

$$\int_{T^*}^t \psi(s,w(s))\Delta s \leq t - T^*.$$

Substituting the above in (2.20), we get

$$tw(t) \leq \frac{1}{t}T^{*2}w(T^*) - \frac{1}{t}\int_{T^*}^t (\sigma(s))^2\Theta(s)\alpha(s,T^*)\Delta s + \frac{t-T^*}{t}.$$

Taking the lim sup of the above as $t \to \infty$, from (2.16), we get

$$r^* \leq -q_* + 1,$$

and then

$$q_* \leq 1 - r^*.$$

Combining this with (2.19), we get

$$q_* \leq \min\{1-r^*, r^*l^* - r_*^2 l_*\}, \tag{2.21}$$

and consequently,

$$q_* \leq \min\{1-r^*, r^*l^*\},$$

which implies that

$$q_* \leq \frac{l^*}{l^*+1},$$

which contradicts (2.18), and the proof is complete. $\square$

Theorem 2.5. *If* $0 \leq p_* \leq \frac{1}{4}$ *and*

$$q_* > \frac{1}{1+l^*}\Big[l^* - l_*\Big(\frac{1}{2} - p_* - \frac{1}{2}\sqrt{1-4p_*}\Big)\Big], \tag{2.22}$$

then (1.1) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1). Proceeding as in the proof of Theorems 2.3 and 2.4, we have that (2.17) and (2.21) hold, respectively. From (2.17), we can get that

$$r_* \geq \varepsilon_* := \frac{1}{2}(1-\sqrt{1-4p_*}),$$

and so using (2.21),

$$q_* \leq \min\{1-r^*, r^*l^* - r_*^2 l_*\} \leq \min\{1-r^*, r^*l^* - \varepsilon_*^2 l_*\}$$

for $\varepsilon_* \leq r^* \leq 1$, where r_* and r^* are as in the proof of Theorem 2.3. Note that

$$1 - r^* = r^* l^* - \varepsilon_*^2 l_* \quad \text{when } r^* = \varepsilon^* := \frac{1}{1+l^*}(1 + \varepsilon_*^2 l_*),$$

and after some easy calculations, so

$$q_* \leq 1 - \varepsilon^* = \frac{1}{1+l^*}\Big[l^* - l_*\Big(\frac{1}{2} - p_* - \frac{1}{2}\sqrt{1-4p_*}\Big)\Big],$$

which contradicts (2.22), and the proof is complete. □

Remark 2.2. A close look at the proof of Theorem 2.4 shows that the inequality

$$q_* \leq r^* l^* - r_*^2 l_*$$

holds, when we replace l^* and l_* by

$$\lambda^* := \limsup_{t\to\infty} \frac{1}{t}\int_T^t \frac{\sigma(s)}{s}\Delta s \quad \text{and} \quad \lambda_* := \liminf_{t\to\infty} \frac{1}{t}\int_T^t \frac{\sigma(s)}{s}\Delta s,$$

respectively. Then Theorems 2.4 and 2.5 hold with l^* and l_* replaced by λ^* and λ_*, respectively.

3. EXAMPLES

In this section, we give some examples to illustrate our main results.

Example 3.1. Consider the delay dynamic equation

$$x^{\Delta\Delta}(t) + ax(t-1) + \frac{b\sin t}{t}x(t-\frac{1}{2}) = 0, \quad t \in (0,\infty)_{\mathbb{T}}, \tag{3.1}$$

where $p_1(t) = a$, $p_2(t) = (b\sin t)/t$ for $a > 2b > 0$, $\tau_1 = 1$ and $\tau_2 = 1/2$. It is easy to check that $p_1(t)$, $p_2(t) \geq 0$ on the interval $[2n\pi, \pi + 2n\pi]_{\mathbb{T}}$, $n = 0, 1, 2, \ldots$, and for all $t \in (0,\infty)_{\mathbb{T}}$,

$$p_1(t) > 0, \quad \frac{\tau_2}{\tau_1}p_1(t) + p_2(t) = \frac{a}{2} + \frac{b\sin t}{t} \geq \frac{a}{2} - b > 0.$$

Let $T^* = 1$ and $v(t) = 1$, we have

$$\begin{aligned} &\limsup_{t\to\infty}\int_1^t \Big[v(s)\Theta(s)\alpha(s,1) - \frac{(v_+^\Delta(s))^2}{4v(s)}\Big]\Delta s \\ &\geq \limsup_{t\to\infty}\int_1^t \Big(\frac{a}{2} - b\Big)\frac{s-\frac{3}{2}}{\sigma(s)-1}\Delta s = \infty. \end{aligned}$$

Then, by Theorem 2.1, (3.1) is oscillatory.

