*Communications in Applied Analysis* **14** (2010), no. 4, 541–566

# NEUMANN-BOUNDARY STABILIZATION OF THE WAVE EQUATION WITH DAMPING CONTROL AND APPLICATIONS

BOUMEDIÈNE CHENTOUF[1] AND AISSA GUESMIA[2]

[1]Department of Mathematics and Statistics, Sultan Qaboos University
P.O. Box 36. Al Khodh 123, Muscat, Sultanate of Oman
*E-mail:* chentouf@squ.edu.om

[2]Metz Laboratory of Mathematics and Applications
University Paul Verlaine-Metz. 57045 Metz, France
*E-mail:* guesmia@univ-metz.fr

**ABSTRACT.** This article is devoted to the boundary stabilization of a non-homogeneous $n$-dimensional wave equation subject to static or dynamic Neumann boundary conditions. Using a linear feedback law involving only a damping term, we provide a simple method and obtain an asymptotic convergence result for the solutions of the considered systems. The method consists in proposing a new energy norm. Then, a similar result is derived for the case of dynamic Neumann boundary conditions with nonlinear damping feedback laws. Finally, the method presented in this work is also applied to several distributed parameter systems such as the Petrovsky system, coupled wave-wave equations and elasticity systems.

**AMS (MOS) Subject Classification.** 35L05, 93C20, 93D15, 93D20.

## 1. INTRODUCTION

Let $\Omega$ be a bounded open connected set in $\mathbb{R}^n$ having a smooth boundary $\Gamma = \partial\Omega$ of class $C^2$. Given a partition $(\Gamma_0, \Gamma_1)$ of $\Gamma$, consider the following wave equation:

$$y_{tt}(x,t) - Ay(x,t) = 0, \quad \text{in } \Omega \times (0,\infty) \tag{1.1}$$

with either static Neumann boundary conditions and initial conditions

$$\begin{cases} \partial_A y(x,t) = 0, & \text{on } \Gamma_0 \times (0,\infty) \\ \partial_A y(x,t) = U(t), & \text{on } \Gamma_1 \times (0,\infty) \\ y(x,0) = y_0(x) \in H^1(\Omega),\ y_t(x,0) = z_0(x) \in L^2(\Omega), & \end{cases} \tag{1.2}$$

Received April 10, 2009 

or dynamical Neumann boundary conditions and initial conditions

$$\begin{cases} m(x)y_{tt}(x,t) + \partial_A y(x,t) = 0, & \text{on } \Gamma_0 \times (0,\infty) \\ M(x)y_{tt}(x,t) + \partial_A y(x,t) = U(t), & \text{on } \Gamma_1 \times (0,\infty) \\ y(x,0) = y_0(x) \in H^1(\Omega),\ y_t(x,0) = z_0(x) \in L^2(\Omega), \\ y_t|_{\Gamma_0}(x,0) = w_0^0(x) \in L^2(\Gamma_0),\ y_t|_{\Gamma_1}(x,0) = w_1^0(x) \in L^2(\Gamma_1), \end{cases} \tag{1.3}$$

where $A = \sum_{i,j=1}^n \partial_i(a_{ij}\partial_j)$, $\partial_A = \sum_{i,j=1}^n a_{ij}\nu_j\partial_j$, $\partial_k = \frac{\partial}{\partial x_k}$, $\nu = (\nu_1, \ldots, \nu_n)$ is the unit normal of $\Gamma$ pointing towards the exterior of $\Omega$ and $a_{ij} \in C^1(\bar{\Omega})$ such that there exists $\alpha_0 > 0$ satisfying

$$a_{ij} = a_{ji},\ \forall i,j = 1,\ldots,n, \quad \sum_{i,j=1}^n a_{ij}\epsilon_i\epsilon_j \geq \alpha_0 \sum_{i=1}^n \epsilon_i^2,\ \forall\ (\epsilon_1,\ldots,\epsilon_n) \in \mathbb{R}^n. \tag{1.4}$$

Moreover

$$\begin{cases} m \in L^\infty(\Gamma_0);\ m(x) \geq m_0 > 0, \forall x \in \Gamma_0; \\ M \in L^\infty(\Gamma_1);\ M(x) \geq M_1 > 0, \forall x \in \Gamma_1. \end{cases} \tag{1.5}$$

Furthermore, $U$ is a feedback law depending only on a damping term, that is,

$$U(t) = -a(x)\, y_t(x,t), \quad (x,t) \in \Gamma_1 \times (0,\infty), \tag{1.6}$$

where the function $a$ satisfies

$$a \in L^\infty(\Gamma_1);\ a(x) \geq a_0 > 0, \forall x \in \Gamma_1. \tag{1.7}$$

Note that $\Gamma_1$ is supposed to be nonempty ($\mathrm{vol}(\Gamma_1) \neq 0$) whereas $\Gamma_0$ may be empty.

In this article, it is proved that the solutions of each of the above closed loop systems (1.1)-(1.2) and (1.6) as well as (1.1), (1.3) and (1.6) asymptotically tend towards a constant depending on the corresponding initial data.

There is a rich literature concerning the stabilization problem of the wave equation with static boundary conditions (1.1)-(1.2) and (1.6) (see [2], [5]-[8], [26], [28]-[31], [33], [35]-[38] and the references therein). In all references cited above, at least one of the following conditions is assumed to be satisfied:

- the equation (1.1) involves also the displacement term $y$.
- the stabilizing feedback law $U(t)$ contains not only a boundary dissipation $y_t$ but also a boundary displacement $y$.
- the first boundary condition in (1.2) involves the displacement term $y$ (the boundary condition (1.2) is replaced, for instance, by $y = 0$ or $\partial_A y + y = 0$ on $\Gamma_0 \times (0,\infty)$).

In other words, the term $y$ is present in the closed-loop system. This type of assumption was impossible to circumvent for the stabilization of the wave equation because most of the authors defined the energy-norm of the system on the state space $H^1(\Omega) \times L^2(\Omega)$ as:

$$E_0(t) = \frac{1}{2} \int_\Omega \left( \sum_{i,j=1}^n a_{ij} \partial_i y \partial_j y + |y_t|^2 \right) dx,$$

which is only a semi-norm in our case. Note that J. Lagnese [27] has proved the energy decay of $E_0(t)$ in the case when $\Gamma_0 = \emptyset$ and $a_{ij} = \delta_{ij}$ for the closed-loop system (1.1)-(1.2) and (1.6). Nevertheless, the proof of this result is very technical and requires a preliminary result (see Theorem 2 in [27]).

We also point out that in the case when $a_{ij} = \delta_{ij}$, $m = 0$ and $M = 1$, it was shown in [39] and [40] that the closed loop system (1.1), (1.3) and (1.6) is strongly stable but under two restrictive conditions:

- the domain $\Omega$ is a bounded open set in $\mathbb{R}^2$ ($n = 2$).
- $y = 0$ on a part of the boundary $\Gamma = \partial\Omega$.

The main contribution of this paper is twofold: (i) to provide an alternative proof of Lagnese's result [27] by means of a simple and direct method and (ii) to extend the results of [13], where the one-dimensional equation is dealt with, and those of [39] and [40] to a general domain in $\mathbb{R}^n$ and a general operator $A$ without any boundary displacement term $y$ in the system. The key idea of the proof is to introduce the following new energy associated to the closed-loop system (1.1)-(1.2) and (1.6):

$$E(t) = \frac{1}{2} \left\{ \int_\Omega \left( \sum_{i,j=1}^n a_{ij} \partial_i y \partial_j y + |y_t|^2 \right) dx + \epsilon \left[ \int_\Omega y_t \, dx + \int_{\Gamma_1} ay \, d\sigma \right]^2 \right\}, \tag{1.8}$$

where $\epsilon > 0$. Note that $E(t)$ is a perturbation of the energy $E_0(t)$ used in literature. As mentioned above, the choice of the new energy $E(t)$ is justified by the fact that the usual energy function $E_0(t)$ only induces a semi-norm on the state space $H^1(\Omega) \times L^2(\Omega)$ for the closed-loop system (1.1)-(1.2) and (1.6). In fact, the constants are solutions of (1.1)-(1.2) and (1.6) and have null energy for $E_0(t)$. Moreover, an easy formal computation shows that

$$\dot{E}(t) = -\int_{\Gamma_1} a \, |y_t|^2 d\sigma \leq 0, \tag{1.9}$$

and thus the energy $E(t)$ is non-increasing.

Concerning the closed-loop system (1.1), (1.3) and (1.6), the following energy is considered:

$$E_d(t) \;=\; \frac{1}{2} \left\{ \int_\Omega \left( \sum_{i,j=1}^n a_{ij} \partial_i y \partial_j y + |y_t|^2 \right) dx + \int_{\Gamma_0} m |y_t|^2 \, d\sigma + \int_{\Gamma_1} M |y_t|^2 \, d\sigma \right.$$

$$+\mu\left[\int_{\Omega} y_t\,dx + \int_{\Gamma_0} m y_t\,d\sigma + \int_{\Gamma_1}\left(M y_t + a y\right)d\sigma\right]^2\Bigg\}, \tag{1.10}$$

where $\mu > 0$. A formal calculation gives

$$\dot{E}_d(t) = -\int_{\Gamma_1} a\,|y_t|^2 d\sigma \le 0, \tag{1.11}$$

and thus the energy $E_d(t)$ is non-increasing.

The main results of this article are:

*(i) For any initial data* $(y_0, z_0) \in H^1(\Omega) \times L^2(\Omega)$, *the solutions of the closed-loop system* (1.1)–(1.2) *and* (1.6) *satisfy:* $(y(t), y_t(t)) \longrightarrow (\chi, 0)$ *in* $H^1(\Omega) \times L^2(\Omega)$ *as* $t \longrightarrow \infty$, *where*

$$\chi = \left(\int_{\Gamma_1} a\,d\sigma\right)^{-1}\left\{\int_{\Omega} z_0\,dx + \int_{\Gamma_1} a y_0\,d\sigma\right\}. \tag{1.12}$$

*(ii) The solutions of the closed-loop system* (1.1), (1.3) *and* (1.6) *stemmed from any initial condition* $(y_0, z_0, w_0^0, w_1^0) \in H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma_0) \times L^2(\Gamma_1)$ *satisfy:*

$$(y(t), y_t(t), y_t|_{\Gamma_0}(t), y_t|_{\Gamma_1}(t)) \longrightarrow (\chi, 0, 0, 0) \text{ as } t \longrightarrow \infty,$$

*where* $\chi$ *is defined by* (1.12).

The nonlinear case of boundary control is also treated in this paper which is organized as follows. In the next section, the well-posedness and the asymptotic convergence of solutions for the wave-static boundary conditions system (1.1)-(1.2) and (1.6) are established. Section 3 is dedicated to the wave equation with dynamical boundary conditions. More precisely, we shall prove that such a system is well-posed in the sense of semigroups theory and its solutions converge asymptotically to a constant for both linear and nonlinear damping controls. Some of the results of these two sections have been partially announced without detailed proofs in [11, 12] for simpler systems. Section 4 is devoted to applications for several systems such as Petrovsky system, coupled wave-wave equations and elasticity systems. Finally, in the last section, some open problems are discussed.

