Communications in Applied Analysis **14** (2010), no. 4, 567–596

ON SOLVABILITY OF OPERATOR INCLUSIONS $\mathbf{x} \in \mathbf{Ax}\,\mathbf{Bx} + \mathbf{Cx}$ IN BANACH ALGEBRAS AND DIFFERENTIAL INCLUSIONS

BAPURAO C. DHAGE

Kasubai, Gurukul Colony, Ahmedpur-413 515
Dist : Latur, Maharashtra, India
E-mail: bcdhage@yahoo.co.in

ABSTRACT. In this paper, some hybrid fixed point theorems concerning the operator inclusions $x \in Ax\,Bx + Cx$ in a Banach algebra are proved. They are applied to some first order ordinary differential inclusions of initial and boundary value problems for proving the existence theorems under mixed Lipschitz and compactness type conditions. Our results includes the multi-valued hybrid fixed point theorems of Dhage [1, 2, 3] as special cases.

AMS (MOS) Subject Classification. 47H10, 34A60

1. INTRODUCTION

Krasnoselskii [4] initiated the study of hybrid fixed point theorems in Banach spaces by combining the well-known metric fixed point theorem of Banach [5, page 16] with the topological fixed point theorem of Schauder [5, page 56] which is known as the Krasnoselskii fixed point theorem in nonlinear analysis. There are several extensions and generalizations of Krasnoselskii's fixed point theorem over the course of time. See Burton [6], O'Regan [7] and the references therein. Similarly, another hybrid fixed point theorem similar to that of Krasnoselskii is proved by the present author [8] in Banach algebras and since then, several extensions and generalizations of this fixed point theorem have also been proved. See Dhage [8, 9, 10] and the references therein. A unified generalization of the above two hybrid fixed point theorems of Krasnoselskii [4] and Dhage [8] has been proved in Dhage [9] and it has been further developed in various directions. See Dhage [10, 11, 12] and the references given therein. It is known that these hybrid fixed theorems have some nice applications to nonlinear integral equations of mixed type that arise in the inversions of certain nonlinear perturbed differential equations. See for example, Krasnoselskii [4], Zeidler [5], Granas and Dugundji [13], Dhage [8] and the references therein.

The fixed point theory for multi-valued mappings is an important topic of set-valued analysis. Several well-known fixed point theorems of single-valued mappings

Received May 10, 2010

such as those of Banach and Schauder have been extended to multi-valued mappings in Banach spaces. The hybrid fixed point theorems of single-valued mappings are also not an exception. Very recently, multi-valued analogues of Krasnoselskii fixed point theorem are obtained in Petrusel [14] and Dhage [3]. In the present paper, we prove some multi-valued analogues of the unified hybrid fixed point theorem of Krasnoselskii [4] and Dhage [9] and its variants due the present author [9, 10, 11, 12, 15], and then we apply them to differential inclusions with initial and periodic boundary conditions to prove the existence results under generalized Lipschitz and Carathéodory conditions.

2. PRELIMINARIES

Before proving our main hybrid fixed point theorems for multi-valued operators in Banach algebras, we give some terminologies useful in the sequel.

Let X be a metric space and let $T : X \to X$. A mapping $T : X \to X$ is called **Lipschitz** if there exists a constant $q > 0$ such that $\|Tx - Ty\| \leq q\|x - y\|$ for all $x, y \in X$. If $q < 1$, then T is called a **contraction** on X with the contraction constant q. Then T is called a **compact** operator if $\overline{T(X)}$ is a compact subset of X. Again, T is called **totally bounded** if for any bounded subset S of X, $T(S)$ is a totally bounded set of X. Further, T is called **completely continuous** if it is a continuous and totally bounded operator on X. Note that every compact operator is totally bounded, but the converse may not be true. However, these two notions are equivalent on a bounded subset of a complete metric space X.

We are interested in the multi-valued analogues of the following types of hybrid fixed point theorems of Dhage [9] involving three operators in Banach algebras.

Theorem 2.1 (Dhage [9])**.** *Let S be a closed, convex and bounded subset of a Banach algebra X and let $A, C : X \to X$, and $B : S \to X$ be three operators such that*

(a) *A and C are Lipschitz with the Lipschitz constants q_1 and q_2 respectively,*
(b) *B is completely continuous, and*
(c) *$AxBy + Cx \in S$ for all $x, y \in S$.*

Then the operator equation $Ax\,Bx + Cx = x$ has a solution, whenever $Mq_1 + q_2 < 1$, where $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\}$.

Note that the above fixed point theorem involves the hypothesis of the complete continuity of the operator B, however, in the case of multi-valued operators we have different types of continuities, namely, lower semi-continuity and upper semi-continuity etc. Here, in this present work, we shall formulate the fixed point theorems for multi-valued mappings for each of these continuity criteria. Below we give some preliminaries of the multi-valued analysis which will be needed in the sequel.