Example 3.2. Consider the delay differential equation

$$x''(t) + \frac{a}{t}x(t-2) + \frac{b\cos\beta t}{t}x(t-1) = 0, \quad t \geq 1, \tag{3.2}$$

where $p_1(t) = a/t$, $p_2(t) = (b\cos\beta t)/t$ for $0 < \beta \le \pi/(12)$ and $a \ge 4b > 0$, $\tau_1 = 2$ and $\tau_2 = 1$. It is easy to check that $p_1(t), p_2(t) \ge 0$ on the interval $[2n\pi/\beta, (1+4n)\pi/(2\beta)]$, $n = 0, 1, 2, \dots$, and for all $t \in [1, \infty)$,

$$p_1(t) > 0, \ \ \frac{\tau_2}{\tau_1}p_1(t) + p_2(t) = \frac{a}{2t} + \frac{b}{t}\cos\beta t \ge \frac{a}{2t} - \frac{b}{t} \ge \frac{b}{t} > 0.$$

Let $T^* = 3$. Then, for all $t \ge 4$,

$$\Theta(t)\alpha(t,3) \ge \frac{b(t-4)}{t(t-3)}.$$

Let $v(t) = 1$. Then $\delta(t,s) = -n(t-s)^{n-1}$, and

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^n}\int_3^t \Big[(t-s)^n v(s)\Theta(s)\alpha(s,3) - \frac{\delta^2(t,s)}{4v(s)(t-s)^n}\Big]ds \\
&\ge \limsup_{t\to\infty} \frac{1}{t^n}\int_3^t (t-s)^{n-2}\Big[\frac{b(s-4)}{s(s-3)}(t-s)^2 - \frac{n^2}{4}\Big]ds \\
&\ge \limsup_{t\to\infty} \frac{1}{t^n}\int_3^t (t-s)^{n-2}\Big[\frac{b}{s}(t-s)^2 - \frac{b}{s(s-3)}(t-s)^2 - \frac{n^2}{4}\Big]ds \\
&= b\limsup_{t\to\infty} \ln t + \text{constant} = \infty.
\end{aligned}$$

Therefore, by Theorem 2.2, (3.2) is oscillatory.

Example 3.3. Consider the delay difference equation

$$\Delta^2 x(t) + \big(\frac{a}{t^2}\sin^2\beta t\big)x(t-2) + \big(\frac{b}{t^2}\cos 2\beta t\big)x(t-1) = 0, \quad t \in [1,\infty)_{\mathbb{N}}, \tag{3.3}$$

where $p_1(t) = (a\sin^2\beta t)/t^2$, $p_2(t) = (b\cos 2\beta t)/t^2$ for $0 < \beta \le \pi/(24)$, and $a \ge 4b > 1$, $\tau_1 = 2$ and $\tau_2 = 1$. It is easy to check that $p_1(t), p_2(t) \ge 0$ on $[n\pi/\beta, (1+4n)\pi/(4\beta)]_{\mathbb{N}}$, $n = 0, 1, 2, \dots$, and for all $t \in [1,\infty)_{\mathbb{N}}$,

$$p_1(t) \ge 0, \ \ \frac{\tau_2}{\tau_1}p_1(t) + p_2(t) = \frac{a}{2t^2}\sin^2\beta t + \frac{b}{t^2}\cos 2\beta t \ge \frac{b}{t^2} > 0.$$

Let $T^* = 3$. Hence, for all $t \ge 4$,

$$\Theta(t)\alpha(t,3) \ge \frac{b(t-4)}{t^2(t-2)},$$

and so

$$\begin{aligned}
\liminf_{t\to\infty} t\int_t^\infty \Theta(s)\alpha(s,3)\Delta s &= b\liminf_{n\to\infty} n\sum_{i=n}^\infty \Big[\frac{1}{i(i-2)} - \frac{4}{i^2(i-2)}\Big] \\
&= \frac{b}{2}\liminf_{n\to\infty} n\Big(\frac{1}{n-2} + \frac{1}{n-1}\Big) = b.
\end{aligned}$$

Therefore, by Theorem 2.3, (3.3) is oscillatory.