## 2. THE WAVE EQUATION WITH STATIC BOUNDARY CONDITIONS

**2.1 Preliminaries and well-posedness of the problem.** In this subsection, we study the existence and uniqueness of the solutions of the closed-loop system (1.1)-(1.2) and (1.6). Consider the state space

$$\Upsilon = H^1(\Omega) \times L^2(\Omega),$$

equipped with the inner product

$$\langle (y,z),(\tilde{y},\tilde{z})\rangle_{\Upsilon} = \int_{\Omega}\left(\sum_{i,j=1}^{n} a_{ij}\partial_i y\partial_j\tilde{y} + z\tilde{z}\right)dx + \epsilon\left(\int_{\Omega} z\,dx + \int_{\Gamma_1} ay\,d\sigma\right)\left(\int_{\Omega}\tilde{z}\,dx + \int_{\Gamma_1} a\tilde{y}\,d\sigma\right), \quad (2.1)$$

where $\epsilon > 0$ is a constant to be determined. The first result is stated in the following proposition:

**Proposition 2.1.** *The state space $\Upsilon = H^1(\Omega)\times L^2(\Omega)$ endowed with the inner product* (2.1) *is a Hilbert space provided that $\epsilon$ is small enough.*

**Proof of Proposition 2.1.** It suffices to show that the norm $\|\cdot\|_{\Upsilon}$ induced by the inner product (2.1) is equivalent to the usual one $\|\cdot\|_{H^1(\Omega)\times L^2(\Omega)}$, that is, prove the existence of two positive constants $K$ and $\tilde{K}$ such that

$$K\|(y,z)\|_{H^1(\Omega)\times L^2(\Omega)} \le \|(y,z)\|_{\Upsilon} \le \tilde{K}\|(y,z)\|_{H^1(\Omega)\times L^2(\Omega)}. \quad (2.2)$$

On one hand, applying Hölder's inequality and using a trace Theorem [1] (see also [32]) and noting that there exists $\alpha_1 > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}\epsilon_i\epsilon_j \le \alpha_1 \sum_{i=1}^{n}\epsilon_i^2,\ \forall\ (\epsilon_1,\ldots,\epsilon_n)\in\mathbb{R}^n, \quad (2.3)$$

yields:

$$\|(y,z)\|_{\Upsilon}^2 \le \int_{\Omega}\left(\alpha_1|\nabla y|^2 + [1+2\epsilon\,\mathrm{vol}(\Omega)]|z|^2\right)dx + 2\epsilon||a||_{\infty}^2 C_1\mathrm{vol}(\Gamma_1)\left(\int_{\Omega}\left(|\nabla y|^2+|y|^2\right)\,dx\right),$$

where $\|\ \|_{\infty} = \|\ \|_{L^{\infty}}$ and $C_1$ is a positive constant depending on $\Omega$ (see [1] or [32]). Therefore the direct inequality of (2.2) holds for a positive constant $\tilde{K}$ depending on $C_1$, $\alpha_1$, $\epsilon$, $a$, $\mathrm{vol}(\Gamma_1)$ and $\mathrm{vol}(\Omega)$. For the reverse inequality, we proceed as follows:

$$\|(y,z)\|_{\Upsilon}^2 = \int_{\Omega}\left(\sum_{i,j=1}^{n} a_{ij}\partial_i y\partial_j y + |z|^2\right)dx + \epsilon\left(\int_{\Omega} z\,dx\right)^2 + \epsilon\left(\int_{\Gamma_1} ay\,d\sigma\right)^2 + 2\epsilon\left(\int_{\Gamma_1} ay\,d\sigma\right)\left(\int_{\Omega} z\,dx\right). \quad (2.4)$$

Obviously for any $\delta > 0$, Young's inequality gives

$$2\left(\int_{\Gamma_1} ay\,d\sigma\right)\left(\int_{\Omega} z\,dx\right) \ge -\delta^{-1}\left(\int_{\Omega} z\,dx\right)^2 - \delta\left(\int_{\Gamma_1} ay\,d\sigma\right)^2. \quad (2.5)$$

Combining (1.4), (2.4) and (2.5), we get

$$\|(y,z)\|_{\Upsilon}^2 \ge \int_{\Omega}\left(\alpha_0|\nabla y|^2 + |z|^2\right)dx + \epsilon(1-\delta^{-1})\left(\int_{\Omega} z\,dx\right)^2 + \epsilon(1-\delta)\left(\int_{\Gamma_1} ay\,d\sigma\right)^2. \quad (2.6)$$

Using a classical compactness argument, one can show the following generalized Poincaré inequality:

$$\int_\Omega |y|^2 dx \leq C_2 \left\{ \int_\Omega |\nabla y|^2 dx + \left( \int_{\Gamma_1} ay\, d\sigma \right)^2 \right\},$$

where $C_2 > 0$ depends on $\Omega$ and $a$. This implies that

$$\left( \int_{\Gamma_1} ay\, d\sigma \right)^2 \geq \frac{1}{C_2} \int_\Omega |y|^2\, dx - \int_\Omega |\nabla y|^2 dx. \tag{2.7}$$

Inserting (2.7) into (2.6), it follows that

$$\begin{aligned} \|(y,z)\|_\Upsilon^2 \;\geq\; & \int_\Omega \left\{ \left[\epsilon(1-\delta)C_2^{-1}\right] |y|^2 + \left[\alpha_0 - \epsilon(1-\delta)\right] |\nabla y|^2 + |z|^2 \right\} dx \\ & + \epsilon(\delta - 1)\delta^{-1} \left( \int_\Omega z\, dx \right)^2, \end{aligned} \tag{2.8}$$

for any $0 < \delta < 1$. Finally, applying again Hölder's inequality to the last term, one can show the existence of a positive constant $K$ depending on $C_2$, $\alpha_0$, $a$, $\delta$ and $\mathrm{vol}(\Omega)$ such that the reverse inequality of (2.2) holds, provided that $0 < \delta < 1$ and $\epsilon$ satisfies the following condition

$$0 < \epsilon < \min \left\{ \frac{\alpha_0}{1-\delta}, \frac{\delta}{(1-\delta)\mathrm{vol}(\Omega)} \right\}.$$

This concludes the proof of Proposition 2.1. $\diamond$

We turn now to the formulation of the closed-loop system (1.1)-(1.2) and (1.6) in an abstract form in $\Upsilon$. Let $z(t) = y_t(t)$ and $\Phi(t) = (y(t), z(t))$. Then, the closed loop system can be written as follows

$$\begin{cases} \Phi_t(t) = \mathbb{T}\Phi(t), \\ \Phi(0) = \Phi_0 = (y_0, z_0), \end{cases} \tag{2.9}$$

where $\mathbb{T}$ is an unbounded linear operator defined by:

$$D(\mathbb{T}) = \left\{ (y,z) \in H^1(\Omega) \times H^1(\Omega); Ay \in L^2(\Omega); \partial_A y = 0 \text{ on } \Gamma_0, \partial_A y + az = 0 \text{ on } \Gamma_1 \right\} \tag{2.10}$$

and for any $(y,z) \in D(\mathbb{T})$,

$$\mathbb{T}(y,z) = (z, Ay). \tag{2.11}$$

Now we are able to state a well-posedness result for the closed-loop system (2.9):

**Lemma 2.2.** *(i) The linear operator $\mathbb{T}$, defined by (2.10)–(2.11), generates a $C_0$ semigroup of contractions $S(t)$ on $\Upsilon = \overline{D(\mathbb{T})}$.*
*(ii) For any initial data $\Phi_0 = (y_0, z_0) \in D(\mathbb{T})$, the system (2.9) admits a unique strong solution $\Phi(t) = (y(t), y_t(t)) = S(t)\Phi_0 \in D(\mathbb{T})$ for all $t \geq 0$ such that $\Phi = (y, y_t) \in C^1(\mathbb{R}^+; \Upsilon) \cap C(\mathbb{R}^+; D(\mathbb{T}))$. Moreover, the function $t \longmapsto \|\mathbb{T}\Phi(t)\|_\Upsilon$ is non-increasing.*
*(iii) For any initial data $\Phi_0 = (y_0, z_0) \in \Upsilon$, the system (2.9) has a unique weak*

*solution* $\Phi(t) = (y(t), y_t(t)) = S(t)\Phi_0 \in \Upsilon$ *for all* $t \geq 0$ *such that* $\Phi = (y, y_t) \in C(\mathbb{R}^+; \Upsilon)$.

**Proof of Lemma 2.2.** (i) Let $\phi = (y, z) \in D(\mathbb{T})$. Using Green formula, one can obtain after a straightforward computation

$$< \mathbb{T}(y,z), (y,z) >_\Upsilon = -\int_{\Gamma_1} a|z|^2 \, d\sigma \leq 0. \tag{2.12}$$

Therefore $-\mathbb{T}$ is monotone. Moreover, using Lax-Milgram Theorem [4], one can prove that $-\mathbb{T}$ is maximal, that is, range$(I - \mathbb{T}) = \Upsilon$. Thus, Lummer-Phillips Theorem [34] leads us to claim that $\mathbb{T}$ generates a $C_0$ semigroup of contractions $S(t)$ on $\Upsilon = \overline{D(\mathbb{T})}$.
(ii)–(iii) These claims are direct consequences of semigroup theory [34]. ⋄

**2.2 Asymptotic behavior.** In this subsection, we will show an asymptotic behavior result for the unique solution of the system (2.9) in $\Upsilon$. To do so, we shall first show the following lemma:

**Lemma 2.3.** *The resolvent operator* $(\lambda I - \mathbb{T})^{-1} : \Upsilon \to \Upsilon$ *is compact for any* $\lambda > 0$ *and hence the canonical embedding* $i : D(\mathbb{T}) \to \Upsilon$ *is compact, where* $D(\mathbb{T})$ *is equipped with the graph norm.*

**Proof of Lemma 2.3.** Based on the proof of Lemma 2.2 and using Sobolev embedding, one deduces that $(I - \mathbb{T})^{-1}$ is compact. The proof of Lemma 2.3 follows then from the well-known result of Kato [24]. ⋄

The first main result of this paper is:

**Theorem 2.4.** *For any initial data* $\Phi_0 = (y_0, z_0) \in \Upsilon$, *the solution* $\Phi(t) = (y(t), y_t(t))$ *of* (2.9) *tends in* $\Upsilon$ *to* $(\chi, 0)$ *as* $t \longrightarrow \infty$, *where*

$$\chi = \left(\int_{\Gamma_1} a \, d\sigma\right)^{-1} \left(\int_\Omega z_0 \, dx + \int_{\Gamma_1} a y_0 \, d\sigma\right).$$