Let X be a Banach space and let $\mathcal{P}(X)$ denote the class of all subsets of X. We set

$$\mathcal{P}_p(X) = \{A \subset X \mid A \text{ is non-empty and has a property } p\}. \tag{2.1}$$

Here, p may be $p =$ closed (in short cl) or $p =$ convex (in short cv) or $p =$ bounded (in short bd) or $p =$ compact (in short cp). Thus $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{cv}(X)$, $\mathcal{P}_{bd}(X)$ and $\mathcal{P}_{cl}(X)$ denote, respectively, the classes of all closed, convex, bounded and compact subsets of X. Similarly, $\mathcal{P}_{cl,bd}(X)$ and $\mathcal{P}_{cv,cp}(X)$ denote, respectively, the classes of closed-bounded and compact-convex subsets of X.

A correspondence $T : X \to \mathcal{P}_p(X)$ is called a **multi-valued** operator or multi-valued mapping on X. A point $u \in X$ is called a **fixed point** of T if $u \in Tu$ and the set of all fixed points of T in X is denoted by $\mathcal{F}_T$.

A multi-valued operator T is called **lower semi-continuous** (in short l.s.c.) if G is any open subset of X, then the weak inverse

$$T^-(G) = \{x \in X \mid Tx \cap G \neq \emptyset\} \tag{2.2}$$

is an open subset of X. Similarly, the multi-valued operator T is called **upper semi-continuous** (in short u.s.c.) if the set

$$T^+(G) = \{x \in X \mid Tx \subset G\}$$

is open set in X for every open set G in X. Finally, T is called **continuous** if it is lower as well as upper semi-continuous on X. A multi-valued map $T : X \to \mathcal{P}_{cp}(X)$ is called **compact** if $\overline{T(X)}$ is a compact subset of X. T is called **totally bounded** if for any bounded subset S of X, $T(S) = \bigcup_{x\in S} Tx$ is a totally bounded subset of X. It is clear that every compact multi-valued operator is totally bounded, but the converse may not be true. However, these two notions are equivalent on a bounded subset of X. Finally, T is called **completely continuous** if it is upper semi-continuous and totally bounded multi-valued operator on X.

Let X be a Banach algebra. Then, for any $A, B \in P_p(X)$, let us denote

$$A \pm B = \{a \pm b \mid a \in A, b \in B\},$$

$$A \circ B = A\,B = \{ab \mid a \in A, b \in B\},$$

and

$$\lambda A = \{\lambda a \mid a \in A\}$$

for $\lambda \in \mathbb{R}$. Similarly, let

$$\|A\| = \{\|a\| \mid a \in A\}$$

and

$$\|A\|_{\mathcal{P}} = \sup\{\|a\| \mid a \in A\}.$$

Let $A, B \in \mathcal{P}_{cl}(X)$ and let $a \in A$. Let

$$D(a, B) = \inf\{\|a - b\| \mid b \in B\}$$

and

$$\rho(A, B) = \sup\{D(a, B) \mid a \in A\}.$$

The function $d_H : \mathcal{P}_{cl}(X) \times \mathcal{P}_{cl,bd}(X) \to \mathbb{R}^+$ defined by

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\} \tag{2.3}$$

is a metric and is called the **Hausdorff metric** on X. It is clear that

$$d_H(0, C) = \|C\|_{\mathcal{P}} = \sup\{\|c\| \mid c \in C\}$$

for any $C \in \mathcal{P}_{cl}(X)$.

Definition 2.2. Let (X, d_X) and (Y, d_Y) be two metric spaces. A multi-valued mapping $F : X \to \mathcal{P}_p(Y)$ is called H-lower semi-continuous at $x_0 \in X$ if and only if for $\epsilon > 0$ there exists $\eta > 0$ such that $F(x_0) \subset V(F(x), \epsilon)$ for all $x \in \mathcal{B}_\eta(x_0)$, where $\mathcal{B}_\eta(x_0)$ is an open ball in X centered at x_0 of radius ϵ and $V(F(x), \epsilon)$ is a closed neighborhood of $F(x)$ in Y. F is called H-lower semi-continuous on X if it is H-lower semi-continuous at each point x_0 of X. Similarly, F is called H-upper semi-continuous at $x_0 \in X$ if and only if for $\epsilon > 0$ there exists $\eta > 0$ such that $F(x) \subset V(F(x_0), \epsilon)$ for all $x \in \mathcal{B}_\eta(x_0)$. F is called H-upper semi-continuous on X if it is H-upper semi-continuous at each point x_0 of X.