Example 3.4. Consider the delay difference equation

$$\Delta_h^2 x(t) + \frac{a}{t^2}x(t-1.5) + \frac{b}{t^2}\cos\beta t\cdot x(t-1) + \frac{b}{t^2}\sin\Big(\beta t - \frac{\pi}{6}\Big)x(t-0.5) = 0, \tag{3.4}$$

where $t \in [0.5, \infty)_{\frac{1}{2}\mathbb{N}}$, $p_1(t) = a/t^2$, $p_2(t) = (b\cos\beta t)/t^2$, $p_3(t) = (b\sin(\beta t - /6))/t^2$ for $0 < \beta \le \pi/(24)$, and $a \ge 4\sqrt{3}b > 0$, $\tau_1 = 1.5$, $\tau_2 = 1$, and $\tau_3 = 0.5$. It is easy to check that $p_1(t)$, $p_2(t)$, and $p_3(t) \ge 0$ on $[\pi/(6\beta) + (2n\pi)/\beta, \pi/(2\beta) + (2n\pi)/(\beta)]_{\frac{1}{2}\mathbb{N}}$, $n = 0, 1, 2, \dots$, and for all $t \in [0.5, \infty)_{\frac{1}{2}\mathbb{N}}$,

$$
\begin{aligned}
&p_1(t) = \frac{a}{t^2} \ge 0,\\
&\frac{\tau_3}{\tau_1 - \tau_2 + \tau_3} p_1(t) + p_2(t) = \frac{a}{2t^2} + \frac{b}{t^2}\cos\beta t \ge \frac{(2\sqrt{3} - 1)b}{t^2} > 0,\\
&\frac{(\tau_1 - \tau_2) \times \tau_3}{(\tau_1 - \tau_3)(\tau_1 - \tau_2 + \tau_3)} p_1(t) + \frac{\tau_1 - \tau_2}{\tau_1 - \tau_3} p_2(t) + p_3(t)\\
&\quad = \frac{1}{4} p_1(t) + \frac{1}{2} p_2(t) + p_3(t) \ge \frac{\sqrt{3}b}{2t^2} > 0.
\end{aligned}
$$

Let $T^* = 2$, then $p_1(t)$, $p_2(t)$, and $p_3(t)$ satisfy the conditions (i) and (ii). Hence, for all $t \ge 2$, we have

$$
\Theta(t)\alpha(t, 2) \ge \frac{\sqrt{3}b(t - 2.5)}{2t^2(t - 1.5)}.
$$

Then,

$$
\begin{aligned}
p_* &= \liminf_{t\to\infty} t \int_t^\infty \Theta(s)\alpha(s, 2)\Delta s\\
&\ge \liminf_{n\to\infty} n \sum_{i=n}^\infty \frac{\sqrt{3}b(i - 2.5)}{4i^2(i - 1.5)}\\
&= \frac{\sqrt{3}b}{4} \liminf_{n\to\infty} n \sum_{i=n}^\infty \left[\frac{1}{i(i - 1.5)} - \frac{2.5}{i^2(i - 1.5)}\right]\\
&\ge \frac{\sqrt{3}b}{4} \liminf_{n\to\infty} n \sum_{i=n}^\infty \left[\frac{1}{i(i - 1)} - \frac{2.5}{n^2(n - 1.5)}\right] = \frac{\sqrt{3}b}{4},
\end{aligned}
$$

and

$$
\begin{aligned}
q_* &= \liminf_{t\to\infty} \frac{1}{t} \int_2^t (\sigma(s))^2 \Theta(s)\alpha(s, 2)\Delta s\\
&\ge \liminf_{n\to\infty} \frac{1}{n} \sum_{i=4}^{n-1} \frac{1}{2}(i + \frac{1}{2})^2 \frac{\sqrt{3}b}{2i^2} \frac{i - 2.5}{i - 1.5}\\
&= \frac{\sqrt{3}b}{4} \liminf_{n\to\infty} \frac{1}{n} \sum_{i=4}^{n-1} \left(1 - \frac{\frac{9}{4}i + \frac{5}{8}}{i^3 - 1.5i^2}\right)\\
&\ge \frac{\sqrt{3}b}{4} \liminf_{n\to\infty} \frac{1}{n} \sum_{i=4}^{n-1} \left[1 - \frac{77}{32i(i - 1.5)}\right]\\
&\ge \frac{\sqrt{3}b}{4} \liminf_{n\to\infty} \frac{1}{n} \sum_{i=4}^{n-1} \left[1 - \frac{77}{32i(i - 2)}\right]\\
&= \frac{\sqrt{3}b}{4} \liminf_{n\to\infty} \left[1 - \frac{1921}{384n} + \frac{77(2n - 3)}{64n(n - 1)(n - 2)}\right] = \frac{\sqrt{3}b}{4}.
\end{aligned}
$$

Clearly, $l_* = l^* = 1$. Therefore,

(1) Let $b > \frac{2\sqrt{3}}{3}$, i.e., $q_* > \frac{l^*}{l^*+1} = \frac{1}{2}$, by Theorem 2.4, (3.4) is oscillatory;

(2) Let $\frac{1}{4} < b \leq \frac{1}{2}$, then (2.12) holds, by Theorem 2.3, (3.4) is oscillatory.

Acknowledgments. The authors would like to express his great appreciation to the referees for their helpful suggestions.

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