**Proof of Theorem 2.4.** Using the density of $D(\mathbb{T}^2)$ in $\Upsilon$ and the contraction of the semigroup $S(t)$, it suffices to prove Theorem 2.4 for smooth initial data $\Phi_0 = (y_0, z_0) \in D(\mathbb{T}^2)$. Let $\Phi(t) = (y(t), y_t(t)) = S(t)\Phi_0$ be the solution of (2.9). It follows from the second part of Lemma 2.2 that the trajectory of solution $\{\Phi(t)\}_{t\geq 0}$ is a bounded set for the graph norm and thus precompact by virtue of Lemma 2.3. Applying LaSalle's principle, we deduce that $\omega(\Phi_0)$ is non empty, compact, invariant under the semigroup $S(t)$ and in addition $S(t)\Phi_0 \longrightarrow \omega(\Phi_0)$ as $t \to \infty$ [23]. In order to prove the result, it suffices to show that $\omega(\Phi_0)$ reduces to $(\chi, 0)$. To this end, let $\tilde{\Phi}_0 = (\tilde{y}_0, \tilde{z}_0) \in \omega(\Phi_0) \subset D(\mathbb{T})$ and $\tilde{\Phi}(t) = (\tilde{y}(t), \tilde{y}_t(t)) = S(t)\tilde{\Phi}_0 \in D(\mathbb{T})$ the unique strong solution of (2.9). Recall that it is well-known that $\|\tilde{\Phi}(t)\|_\Upsilon$ is constant [23] and thus $\frac{d}{dt}\left(\|\tilde{\Phi}(t)\|_\Upsilon^2\right) = 0$, i.e.,

$$\langle \mathbb{T}\tilde{\Phi}, \tilde{\Phi}\rangle_\Upsilon = -\int_{\Gamma_1} a|z|^2\, d\sigma = 0. \tag{2.13}$$

This implies that $\tilde{z} = \tilde{y}_t = 0$ on $\Gamma_1$ and therefore $\tilde{y}$ is solution of the following system:

$$\begin{cases} \tilde{y}_{tt} - A\tilde{y} = 0, & \text{in } \Omega \\ \partial_A \tilde{y} = 0, & \text{on } \Gamma_0 \\ \tilde{y}_t = \partial_A \tilde{y} = 0, & \text{on } \Gamma_1 \\ \tilde{y}(0) = \tilde{y}_0;\ \tilde{y}_t(0) = \tilde{z}_0, & \text{in } \Omega \\ \tilde{y} \in H^1(\Omega);\ A\tilde{y} \in L^2(\Omega), \end{cases} \tag{2.14}$$

and $\tilde{z} = \tilde{y}_t$ is solution of

$$\begin{cases} \tilde{z}_{tt} - A\tilde{z} = 0, & \text{in } \Omega \\ \partial_A \tilde{z} = 0, & \text{on } \Gamma_0 \\ \tilde{z} = \partial_A \tilde{z} = 0, & \text{on } \Gamma_1 \\ \tilde{z}(0) = \tilde{z}_0,\ \tilde{z}_t(0) = A\tilde{y}_0, & \text{in } \Omega. \end{cases} \tag{2.15}$$

Obviously, to deduce the desired result, it suffices to show that $\tilde{y} = \mathit{constant}$ is the only solution of (2.14). To do so, we first use the standard Holmgren's uniqueness theorem for the system (2.15) to conclude that $\tilde{z} \equiv 0$. Thus the system (2.14) is reduced to an elliptic problem:

$$\begin{cases} A\tilde{y} = 0, & \text{in } \Omega \\ \partial_A \tilde{y} = 0, & \text{on } \Gamma_0 \\ \partial_A \tilde{y} = 0, & \text{on } \Gamma_1, \end{cases}$$

which clearly yields that $\tilde{y} \equiv$ constant. Thus, we have proved that for any $\tilde{\Phi}_0 = (\tilde{y}_0, \tilde{z}_0) \in \omega(\Phi_0) \subset D(\mathbb{T})$, the solution $\tilde{\Phi}(t) = (\tilde{y}(t), \tilde{y}_t(t)) = S(t)\tilde{\Phi}_0 \in D(\mathbb{T})$ satisfies $(\tilde{y}(t), \tilde{y}_t(t)) = (\chi, 0)$, for any $t \geq 0$, where $\chi$ is a constant. In particular, $\tilde{\Phi}_0 = (\tilde{y}_0, \tilde{z}_0) = (\chi, 0)$ and hence the $\omega$-limit set $\omega(\Phi_0)$ consists of constants $(\chi, 0)$. Now, we shall find the explicit expression of the constant $\chi$. Let $(\chi, 0) \in \omega(\Phi_0)$. This implies that there exists $\{t_n\} \to \infty$, as $n \to \infty$ such that

$$\Phi(t_n) = (y(t_n), y_t(t_n)) = S(t_n)\Phi_0 \longrightarrow (\chi, 0) \tag{2.16}$$

in the state space $\Upsilon = H^1(\Omega) \times L^2(\Omega)$. Furthermore, any solution of the closed-loop system (1.1)-(1.2) and (1.6) stemmed from $\Phi_0 = (y_0, z_0)$ verifies

$$\frac{d}{dt}\left\{\int_\Omega y_t(x,t)\, dx + \int_{\Gamma_1} a y(x,t)\, d\sigma\right\} = 0.$$

(This can be obtained by integrating $y_{tt}(x,t) - Ay(x,t) = 0$ with respect to $x$, then using Green formula and finally using the boundary conditions (1.2) and (1.6).) Hence $\int_\Omega y_t(x,t)\,dx + \int_{\Gamma_1} ay(x,t)\,d\sigma$ is constant and so

$$\begin{aligned}\int_\Omega y_t(x,t)\,dx + \int_{\Gamma_1} ay(x,t)\,d\sigma &= \int_\Omega y_t(x,0)\,dx + \int_{\Gamma_1} ay(x,0)\,d\sigma \\ &= \int_\Omega z_0(x)\,dx + \int_{\Gamma_1} ay_0(x)\,d\sigma. \end{aligned} \tag{2.17}$$

Finally, let $t = t_n$ in (2.17) with $n \to \infty$ and use (2.16) to obtain:

$$\chi = \left(\int_{\Gamma_1} a\,d\sigma\right)^{-1}\left(\int_\Omega z_0\,dx + \int_{\Gamma_1} ay_0\,d\sigma\right).$$

This achieves the proof of the theorem. $\diamond$

## 3. THE WAVE EQUATION WITH DYNAMICAL BOUNDARY CONDITIONS

In this section, we treat the case when the boundary conditions are dynamical. Indeed, we shall consider both linear and nonlinear damping control.

### 3.1 Dynamical boundary conditions with a linear damping control.

**3.1.1 Well-posedness of the problem.** In this part, we study the well-posedness of the problem (1.1), (1.3) and (1.6). To do so, let us consider the state space

$$\Upsilon_d = H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma_0) \times L^2(\Gamma_1),$$

equipped with the inner product

$$\begin{aligned}\Big\langle (y,z,w_0,w_1),(\tilde{y},\tilde{z},\tilde{w}_0,\tilde{w}_1)\Big\rangle_{\Upsilon_d} &= \int_\Omega \left(\sum_{i,j=1}^n a_{ij}\partial_i y \partial_j \tilde{y} + z\tilde{z}\right)dx + \int_{\Gamma_0} mw_0\tilde{w}_0\,d\sigma + \\ \int_{\Gamma_1} Mw_1\tilde{w}_1\,d\sigma + \mu&\left(\int_\Omega z\,dx + \int_{\Gamma_0} mw_0\,d\sigma + \int_{\Gamma_1}(Mw_1 + ay)\,d\sigma\right) \\ &\times\left(\int_\Omega \tilde{z}\,dx + \int_{\Gamma_0} m\tilde{w}_0\,d\sigma + \int_{\Gamma_1}(M\tilde{w}_1 + a\tilde{y})\,d\sigma\right),\end{aligned} \tag{3.1}$$

where $\mu > 0$ is a constant to be determined. We have the following proposition:

**Proposition 3.1.** *The state space $\Upsilon_d$ equipped with the inner product* (3.1) *is a Hilbert space provided that $\mu$ is small enough.*

**Proof of Proposition 3.1.** We just need to show that the norm $\|\cdot\|_{\Upsilon_d}$ induced by the inner product (3.1) is equivalent to the usual one denoted by $\|\cdot\|$, that is, prove the existence of two positive constants $K$ and $\tilde{K}$ such that

$$K\|(y,z,w_0,w_1)\| \le \|(y,z,w_0,w_1)\|_{\Upsilon_d} \le \tilde{K}\|(y,z,w_0,w_1)\|. \tag{3.2}$$

On one hand, using (1.5) and applying Cauchy-Schwartz inequality yields:

$$\begin{aligned}\|(y,z,w_0,w_1)\|^2_{\Upsilon_d} &\leq \int_\Omega \left(\sum_{i,j=1}^n a_{ij}\partial_i y \partial_j y + |z|^2\right) dx + \|m\|_\infty \int_{\Gamma_0} |w_0|^2\, d\sigma \\ &+\|M\|_\infty \int_{\Gamma_1} |w_1|^2\, d\sigma + 4\mu \left(\int_\Omega z\, dx\right)^2 + 4\mu \left(\int_{\Gamma_0} m w_0\, d\sigma\right)^2 \\ &+4\mu \left(\int_{\Gamma_1} M w_1\, d\sigma\right)^2 + 4\mu \left(\int_{\Gamma_1} a y\, d\sigma\right)^2. \qquad (3.3)\end{aligned}$$