Note that every Lipschitz multi-valued operator $F : X \to \mathcal{P}_p(Y)$ is H-lower semi-continuous as well as H-upper semi-continuous on X.

Remark 2.3. It is known that every upper semi-continuous operator F on X is H-upper semi-continuous on X, but the converse may not be true. However, converse holds, if the multi-valued mapping F is compact-valued on X.

Lemma 2.4 (Dhage [20]). *Let X be a Banach algebra. If $A, B, C \in \mathcal{P}_{bd,cl}(X)$, then $\rho(AC, BC) \leq d_H(0, C)\, \rho(A, B)$ and $d_H(AC, BC) \leq d_H(0, C)\, d_H(A, B)$.*

Definition 2.5. A multi-valued mapping $T : X \to \mathcal{P}_{cl}(X)$ is called Lipschitz if there exists a real number $q > 0$ such that

$$d_H(Tx, Ty) \leq q\|x - y\| \tag{2.4}$$

for all $x, y \in X$. The real number q is called the Lipschitz constant of T on X. If $q < 1$, then T is called a multi-valued contraction on X with the contraction constant q.

The following fixed point theorem for the multi-valued contraction mappings appears in Covitz and Nadler [21].

Theorem 2.6. *Let (X, d) be a complete metric space and let $T : X \to \mathcal{P}_{cl}(X)$ be a multi-valued contraction. Then the fixed point set $\mathcal{F}_T$ of T is a non-empty and closed subset of X.*

Remark 2.7. Note that if the multi-valued mapping T in Theorem 2.6 has compact values, then the fixed point set $\mathcal{F}_T$ of T is a non-empty and compact subset of X.

3. MULTI-VALUED HYBRID FIXED POINT THEORY

Before going to the main fixed point results, we state some lemmas useful in the sequel.

Lemma 3.1 (Lim [25]). *Let (X, d) be a complete metric space and let $T_1, T_2 : X \to \mathcal{P}_{bd,cl}(X)$ be two multi-valued contractions with the same contraction constant k. Then*

$$\rho(\mathcal{F}_{T_1}, \mathcal{F}_{T_2}) \leq \frac{1}{1-k} \sup_{x \in X} \rho(T_1(x), T_2(x)). \tag{3.1}$$

Lemma 3.2. *Let $A : X \to \mathcal{P}_{bd}(X)$ be a multi-valued Lipschitz operator. Then for any bounded subset S of X, we have that $A(S)$ is bounded.*

Proof. Let S be a bounded subset of the Banach space X. Then there is constant $r > 0$ such that $\|x\| \leq r$ for all $x \in S$. Since A is Lipschitz, we have

$$\begin{aligned}
\|Ax\|_{\mathcal{P}} &= d_H(Ax, 0) \\
&\leq d_H(Ax, A0) + d_H(A0, 0) \\
&\leq q\,\|x\| + \|A0\|_{\mathcal{P}} \\
&\leq q\,r + \|A0\|_{\mathcal{P}} \\
&= \delta
\end{aligned}$$

for all $x \in S$. Hence $A(S)$ is bounded. □

Now we state a key result due to Rybinski [24] that will be useful in the sequel.

Theorem 3.3. *Let S be a nonempty and closed subset of a Banach space X and let Y be a metric space. Assume that the multi-valued operator $F : S \times Y \to \mathcal{P}_{cl,cv}(S)$ satisfies*

(a) *$d_H(F(x_1, y), F(x_2, y)) \leq q\|x_1 - x_2\|$ for all $(x_1, y), (x_2, y) \in S \times Y$, where $q < 1$,*
(b) *for every $x \in S$, $F(x, \cdot)$ is lower semi-continuous (briefly l.s.c.) on Y.*

Then there exists a continuous mapping $f : S \times Y \to S$ such that $f(x, y) \in F(f(x, y), y)$ for each $(x, y) \in S \times Y$.

Now we are in a position to formulate the multi-valued hybrid fixed point theorems of this paper. Our first fixed point theorem for multi-valued mappings in a Banach algebra is the following.

Theorem 3.4. *Let S be a closed convex and bounded subset of the Banach algebra X and let $A, C : X \to \mathcal{P}_{cl,cv,bd}(X)$ and $B : S \to \mathcal{P}_{cp,cv}(X)$ be three multi-valued operators such that*

(a) *A and C are multi-valued Lipschitz operators with the Lipschitz constants q_1 and q_2, respectively,*
(b) *B is l.s.c. and compact,*
(c) *$Ax\,By + Cx \in \mathcal{P}_{cl,cv}(S)$ for all $x, y \in S$, and*
(d) *$q_1 M + q_2 < 1$, where $M = \| \cup B(S)\|_{\mathcal{P}} = \sup\{\|B(x)\|_{\mathcal{P}} \mid x \in S\}$.*

Then the operator inclusion $x \in Ax\,Bx + Cx$ has a solution.