Then we apply Hölder's inequality for the four last terms of the right-hand side of (3.3) and use (1.5), (1.7) to get

$$\begin{aligned}\|(y,z,w_0,w_1)\|^2_{\Upsilon_d} &\leq \int_\Omega \left[\alpha_1 |\nabla y|^2 + (1+4\mu\, \mathrm{vol}(\Omega))|z|^2\right] dx + C_m \int_{\Gamma_0} |w_0|^2\, d\sigma \\ &+C_M \int_{\Gamma_1} |w_1|^2\, d\sigma + 4\mu \|a\|^2_\infty \mathrm{vol}(\Gamma_1) \left(\int_{\Gamma_1} |y|^2\, d\sigma\right),\end{aligned}$$

where $C_m = \|m\|_\infty \left[1+4\mu\|m\|_\infty \mathrm{vol}(\Gamma_0)\right]$, $C_M = \|M\|_\infty \left[1+4\mu\|M\|_\infty \mathrm{vol}(\Gamma_1)\right]$ and $\alpha_1$ is defined in (2.3). On the other hand, using a trace Theorem [32], the above inequality becomes

$$\begin{aligned}\|(y,z,w_0,w_1)\|^2_{\Upsilon_d} &\leq \left(\alpha_1 + 4\mu\|a\|^2_\infty C_1 \mathrm{vol}(\Gamma_1)\right) \int_\Omega |\nabla y|^2\, dx \\ &+4\mu\|a\|^2_\infty C_1 \mathrm{vol}(\Gamma_1) \int_\Omega |y|^2\, dx + (1+4\mu\, \mathrm{vol}(\Omega)) \int_\Omega |z|^2\, dx \\ &+C_m \int_{\Gamma_0} |w_0|^2\, d\sigma + C_M \int_{\Gamma_1} |w_1|^2\, d\sigma,\end{aligned}$$

where $C_1$ is a positive constant depending on $\Omega$ (see [32]). Therefore

$$\|(y,z)\|^2_{\Upsilon_d} \leq \tilde{K} \|(y,z)\|^2_{H^1(\Omega)\times L^2(\Omega)},$$

where $\tilde{K}$ is a positive constant depending on $\alpha_1$, $\mu$, $\|a\|_\infty$, $\|m\|_\infty$, $\|M\|_\infty$, $\mathrm{vol}(\Gamma_i)$, $i=0,1$ and $\mathrm{vol}(\Omega)$. For the reverse inequality, we proceed as follows:

$$\begin{aligned}\|(y,z,w_0,w_1)\|^2_{\Upsilon_d} &= \int_\Omega \left(\sum_{i,j=1}^n a_{ij}\partial_i y \partial_j y + |z|^2\right) dx + \int_{\Gamma_0} m|w_0|^2\, d\sigma + \int_{\Gamma_1} M|w_1|^2\, d\sigma \\ &+\mu \left[\int_\Omega z\, dx + \int_{\Gamma_0} m w_0\, d\sigma + \int_{\Gamma_1} (M w_1 + a y)\, d\sigma\right]^2 \qquad (3.4) \\ &+\mu \left[\int_{\Gamma_1} a y\, d\sigma\right]^2 + 2\mu \left[\int_{\Gamma_1} a y\, d\sigma\right] \\ &\quad \times \left[\int_\Omega z\, dx + \int_{\Gamma_0} m w_0\, d\sigma + \int_{\Gamma_1} (M w_1 + a y)\, d\sigma\right].\end{aligned}$$

Applying Cauchy-Schwartz inequality to the last term yields

$$\begin{aligned}
&2\left[\int_{\Gamma_1} ay\,d\sigma\right]\left[\int_\Omega z\,dx+\int_{\Gamma_0} mw_0\,d\sigma+\int_{\Gamma_1}(Mw_1+ay)\,d\sigma\right]\\
&\geq -\delta^{-1}\left(\int_\Omega z\,dx+\int_{\Gamma_0} mw_0\,d\sigma+\int_{\Gamma_1}(Mw_1+ay)\,d\sigma\right)^2-\delta\left(\int_{\Gamma_1} ay\,d\sigma\right)^2,
\end{aligned}\tag{3.5}$$

for any $\delta>0$. Now, using (1.4), (1.5) and combining (3.4) and (3.5), we get

$$\begin{aligned}
&\|((y,z,w_0,w_1)\|^2_{\Upsilon_d}\geq \int_\Omega\left(\alpha_0|\nabla y|^2+|z|^2\right)dx+m_0\int_{\Gamma_0}|w_0|^2\,d\sigma+M_1\int_{\Gamma_1}|w_1|^2\,d\sigma\\
&+\mu\left(1-\delta^{-1}\right)\left(\int_\Omega z\,dx+\int_{\Gamma_0} mw_0\,d\sigma+\int_{\Gamma_1}(Mw_1+ay)\,d\sigma\right)^2\\
&+\mu(1-\delta)\left(\int_{\Gamma_1} ay\,d\sigma\right)^2.
\end{aligned}\tag{3.6}$$

Choosing $0<\delta<1$ and applying Cauchy-Schwartz and Hölder's inequalities for the term $\left(\int_\Omega z\,dx+\int_{\Gamma_0} mw_0\,d\sigma+\int_{\Gamma_1}(Mw_1+ay)\,d\sigma\right)^2$ of the right-hand side of (3.6), we obtain

$$\begin{aligned}
\|(y,z,w_0,w_1)\|^2_{\Upsilon_d} \;\geq\; &\int_\Omega \alpha_0|\nabla y|^2\,dx+\left(1-3\mu(1-\delta)\delta^{-1}\operatorname{vol}(\Omega)\right)\int_\Omega|z|^2\,dx\\
&+m_0\left(1-3\mu m_0(1-\delta)\delta^{-1}\operatorname{vol}(\Gamma_0)\right)\int_{\Gamma_0}|w_0|^2\,d\sigma\\
&+M_1\left(1-3\mu M_1(1-\delta)\delta^{-1}\operatorname{vol}(\Gamma_1)\right)\int_{\Gamma_1}|w_1|^2\,d\sigma\\
&+\mu(1-\delta)\left(\int_{\Gamma_1} ay\,d\sigma\right)^2.
\end{aligned}\tag{3.7}$$

Inserting (2.7) into (3.7) yields

$$\begin{aligned}
\|(y,z,w_0,w_1)\|^2_{\Upsilon_d} \;\geq\; &\mu(1-\delta)a_0^2C_2^{-1}\int_\Omega|y|^2\,dx+\left(\alpha_0-\mu(1-\delta)a_0^2\right)\int_\Omega|\nabla y|^2\,dx\\
&+\left(1-3\mu(1-\delta)\delta^{-1}\operatorname{vol}(\Omega)\right)\int_\Omega|z|^2\,dx\\
&+m_0\left(1-3\mu m_0(1-\delta)\delta^{-1}\operatorname{vol}(\Gamma_0)\right)\int_{\Gamma_0}|w_0|^2\,d\sigma\\
&+M_1\left(1-3\mu M_1(1-\delta)\delta^{-1}\operatorname{vol}(\Gamma_1)\right)\int_{\Gamma_1}|w_1|^2\,d\sigma,
\end{aligned}$$

for any $0<\delta<1$. Finally, provided that $\mu$ satisfies the following condition

$$0<\mu<\min\left(\frac{\alpha_0}{(1-\delta)a_0^2},\frac{\delta}{3(1-\delta)\operatorname{vol}(\Omega)},\frac{\delta}{3(1-\delta)m_0\operatorname{vol}(\Gamma_0)},\frac{\delta}{3(1-\delta)M_1\operatorname{vol}(\Gamma_1)}\right),$$

one can provide a positive constant $K$ depending on $\alpha_0$, $m_0, a_0, M_1$, $\delta$, $C_2$, $\operatorname{vol}(\Omega)$ and $\operatorname{vol}(\Gamma_i)$, $i=0,1$ such that the left inequality in (3.2) holds. This concludes the proof of Proposition 3.1. ◇

We turn now to the formulation of the closed-loop system (1.1), (1.3) and (1.6) in an abstract form on $\Upsilon_d$. Setting $z = y_t, w_0 = z|_{\Gamma_0}, w_1 = z|_{\Gamma_1}$ and $\Phi(t) = (y(t), z(t), w_0(t), w_1(t))$, the closed loop system can be written into the following form:

$$\begin{cases} \Phi_t(t) = \mathbb{T}_d \Phi(t), \\ \Phi(0) = \Phi_0 = (y_0, z_0, w_0^0, w_1^0), \end{cases} \tag{3.8}$$

where $\mathbb{T}_d$ is an unbounded linear operator defined by:

$$\begin{gathered} D(\mathbb{T}_d) = \{(y, z, w_0, w_1) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Gamma_0) \times L^2(\Gamma_1); \\ Ay \in L^2(\Omega),\ w_0 = z|_{\Gamma_0},\ w_1 = z|_{\Gamma_1}\}, \end{gathered} \tag{3.9}$$

and for any $(y, z, w_0, w_1) \in D(\mathbb{T}_d)$,

$$\mathbb{T}_d(y, z, w_0, w_1) = \left( z, Ay, -\frac{1}{m}\partial_A y, -\frac{1}{M}\left(a w_1 + \partial_A y\right) \right). \tag{3.10}$$

The well-posedness result for the closed-loop system (3.8) is:

**Lemma 3.2.** *(i) The linear operator $\mathbb{T}_d$, defined by (3.9)–(3.10), generates a $C_0$ semigroup of contractions $S_d(t)$ on $\Upsilon_d = \overline{D(\mathbb{T}_d)}$.*
*(ii) For any initial data $\Phi_0 = (y_0, z_0, w_0^0, w_1^0) \in D(\mathbb{T}_d)$, the system (3.8) admits a unique strong solution $\Phi(t) = (y(t), y_t(t), w_0(t), w_1(t)) = S_d(t)\Phi_0 \in D(\mathbb{T}_d)$ for all $t \geq 0$ satisfying $\Phi \in C^1(\mathbb{R}^+; \Upsilon_d) \cap C(\mathbb{R}^+; D(\mathbb{T}_d))$. Moreover, the function $t \longmapsto \|\mathbb{T}_d \Phi(t)\|_{\Upsilon_d}$ is non-increasing.*
*(iii) For any initial data $\Phi_0 = (y_0, z_0, w_0^0, w_1^0) \in \Upsilon_d$, the system (3.8) has a unique weak solution $\Phi(t) = (y(t), y_t(t), w_0(t), w_1(t)) = S_d(t)\Phi_0 \in \Upsilon_d$ for all $t \geq 0$ verifying $\Phi \in C(\mathbb{R}^+; \Upsilon_d)$.*

**Proof of Lemma 3.2.** (i) Let $\phi = (y, z, w_0, w_1) \in D(\mathbb{T}_d)$. Using Green formula, one can obtain after a straightforward computation

$$\langle \mathbb{T}_d(y, z, w_0, w_1), (y, z, w_0, w_1)\rangle_{\Upsilon_d} = -\int_{\Gamma_1} a|w_1|^2 \, d\sigma \leq 0. \tag{3.11}$$

Therefore $-\mathbb{T}_d$ is monotone. Now, given $(f, g, \xi, \eta) \in \Upsilon_d$, we seek $(y, z, w_0, w_1) \in D(\mathbb{T}_d)$ solution of the equation $(I - \mathbb{T}_d)(y, z, w_0, w_1) = (f, g, \xi, \eta)$, that is,

$$\begin{cases} y - z = f, & \text{in } \Omega \\ z - Ay = g, & \text{in } \Omega \\ w_0 + \dfrac{1}{m}\partial_A y = \xi, & \text{on } \Gamma_0, \\ w_1 + \dfrac{1}{M}\left(a w_1 + \partial_A y\right) = \eta, & \text{on } \Gamma_1. \end{cases}$$

Then eliminating $z$ and using $w_0 = z|_{\Gamma_0}$, $w_1 = z|_{\Gamma_1}$, we find that $y$ satisfies the system

$$\begin{cases} Ay - y = -(f+g) \in L^2(\Omega), & \text{in } \Omega \\ y + \dfrac{1}{m}\partial_A y = \xi, & \text{on } \Gamma_0 \\ \left(1+\dfrac{a}{M}\right) y + \dfrac{1}{M}\partial_A y = \left(1+\dfrac{a}{M}\right) f + \eta, & \text{on } \Gamma_1. \end{cases} \tag{3.12}$$

Using Green formula, one can prove that the system (3.12) is equivalent to the following variational equation:

$$\begin{aligned} &\int_\Omega \left( y\psi + \sum_{i,j=1}^n a_{ij}\partial_i y \partial_j \psi \right) dx + \int_{\Gamma_0} my\psi \, d\sigma + \int_{\Gamma_1} (M+a) y\psi \, d\sigma \\ &= \int_\Omega (f+g)\,\psi \, dx + \int_{\Gamma_0} m\xi\psi \, d\sigma + \int_{\Gamma_1} (M\eta + (M+a)f)\,\psi \, d\sigma, \end{aligned} \tag{3.13}$$

for any $\psi \in H^1(\Omega)$. Thanks to Lax-Milgram Theorem [4], one can prove that (3.13) admits a unique solution $y \in H^1(\Omega)$. Then defining $z = y - f$, $w_0 = z|_{\Gamma_0}$ and $w_1 = z|_{\Gamma_1}$, we find that the element $(y, z, w_0, w_1) \in D(\mathbb{T}_d)$ and is solution of the equation $(I - \mathbb{T}_d)(y, z, w_0, w_1) = (f, g, \xi, \eta)$. Thus $-\mathbb{T}$ is maximal, that is, range$(I - \mathbb{T}_d) = \Upsilon_d$. Finally, Lummer-Phillips theorem [34] permits us to claim that $\mathbb{T}_d$ generates a $C_0$ semigroup of contractions $S_d(t)$ on $\Upsilon_d = \overline{D(\mathbb{T}_d)}$.

(ii)–(iii) These claims follow from semigroups theory [34]. $\diamond$

**3.1.2 Asymptotic behavior for the solution of (3.8).** We will now show an asymptotic behavior result for the unique solution of (3.8) in $\Upsilon_d$. To do so, one can use Lemma 3.2 and Sobolev embedding to show that the canonical embedding $i : D(\mathbb{T}_d) \to \Upsilon_d$ is compact, where $D(\mathbb{T}_d)$ is equipped with the graph norm. Therefore $(I - \mathbb{T}_d)^{-1}$ is compact and the spectrum of $\mathbb{T}_d$ consists of only isolated eigenvalues with finite multiplicity [24].

The second main result of this paper is:

**Theorem 3.3.** *For any initial data $\Phi_0 = (y_0, z_0, w_0^0, w_1^0) \in \Upsilon_d$, the solution $\Phi(t) = (y(t), y_t(t), w_0(t), w_1(t))$ of (3.8) tends in $\Upsilon_d$ to $(\chi, 0, 0, 0)$ as $t \longrightarrow \infty$, where*

$$\chi = \left( \int_{\Gamma_1} a \, d\sigma \right)^{-1} \left\{ \int_\Omega z_0 \, dx + \int_{\Gamma_1} a y_0 \, d\sigma \right\}.$$

**Proof of Theorem 3.3.** By a standard argument of density of $D(\mathbb{T}_d^2)$ in $\Upsilon_d$ and the contraction of the semigroup $S_d(t)$, it suffices to prove Theorem 3.3 for smooth initial data $\Phi_0 = (y_0, z_0, w_0^0, w_1^0) \in D(\mathbb{T}^2)$. Let $\Phi(t) = (y(t), y_t(t), w_0(t), w_1(t)) = S_d(t)\Phi_0$ be the solution of (3.8). It follows from Lemma 3.2 (ii) that the trajectory of solution $\{\Phi(t)\}_{t\geq 0}$ is a bounded set for the graph norm and thus precompact by virtue of the compactness of the operator $(I - \mathbb{T}_d)^{-1}$. Applying LaSalle's principle, we deduce that $\omega(\Phi_0)$ is non empty, compact, invariant under the semigroup $S_d(t)$

and in addition $S_d(t)\Phi_0 \longrightarrow \omega(\Phi_0)$ as $t \to \infty$ [23]. In order to prove the strong stability, it suffices to show that $\omega(\Phi_0)$ reduces to $(\chi, 0, 0, 0)$. To this end, let $\tilde{\Phi}_0 = (\tilde{y}_0, \tilde{z}_0, \tilde{w}_0, \tilde{w}_1) \in \omega(\Phi_0) \subset D(\mathbb{T}_d)$ and $\tilde{\Phi}(t) = (\tilde{y}(t), \tilde{y}_t(t), \tilde{w}_0(t), \tilde{w}_1(t)) = S_d(t)\tilde{\Phi}_0 \in D(\mathbb{T}_d)$ the unique strong solution of (3.8). Recall that it is well-known that $\|\tilde{\Phi}(t)\|_{\Upsilon_d} = \sqrt{2E_d(t)}$ (see (1.10)) is constant [23] and thus $\frac{d}{dt}\left(\|\tilde{\Phi}(t)\|^2_{\Upsilon_d}\right) = 0$, i.e,

$$\dot{E}_d(t) = \langle \mathbb{T}_d\tilde{\Phi}, \tilde{\Phi}\rangle_{\Upsilon_d} = 0. \tag{3.14}$$

This, together with (3.11) (see also (1.11)), implies that $w_1 = y_t|_{\Gamma_1} = 0$ and therefore $\tilde{y}$ is solution of the following system:

$$\begin{cases} \tilde{y}_{tt} - A\tilde{y} = 0, & \text{in}\,\Omega \\ m\tilde{y}_{tt} + \partial_A\tilde{y} = 0, & \text{on}\,\Gamma_0, \\ \tilde{y}_t = \partial_A\tilde{y} = 0, & \text{on}\,\Gamma_1, \\ \tilde{y}(0) = \tilde{y}_0;\ \tilde{y}_t(0) = \tilde{z}_0, & \text{in}\,\Omega \\ \tilde{y} \in H^1(\Omega);\ A\tilde{y} \in L^2(\Omega). \end{cases} \tag{3.15}$$

A straightforward computation shows that $\tilde{z} = \tilde{y}_t$ is solution of

$$\begin{cases} \tilde{z}_{tt} - A\tilde{z} = 0, & \text{in}\,\Omega \\ \tilde{z} = \partial_A\tilde{z} = 0, & \text{on}\,\Gamma_0, \\ \tilde{z} = \partial_A\tilde{z} = 0, & \text{on}\,\Gamma_1, \\ \tilde{z}(0) = \tilde{z}_0;\ \tilde{z}_t(0) = A\tilde{y}_0. & \text{in}\,\Omega. \end{cases} \tag{3.16}$$

Obviously, to deduce the desired result, it suffices to show that $\tilde{y} =$ constant is the only solution of (3.15). To do so, we first use the standard Holmgren's uniqueness theorem for the system (3.16) to conclude that $\tilde{z} = 0$. Thus the system (3.15) is reduced to an elliptic problem:

$$\begin{cases} A\tilde{y} = 0, & \text{in}\,\Omega \\ \partial_A\tilde{y} = 0, & \text{on}\,\Gamma_0, \\ \partial_A\tilde{y} = 0, & \text{on}\,\Gamma_1, \end{cases}$$

which clearly yields that $\tilde{y}$ is constant. This, together with (3.14), implies the desired result in the same way as for the proof of Theorem 2.4. $\diamond$

**Remark 3.4.** Integrating with respect to $x$ and $t$ and using Green formula for the closed loop system (1.1)-(1.2) and (1.6) (resp. (1.1), (1.3) and (1.6)), we obtain the following identity:

$$\int_\Omega y_t\,dx + \int_{\Gamma_1} ay\,d\sigma = \int_\Omega z_0\,dx + \int_{\Gamma_1} ay_0\,d\sigma.$$

Therefore, if the initial values ($y_0$ and $z_0$) satisfy the additional condition

$$\int_\Omega z_0\,dx + \int_{\Gamma_1} ay_0\,d\sigma = 0,$$

then the constant $\chi$ of Theorem 2.4 (resp. Theorem 3.3) is zero. In other words, the energy defined by (1.8) (resp. (1.10)) tends to 0 as $t \longrightarrow \infty$.

**3.2 Dynamical boundary conditions with a nonlinear damping control.** The aim of this subsection is to extend the previous results to the case of nonlinear feedback control. For sake of simplicity and without loss of generality, we shall consider the system with constant coefficients

$$\begin{cases} y_{tt}(x,t) - \Delta y(x,t) = 0, & \text{in } \Omega \times (0,\infty) \\ m y_{tt}(x,t) + \partial_\nu y(x,t) = 0, & \text{on } \Gamma_0 \times (0,\infty) \\ M y_{tt}(x,t) + \partial_\nu y(x,t) = -f(y_t(x,t)), & \text{on } \Gamma_1 \times (0,\infty) \\ y(x,0) = y_0(x) \in H^1(\Omega),\ y_t(x,0) = z_0(x) \in L^2(\Omega), & \\ y_t|_{\Gamma_0}(x,0) = w_0^0(x) \in L^2(\Gamma_0),\ y_t|_{\Gamma_1}(x,0) = w_1^0(x) \in L^2(\Gamma_1), & \end{cases} \tag{3.17}$$

where $f$ satisfies the classical assumptions, namely, $f$ is a non-decreasing continuous function such that $f(0) = 0$. Here and in the sequel, $\partial_\nu$ denotes the normal derivative.

Then, one can check that, although with such a function $f$, the energy defined in (1.10) is not necessarily non-increasing. Hence a new methodology should be adopted. Indeed, we shall consider, as in the previous subsection, our state space

$$\Upsilon_d = H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma_0) \times L^2(\Gamma_1),$$

but equipped with the new inner product

$$\begin{aligned} &\Big\langle (y,z,w_0,w_1), (\tilde{y},\tilde{z},\tilde{w}_0,\tilde{w}_1) \Big\rangle_{\Upsilon_d} = \int_\Omega (\nabla y \nabla \tilde{y} + z\tilde{z})\, dx + \int_{\Gamma_0} m w_0 \tilde{w}_0 \, d\sigma \\ &+M \int_{\Gamma_1} w_1 \tilde{w}_1 \, d\sigma + \rho \int_{\Gamma_1} y\tilde{y}\, d\sigma, \end{aligned} \tag{3.18}$$

where $\rho$ is any positive constant. Clearly, the norm induced by this inner product is equivalent to the usual one. Then one writes the system (3.17) in the space $\Upsilon_d$ as follows:

$$\begin{cases} \Phi_t(t) + (B+P)\,\Phi(t) = 0, \\ \Phi(0) = \Phi_0 = (y_0, z_0, w_0^0, w_1^0), \end{cases} \tag{3.19}$$

where $\Phi(t) = (y(t), z(t), w_0(t), w_1(t))$, $z = y_t$, $w_0 = z|_{\Gamma_0}$, $w_1 = z|_{\Gamma_1}$ and $B$ is a nonlinear operator defined by:

$$\begin{aligned} D(B) = \{ &(y,z,w_0,w_1) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Gamma_0) \times L^2(\Gamma_1); \\ &\Delta y \in L^2(\Omega),\ w_0 = z|_{\Gamma_0},\ w_1 = z|_{\Gamma_1} \}, \end{aligned}$$

and for any $(y,z,w_0,w_1) \in D(B)$,

$$B(y,z,w_0,w_1) = \left( -z, -\Delta y, \frac{1}{m}\partial_\nu y, \frac{1}{M}\left(f(w_1) + \partial_\nu y + \rho y\right) \right).$$

Moreover, $P$ is a linear Lipschitz compact operator on $\Upsilon_d$ such that

$$P(y, z, w_0, w_1) = (0, 0, 0, -\frac{\rho}{M} y).$$

Our objective is to show that $B$ is a nonlinear maximal monotone operator in $\Upsilon_d$ [3]. To do so, a straightforward computation gives:

$$\begin{aligned}&\langle B(y, z, w_0, w_1) - B(\tilde{y}, \tilde{z}, \tilde{w}_0, \tilde{w}_1), (y, z, w_0, w_1) - (\tilde{y}, \tilde{z}, \tilde{w}_0, \tilde{w}_1)\rangle_{\Upsilon_d} \\ &\quad = \int_{\Gamma_1} \left(f(w_1) - f(\tilde{w}_1)\right)(w_1 - \tilde{w}_1)\, d\sigma \geq 0,\end{aligned} \tag{3.20}$$

for any $\phi = (y, z, w_0, w_1)$, $(\tilde{y}, \tilde{z}, \tilde{w}_0, \tilde{w}_1) \in D(B)$. Thus, the operator $B$ is monotone in $\Upsilon_d$.

Next, we are going to show that $B$ is maximal in $\Upsilon_d$. For this, given $(u, v, \xi, \eta) \in \Upsilon_d$, we seek a solution $(y, z, w_0, w_1) \in D(B)$ of the equation $(I + B)(y, z, w_0, w_1) = (u, v, \xi, \eta)$, i.e.,

$$\begin{cases} y - \Delta y = u + v, & \text{in}\, \Omega \\ y + \dfrac{1}{m}\partial_\nu y = u + \xi, & \text{on}\, \Gamma_0, \\ \left(1 + \dfrac{\rho}{M}\right) y + \dfrac{1}{M}\left(f(y - u) + \partial_\nu y\right) = u + \eta, & \text{on}\, \Gamma_1, \end{cases} \tag{3.21}$$

with $z = y - u$, in $\Omega$, $w_0 = z|_{\Gamma_0}$ and $w_1 = z|_{\Gamma_1}$. Now let us define the function $J$ on $H^1(\Omega)$ by

$$\begin{aligned}J(\psi) &= \frac{1}{2}\int_\Omega \left(\psi^2 + (\nabla\psi)^2\right)\, dx + m\int_{\Gamma_0} \psi^2\, d\sigma + (\rho + M)\int_{\Gamma_1} \psi^2\, d\sigma \\ &+ \int_{\Gamma_1} F\left(\psi - u\right)\, d\sigma - \int_\Omega (u + v)\psi\, dx - m\int_{\Gamma_0} (u + \xi)\psi\, d\sigma - M\int_{\Gamma_1} (u + \eta)\psi\, d\sigma,\end{aligned}$$

where

$$F(x) = \int_0^x f(s)ds,\ \forall x \in \mathbb{R}.$$

From the assumptions on $f$, we deduce that $J$ is convex, coercive and continuous in $H^1(\Omega)$. Hence by a minimization theorem [41], there exists a function $y \in H^1(\Omega)$ such that

$$J(y) = \inf_{\psi \in H^1(\Omega)} J(\psi).$$

This implies that the function $\Theta : \lambda \longrightarrow \Theta(\lambda) = J(y + \lambda\psi)$ admits a minimum at $\lambda = 0$ and thus

$$\frac{d}{d\lambda}\left(J(y + \lambda\psi)\right)|_{\lambda=0} = 0, \qquad \forall\psi \in H^1(0, 1).$$

This means that for any $\psi \in H^1(0, 1)$, the element $y$ is a weak solution of the system (3.21). Then one can show that $y$ is indeed the unique solution (see [9, 10] and the references therein for similar arguments for one-dimensional systems). Therefore the operator $B$ is maximal monotone on $\Upsilon_d$.

This, together with the fact that $P$ is a linear Lipschitz compact operator on $\Upsilon_d$, implies that the operator $-(B+P)$ generates a $C_0$-semigroup $\tilde{S}(t)$ on $\Upsilon_d$ (see Remark 3.14, p. 106 in [3]).

Now, using the semigroups theory of nonlinear operators (see for instance [3]), one can claim that for any initial data $(y_0, z_0, w_0^0, w_1^0) \in D(B)$, the system (3.17) (or (3.19)) admits a unique strong solution $(y, z, w_0, w_1) = \tilde{S}(t)(y_0, z_0, w_0^0, w_1^0) \in D(B)$ such that

$$\frac{d}{dt}(y, z, w_0, w_1) \in L^\infty(\mathbb{R}^+; \Upsilon_d).$$

In turn, for any initial data $(y_0, z_0, w_0^0, w_1^0) \in \Upsilon_d$, the system (3.17) (or (3.19)) has a unique weak $(y, z, w_0, w_1) = \tilde{S}(t)(y_0, z_0, w_0^0, w_1^0) \in \Upsilon_d$ such that

$$(y, z, w_0, w_1) \in C(\mathbb{R}^+; \Upsilon_d).$$

This yields in particular

$$\int_0^\infty w_1(t) f(w_1(t)) dt < \infty. \tag{3.22}$$

Integrating the equation (3.17) and assuming that the function $f$ is differentiable, it follows that

$$\begin{aligned}
&\int_\Omega y_t(x,t)\,dx + m\int_{\Gamma_0} y_t(x,t)\,d\sigma + \int_{\Gamma_1}\left(My_t(x,t) + f'(0)y(x,t)\right)d\sigma = \int_\Omega y_t(x,0)\,dx \\
&+m\int_{\Gamma_0} y_t(x,0)\,d\sigma + \int_{\Gamma_1}\left(My_t(x,0) + f'(0)y(x,0)\right)d\sigma \\
&+\int_0^t\int_{\Gamma_1}\left(f'(0)y_t(x,s) - f(y_t(x,s))\right)d\sigma.
\end{aligned} \tag{3.23}$$

Then, suppose that there exists a positive constant $K$ such that $f$ satisfies the following hypothesis:

$$|f'(0)s - f(s)| \le Ksf(s), \tag{3.24}$$

for any $s$ in some neighborhood of 0. This leads us to claim that the solution $(y, z, w_0, w_1) = (y, y_t, y_t|_{\Gamma_0}, y_t|_{\Gamma_1})$ stemmed from any initial condition $(y_0, z_0, w_0^0, w_1^0)$ in $\Upsilon_d$ satisfies the following:

$$\text{the function } t \mapsto (y, y_t, y_t|_{\Gamma_0}, y_t|_{\Gamma_1}) \text{ is bounded in } \Upsilon_d.$$

This, together with (3.22)-(3.24), implies that as $t \to \infty$, the solution

$$(y(t), y_t(t), y_t|_{\Gamma_0}(t), y_t|_{\Gamma_1}(t)) \to (\tilde{\chi}, 0, 0, 0),$$

where

$$\begin{aligned}
\tilde{\chi} = (f'(0)\mathrm{vol}\,(\Gamma_1))^{-1}&\left\{\int_\Omega z_0\,dx + m\int_{\Gamma_0} w_0^0\,d\sigma + \int_{\Gamma_1}\left(Mw_1^0 + f'(0)y_0\right)d\sigma\right. \\
&\left.+\int_0^\infty\int_{\Gamma_1}\Big(f'(0)w_1(s) - f(w_1(s))\Big)d\sigma\right\}.
\end{aligned}$$

## 4. APPLICATIONS TO OTHER SYSTEMS

The method presented in the previous sections can be applied for a large class of distributed systems (where the classical energy defines only a semi-norm in the state space) to prove that the solution converges to an equilibrium point (when the time goes to infinity) which can be determined. We give here some particular applications to Petrovsky system, coupled wave-wave equations and elasticity systems. There are many results concerning the stability of this type of systems (see [14]-[22] and the references therein) with different controls (linear, nonlinear, internal, boundary, of memory,...) and different boundary conditions (Dirichlet, Neumann,...). In all these works, the considered contexts of systems guarantee that the classical energy defines a norm on the state space. This property is not valid in the case of our applications.

**1. Petrovsky system.** Let $\Omega$ be a bounded open connected set in $\mathbb{R}^n$ having a smooth boundary $\Gamma = \partial\Omega$ of class $C^4$ with a partition $(\Gamma_0, \Gamma_1)$. We consider the following Petrovsky system:

$$\begin{cases} y_{tt}(x,t) + \Delta^2 y(x,t) = 0, & \text{in } \Omega \times (0,\infty) \\ \partial_\nu y(x,t) = 0, & \text{on } \Gamma \times (0,\infty) \\ \partial_\nu \Delta y(x,t) = 0, & \text{on } \Gamma_0 \times (0,\infty) \\ \partial_\nu \Delta y(x,t) = a y_t(x,t), & \text{on } \Gamma_1 \times (0,\infty) \\ y(x,0) = y_0(x),\ y_t(x,0) = z_0(x), & \text{in } \Omega \end{cases} \tag{4.1}$$

where $a$ is defined by (1.7). The classical energy is defined by

$$\tilde{E}_0(t) = \frac{1}{2}\int_\Omega \left(|\Delta y|^2 + |y_t|^2\right) dx,$$

which is only a semi-norm for $(y, y_t)$ in the space

$$\Upsilon_p = H^2(\Omega) \times L^2(\Omega).$$

The new energy associated to our system is defined by

$$E_p(t) = \tilde{E}_0(t) + \epsilon \left[\int_\Omega y_t\, dx + \int_{\Gamma_1} a y\, d\sigma\right]^2$$

which is a norm in $\Upsilon_p$ equivalent to the usual one of $H^2(\Omega) \times L^2(\Omega)$, for $\epsilon > 0$ small enough. An easy formal computation shows that

$$\dot{E}_p(t) = -\int_{\Gamma_1} a\, |y_t|^2 d\sigma \le 0,$$

and thus the energy $E_p(t)$ is non-increasing. Using LaSalle's principle and following the arguments used before, we obtain that, for any initial data $\Upsilon_p$, the solutions of the system (4.1) satisfy: $(y(t), y_t(t)) \longrightarrow (\chi, 0)$ in $\Upsilon_p$ as $t \longrightarrow \infty$, where

$$\chi = \left(\int_{\Gamma_1} a\, d\sigma\right)^{-1} \left\{\int_\Omega z_0\, dx + \int_{\Gamma_1} a y_0\, d\sigma\right\}.$$

**Remark 4.1.** One can consider dynamical boundary conditions as (1.3); that is

$$\begin{cases} \partial_\nu y(x,t) = 0, & \text{on } \Gamma \times (0,\infty) \\ -m(x) y_{tt}(x,t) + \partial_\nu \Delta y(x,t) = 0, & \text{on } \Gamma_0 \times (0,\infty) \\ -M(x) y_{tt}(x,t) + \partial_\nu \Delta y(x,t) = a y_t(x,t), & \text{on } \Gamma_1 \times (0,\infty) \\ y(x,0) = y_0(x) \in H^1(\Omega),\ y_t(x,0) = z_0(x) \in L^2(\Omega), & \\ y_t|_{\Gamma_0}(x,0) = w_0^0(x) \in L^2(\Gamma_0),\ y_t|_{\Gamma_1}(x,0) = w_1^0(x) \in L^2(\Gamma_1), & \end{cases} \tag{4.2}$$

where $m$ and $M$ are defined by (1.5). We obtain that $(y(t), y_t(t), y_t|_{\Gamma_0}(t), y_t|_{\Gamma_1}(t)) \longrightarrow (\chi, 0, 0, 0)$ in $H^2(\Omega) \times L^2(\Omega) \times L^2(\Gamma_0) \times L^2(\Gamma_1)$ as $t \longrightarrow \infty$ where $\chi$ is defined above.

Furthermore, one can propose on $\Gamma_1$ a nonlinear damping term $f(y_t(x,t))$ for the system (4.2) and use the same arguments as in subsection 3.2 to obtain very similar results.

**2. Coupled wave-wave equations.** We consider the following coupled system:

$$\begin{cases} y_{tt}(x,t) + Ay(x,t) + b u_{tt} = 0, & \text{in } \Omega \times (0,\infty) \\ u_{tt}(x,t) + Bu(x,t) + b y_{tt} = 0, & \text{in } \Omega \times (0,\infty) \\ \partial_A y(x,t) = \partial_B u(x,t) = 0, & \text{on } \Gamma_0 \times (0,\infty) \\ \partial_A y(x,t) = -a_1 y_t(x,t), & \text{on } \Gamma_1 \times (0,\infty) \\ \partial_B u(x,t) = -a_2 u_t(x,t), & \text{on } \Gamma_1 \times (0,\infty) \\ y(x,0) = y_0(x),\ y_t(x,0) = z_0(x), & \text{in } \Omega \\ u(x,0) = u_0(x),\ u_t(x,0) = v_0(x), & \text{in } \Omega, \end{cases} \tag{4.3}$$

where

$$A = \textstyle\sum_{i,j=1}^n \partial_i(a_{ij}\partial_j),\ B = \sum_{i,j=1}^n \partial_i(b_{ij}\partial_j),$$
$$\partial_A = \textstyle\sum_{i,j=1}^n a_{ij}\nu_j\partial_j,\ \partial_B = \sum_{i,j=1}^n b_{ij}\nu_j\partial_j,\ a_{ij},\ b_{ij} \in C^1(\bar\Omega)$$

such that there exist $\alpha_0$, $\beta_0 > 0$ satisfying

$$a_{ij} = a_{ji}, b_{ij} = b_{ji},\ \forall i,j = 1,\ldots,n,$$

$$\sum_{i,j=1}^n a_{ij}\epsilon_i\epsilon_j \geq \alpha_0 \sum_{i=1}^n \epsilon_i^2,\ \sum_{i,j=1}^n b_{ij}\epsilon_i\epsilon_j \geq \beta_0 \sum_{i=1}^n \epsilon_i^2,\ \forall\ (\epsilon_1,\ldots,\epsilon_n) \in \mathbb{R}^n.$$

Moreover, $a_1$ and $a_2$ are defined as for the function $a$ in (1.7), and $b \in L^\infty(\Omega)$ satisfies $\|b\|_\infty < 1$. The classical energy is defined by

$$\hat{E}_0(t) \;=\; \frac{1}{2}\int_\Omega \left(\sum_{i,j=1}^n a_{ij}\partial_i y \partial_j y + \sum_{i,j=1}^n b_{ij}\partial_i u \partial_j u\right) dx$$

$$+ \quad \frac{1}{2}\int_{\Omega}\left(|y_t|^2+|u_t|^2+2by_tu_t\right)dx$$

which is only a semi-norm for $(y, u, y_t, u_t)$ in the space

$$\Upsilon_w = (H^1(\Omega))^2 \times (L^2(\Omega))^2.$$

The new energy associated to our system is defined by

$$E_w(t) = \hat{E}_0(t) + \epsilon\left[\int_{\Omega}(b+1)(y_t+u_t)\,dx + \int_{\Gamma_1}(a_1y+a_2u)\,d\sigma\right]^2$$

which is a norm in $\Upsilon_w$ equivalent to the usual one of $(H^1(\Omega))^2 \times (L^2(\Omega))^2$ for $\epsilon > 0$ small enough (note that, thanks to the fact that $\|b\|_\infty < 1$, the expression

$$\int_{\Omega}\left(|y_t|^2+|u_t|^2+2by_tu_t\right)dx$$

defines a norm for $(y_t, u_t)$ which is equivalent to that of $(L^2(\Omega))^2$). A formal computation gives that

$$\dot{E}_w(t) = -\int_{\Gamma_1}(a_1\,|y_t|^2 + a_2\,|u_t|^2)d\sigma \leq 0,$$

and thus the energy $E_w(t)$ is non-increasing. Using LaSalle's principle and following the arguments used before, we obtain that, for any initial data $(y_0, u_0, z_0, v_0) \in \Upsilon_w$, the solutions of (4.3) satisfy:

$$(y(t), u(t), y_t(t), u_t(t)) \longrightarrow (\chi_1, \chi_2, 0, 0)$$

in $\Upsilon_w$ as $t \longrightarrow \infty$, where

$$\chi_1\left(\int_{\Gamma_1}a_1\,d\sigma\right) + \chi_2\left(\int_{\Gamma_1}a_2\,d\sigma\right) = \left\{\int_{\Omega}(1+b)(z_0+v_0)\,dx + \int_{\Gamma_1}(a_1y_0+a_2u_0)\,d\sigma\right\}.$$

If $A = B$, $a_1 = a_2$ and $(y_0, z_0) = (u_0, v_0)$, then it follows from the symmetry that

$$\chi_1 = \chi_2 = \left(\int_{\Gamma_1}a_1\,d\sigma\right)^{-1}\left\{\int_{\Omega}(1+b)z_0\,dx + \int_{\Gamma_1}a_1y_0\,d\sigma\right\}.$$

**Remark 4.2.** (i) One can consider dynamical boundary conditions as in (1.3) and then obtain the same result with the constants $\chi_1$ and $\chi_2$ defined above.

(ii) We can consider Neumann or dynamical boundary conditions only for $y$, and the homogeneous Dirichlet one for $u$ (or the reverse). In this case, we get

$$(y(t), u(t), y_t(t), u_t(t)) \longrightarrow (\chi, 0, 0, 0),$$

where

$$\chi = \left(\int_{\Gamma_1}a_1\,d\sigma\right)^{-1}\left\{\int_{\Omega}(1+b)z_0\,dx + \int_{\Gamma_1}a_1y_0\,d\sigma\right\}.$$

(iii) Similar results can obtained for a coupled Petrovsky-Petrovsky or wave-Petrovsky system with Neumann or dynamical boundary conditions (as in (1.2) or (1.3) for the wave equation, and as in (4.1) or (4.2) for Petrovsky one).

(iv) In the case of dynamical boundary conditions, it is easy to check that a nonlinear boundary damping term may be used to obtain similar results to those of subsection 3.2.

**3. Elasticity systems.** We consider the following elasticity system:

$$\begin{cases} y_{itt}(x,t) - \sum_{j=1}^n \sigma_{ij,j}(x,t) = 0, & \text{in } \Omega \times (0,\infty),\ \forall i = 1,\dots,n \\ \sum_{j=1}^n \sigma_{ij}\nu_j = 0, & \text{on } \Gamma_0 \times (0,\infty),\ \forall i = 1,\dots,n \\ \sum_{j=1}^n \sigma_{ij}\nu_j = -a_i y_{it}(x,t), & \text{on } \Gamma_1 \times (0,\infty),\ \forall i = 1,\dots,n \\ y_i(x,0) = y_i^0(x),\ y_{it}(x,0) = z_i^0(x), & \text{in } \Omega,\ \forall i = 1,\dots,n. \end{cases} \tag{4.4}$$

Here $y = (y_1,\dots,y_n) : \Omega \to \mathbb{R}^n$ is the solution, the $a_i$ are defined as $a$ in (1.7), $\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j}$, $\sigma_{ij} = \sum_{k,l=1}^n a_{ijkl}\varepsilon_{kl}$, $\varepsilon_{ij} = \frac{1}{2}(y_{i,j} + y_{j,i})$, $y_{i,j} = \frac{\partial y_i}{\partial x_j}$, $y_{j,i} = \frac{\partial y_j}{\partial x_i}$ and $a_{ijkl} \in C^1(\bar{\Omega})$ such that there exists $\alpha_0 > 0$ satisfying

$$a_{ijkl} = a_{klij} = a_{jikl},\ \forall i,j,k,l = 1,\dots,n,$$

$$\sum_{i,j,k,l=1}^n a_{ijkl}\epsilon_{ij}\epsilon_{kl} \geq \alpha_0 \sum_{i,j=1}^n \epsilon_{ij}\epsilon_{ij}$$

for all symmetric tensor $\epsilon_{ij}$. For more details concerning these systems, see [14]-[17] and the references therein. The classical energy of this system is defined by

$$\bar{E}_0(t) = \frac{1}{2}\int_\Omega \left( \sum_{i,j=1}^n \sigma_{ij}\varepsilon_{ij} + \sum_{i=1}^n |y_{it}|^2 \right) dx,$$

which is only a semi-norm for $(y, y_t)$ in the space

$$\Upsilon_e = (H^1(\Omega))^n \times (L^2(\Omega))^n.$$

The new energy associated to our system is defined by

$$E_e(t) = \bar{E}_0(t) + \epsilon \left[ \int_\Omega \sum_{i=1}^n y_{it}\, dx + \int_{\Gamma_1} \sum_{i=1}^n a_i y_i \, d\sigma \right]^2$$

which is a norm in $\Upsilon_e$ equivalent to the usual one of $(H^1(\Omega))^n \times (L^2(\Omega))^n$ for $\epsilon > 0$ small enough (here one needs to apply Korn inequality). We also have

$$\dot{E}_e(t) = -\int_{\Gamma_1} \sum_{i=1}^n a_i \, |y_{it}|^2 d\sigma \leq 0,$$

and thus the energy $E_e(t)$ is non-increasing. Using LaSalle's principle and following the arguments used before, we obtain that, for any initial data $(y_0, z_0) \in \Upsilon_e$, the solutions of (4.4) satisfy: $(y(t), y_t(t)) \longrightarrow (\chi, 0)$ in $\Upsilon_e$ as $t \longrightarrow \infty$, where $\chi = (\chi_1,\dots,\chi_n)$ and

$$\sum_{i=1}^n \chi_i \left( \int_{\Gamma_1} a_i \, d\sigma \right) = \left\{ \int_\Omega \sum_{i=1}^n z_i^0 \, dx + \int_{\Gamma_1} \sum_{i=1}^n a_i y_i^0 \, d\sigma \right\}.$$

If $a_i = a_j$, $y_i^0 = y_j^0$ and $z_i^0 = z_j^0$ for all $i, j = 1, \dots, n$, then, by symmetry, we have

$$\chi_i = \left( \int_{\Gamma_1} a_1 \, d\sigma \right)^{-1} \left\{ \int_\Omega z_1^0 \, dx + \int_{\Gamma_1} a_1 y_1^0 \, d\sigma \right\},$$

for all $i = 1, \dots n$.

**Remark 4.3.** We can consider Neumann conditions for $y_i$, $i = 1, \dots, r$, dynamical boundary conditions for $y_i$, $i = r+1, \dots, p$, and the homogeneous Dirichlet ones for $y_i$, $i = p+1, \dots, n$, where $0 \le r \le p \le n$; that is

$$\begin{cases} \sum_{j=1}^n \sigma_{ij}\nu_j = 0, & \text{on } \Gamma_0 \times (0,\infty),\ \forall i = 1, \dots, r \\ \sum_{j=1}^n \sigma_{ij}\nu_j = -a_i y_{it}(x,t), & \text{on } \Gamma_1 \times (0,\infty),\ \forall i = 1, \dots, r \\ \sum_{j=1}^n \sigma_{ij}\nu_j + m_i y_{itt} = 0, & \text{on } \Gamma_0 \times (0,\infty),\ \forall i = r+1, \dots, p \\ \sum_{j=1}^n \sigma_{ij}\nu_j + M_i y_{itt} = -a_i y_{it}(x,t), & \text{on } \Gamma_1 \times (0,\infty),\ \forall i = r+1, \dots, p \\ y_i = 0, & \text{on } \Gamma \times (0,\infty),\ \forall i = p+1, \dots, n. \end{cases} \tag{4.5}$$

In this case, the energy of the system will tend to $\chi = (\chi_1, \dots, \chi_n)$, where $\chi_i = 0$ for $i = p+1, \dots, n$, and

$$\sum_{i=1}^p \chi_i \left( \int_{\Gamma_1} a_i \, d\sigma \right) = \left\{ \int_\Omega \sum_{i=1}^n z_i^0 \, dx + \int_{\Gamma_1} \sum_{i=1}^n a_i y_i^0 \, d\sigma \right\}.$$

Moreover, the method adopted in subsection 3.2 can also be used to treat the case when the damping boundary control is nonlinear.

**Remark 4.4.** Clearly, one can check that all the results stated in this work remain valid if the damping control $ay_t$ is distributed, i.e., the equation (1.1) is replaced by

$$y_{tt}(x,t) - Ay(x,t) + a(x)y_t(x,t) = 0, \qquad \text{in } \Omega \times (0,\infty).$$

In fact, in this case, the Neumann boundary conditions (static or dynamical) are homogeneous and one just needs to change, in the energy norm, the integral term $\int_{\Gamma_1} a(x)y(x,t)\, d\sigma$ to $\int_\Omega a(x)y(x,t)\, dx$ and do the appropriate modifications.

## 5. OPEN PROBLEMS

1. We have proved using a simple approach that the energy of each of the considered systems converges to a constant. It would be desirable to use the same approach to provide the decay rate as done in [27] for the wave equation.

2. We have tried, without much success, to obtain an explicit expression of the constant $\chi$ for the simple case of a wave equation with dynamical boundary conditions and a nonlinear damping control. Therefore, it would be interesting to have a more profound result than that of subsection 3.2. Also, the case of the wave equation with static boundary conditions needs to be investigated if the damping control is nonlinear.

3. In the case of coupled systems, we have considered the same partition $(\Gamma_0, \Gamma_1)$ of $\Gamma$ for both equations. One could treat the case of different partitions of $\Gamma$.

## ACKNOWLEDGMENTS

These results were obtained during the visit of the first author in June 2008 to Paul Verlaine - Metz University and achieved during the visit of the second author in December 2009 to Sultan Qaboos University. This work was supported by Sultan Qaboos University (Grant IG/SCI/DOMS/08/02). The authors thank both universities for their support and hospitality.

## REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York-London, 1975.

[2] C. Bardos, G. Lebau and J. Rauch, Sharp sufficient conditions for the observation, controllability and stabilization of waves from the boundary, *SIAM J. Control Optim.* **30** (1992), 1024–1064.

[3] H. Brezis, *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, London, 1973.

[4] H. Brezis, *Analyse Fonctionnelle, Théorie et Applications,* Masson, Paris, 1992.

[5] G. Chen, Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain, *J. Math. Pures Appl.* **58** (1979), 249–274.

[6] G. Chen, Control and stabilization for the wave equation in a bounded domain, Part I, *SIAM J. Control Optim.* **17** (1979), 66–81.

[7] G. Chen, Control and stabilization for the wave equation in a bounded domain, Part II, *SIAM J. Control Optim.* **19** (1981), 114–122.

[8] G. Chen, A note on boundary stabilization of the wave equations, *SIAM J. Control Optim.* **19** (1981), 106–113.

[9] B. Chentouf and J. F. Couchouron, Nonlinear feedback stabilization of a rotating body-beam without damping, *ESAIM: Control, Optimization and Calculus of Variations* **4** (1999), 515–535.

[10] B. Chentouf, C. Z. Xu and G. Sallet: On the stabilization of a vibrating equation, *Nonlinear Analysis: Theory, Methods & Applications* **39** (2000), 537–558.

[11] B. Chentouf and M. S. Boudellioua, A new approach to stabilization of the wave equation with boundary damping control, *Sultan Qaboos University. Science* **9** (2004), 33–40.

[12] B. Chentouf and M. S. Boudellioua, On the stabilization of the wave equation with dynamical control. In: *Proc. 16th International Symposium on Mathematical Theory of Networks and Systems,* 2004, Leuven, Belgium, 6 pages.

[13] F. Conrad, G. O'Dowd and F. Z. Saouri, Asymptotic behavior for a model of flexible cable with tip masses, *Asymptot. Anal.* **30** (2002), 313–330.

[14] A. Guesmia, On linear elasticity systems with variable coefficients, *Kyushu J. Math.* **52** (1998), 227–248.

[15] A. Guesmia, Existence globale et stabilisation interne non linéaire d'un système d'élasticité, *Portugal. Math.* **55** (1998), 333–347.

[16] A. Guesmia, Existence globale et stabilisation frontière non linéaire d'un système d'élasticité, *Portugal. Math.* **56** (1999), 361–379.

[17] A. Guesmia, On the decay estimates for elasticity systems with some localized dissipations, *Asymptot. Anal.* **22** (2003), 1–13.

[18] A. Guesmia, Existence globale et stabilisation interne non linéaire d'un système de Petrovsky, *Bull. Belg. Math. Soc.* **5** (1998), 583–594.

[19] A. Guesmia, Energy decay for a damped nonlinear coupled system, *J. Math. Anal. Appl.* **239** (1999), 38–48.

[20] A. Guesmia, Quelques résultats de stabilisation indirecte des systèmes couplés non dissipatifs, *Bull. Belg. Math. Soc.* **15** (2008), 479–497.

[21] A. Guesmia, A new approach of stabilization of nondissipative distributed systems, *SIAM J. Control Optim.* **42** (2003), 24–52.

[22] A. Guesmia and S. Messaoudi, On the boundary stabilization of a compactly coupled system of nonlinear wave equations, *Intern. J. Evolut. Eq.* **1** (2005), 211–224.

[23] A. Haraux, *Systèmes Dynamiques Dissipatifs et Applications,* Paris, 1991.

[24] T. Kato, *Perturbation theory of linear Operators,* Springer, New York, 1976.

[25] V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*, Masson and John Wiley, 1994.

[26] P. D. Lax, C. S. Morawetz and R.S. Philips, Exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle, Comm. Pure Appl. Math. **16** (1963), 447–486.

[27] J. Lagnese, Decay of solutions of the wave equation in a bounded region with boundary dissipation, *J. Diff. Eq.* **50** (1983), 163–182.

[28] J. Lagnese, Note on boundary stabilization of wave equations, SIAM J. Control Optim. **26** (1988), 1250–1256.

[29] J. L. Lions, Exact controllability, stabilization and perturbations for distributed systems, *SIAM Review* **30** (1988), 1–68.

[30] J. L. Lions, *Contrôlabilité Exacte et Stabilisation de Systèmes Ditribués,* Masson, Paris, 1988.

[31] A. Majda, Disappearing solutions for the dissipative wave equations, *Indiana Univ. Math. J.* **24** (1975), 1119–1133.

[32] V. Mikhaïlov, *Equations aux Dérivées Partielles,* Mir. Moscou, 1980.

[33] C. S. Morawetz, Decay of solutions of the exterior problem for the wave equations, *Comm. Pure Appl. Math.* **28** (1975), 229–264.

[34] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[35] J. P. Quinn and D. L. Russell, Asymptotic stability and energy decay rate for solutions of hyperbolic equations with boundary damping, *Proc. Roy. Soc. Edinburgh Sect. A* **77** (1977), 97–127.

[36] J. Rauch and M. Taylor, Exponential decay of solutions to hyperbolic equations in bounded domain, *Indiana. Univ. Math. J.* **24** (1974), 79–86.

[37] D. L. Russell, Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions, *SIAM Review* **20** (1978), 639–739.

[38] R. Triggiani, Wave equation on a bounded domain with boundary dissipation: an operator approach, *J. Math. Anal. Appl.* **137** (1989), 438–461.

[39] Y. C. You and E. B. Lee, Controllability and stabilization of two-dimensional elastic vibration with dynamical boundary control, *Control of Partial Differential Equations*, edit. by A. Bermudez, Springer-Verlag, New York, pp. 297–308, 1988.

[40] Y. C. You and E. B. Lee, Dynamical boundary control of two-dimensional wave equations: vibrating membrane on general domain, *IEEE Transactions on Automatic Control* **34** (1989), 1181–1185.

[41] A. Zeidler, *Non Linear Functional Analysis and its Applications*, Springer Verlag, New York, 1986.