Proof. Define a multi-valued operator $T : S \times S \to \mathcal{P}_{cl,cv}(S)$ by

$$T(x, y) = Ax\,By + Cx, \tag{3.2}$$

for $x, y \in S$. We will show that $T(x, y)$ is multi-valued contraction in x for each fixed $y \in X$. Let $x_1, x_2 \in X$ be arbitrary. Then by Lemma 3.2,

$$\begin{aligned} d_H(T(x_1, y), T(x_2, y)) &= d_H(A(x_1)B(y) + C(x_1), A(x_2)B(y) + C(x_1)) \\ &\leq d_H(Ax_1, Ax_2)\, d_H(0, By) + d_H(Cx_1, Cx_2) \\ &\leq q_1\|x_1 - x_2\|\ \|B(S)\|_{\mathcal{P}} + q_2\|x_1 - x_2\| \\ &\leq q_1 M\|x_1 - x_2\| + q_2\|x_1 - x_2\| \\ &= q\|x_1 - x_2\| \end{aligned}$$

where, $q = (q_1 M + q_2) < 1$. This shows that the multi-valued operator $T_y(\cdot) = T(\cdot, y)$ is a multi-valued contraction on S with the contraction constant q. Hence an application of the Covitz-Nadler fixed point theorem yields that the fixed point set

$$\mathcal{F}_{T_y} = \{x \in S \mid\ x \in A(x)B(y) + C(x)\} \tag{3.3}$$

is a nonempty and closed subset of S for each $y \in S$.

Now the operator $T(x, y)$ satisfies all the conditions of Theorem 3.3 and hence an application of it yields that there exists a continuous mapping $f : S \times S \to S$ such that $f(x, y) \in A(f(x, y))\,B(y) + C(f(x, y))$. Let us define $H(y) = \mathcal{F}_{T_y}$, $H : S \to \mathcal{P}_{cl}(S)$. Let us consider the single-valued operator $h : S \to S$ defined by $h(x) = f(x, x)$, for each $x \in S$. Then h is a continuous mapping having the property that

$$h(x) = f(x, x) \in A(f(x, x))\,B(x) + C(f(x, x)) = A(h(x))\,B(x) + C(h(x)) \tag{3.4}$$

for each $x \in S$.

Now, we will prove that h is compact on S. To do this, it is sufficient to show that H is compact on S. By Lemma 3.2, there exists a constant $\delta > 0$ such that

$\|Ax\| \leq \delta$ for all $x \in S$. Let $\epsilon > 0$. Since B is compact on S, $B(S)$ is compact. Then there exists $Y = \{y_1, \ldots, y_n\} \subset X$ such that

$$\begin{aligned} B(S) &\subset \{w_1, \ldots, w_n\} + \mathcal{B}\Big(0, \frac{1-(q_1M+q_2)}{\delta}\epsilon\Big) \\ &\subset \bigcup_{i=1}^{n} B(y_i) + \mathcal{B}\Big(0, \frac{1-(q_1M+q_2)}{\delta}\epsilon\Big), \end{aligned}$$

where $w_i \in B(y_i)$ for each $i = 1, 2, \ldots, n$, and $\mathcal{B}(a, r)$ is an open ball in X centered at $a \in X$ of radius r. It, therefore, follows that, for each $y \in S$,

$$B(y) \subset \bigcup_{i=1}^{n} B(y_i) + \mathcal{B}\left(0, \frac{1-(q_1M+q_2)}{\delta}\epsilon\right),$$

and hence there exists an element $y_k \in Y$ such that

$$\rho(B(y), B(y_k)) < \frac{1-(q_1M+q_2)}{\delta}\epsilon.$$

Then

$$\begin{aligned} \rho(H(y), H(y_k)) &= \rho(\mathcal{F}_{T_y}, \mathcal{F}_{T_{y_k}}) \\ &\leq \frac{1}{1-(q_1M+q_2)} \sup_{x \in S} \rho(T_y(x), T_{y_k}(x)) \\ &= \frac{1}{1-(q_1M+q_2)} \sup_{x \in Y} \rho(A(x)B(y) + C(x), A(x)B(y_k) + C(x)) \\ &\leq \frac{1}{1-(q_1M+q_2)} \sup_{x \in Y} \rho(0, Ax)\rho(B(y), B(y_k)) \\ &< \frac{\delta}{1-(q_1M+q_2)} \frac{1-(q_1M+q_2)}{\delta}\epsilon \\ &= \epsilon. \end{aligned} \tag{3.5}$$

Thus, for each $u \in H(y)$ there exists $v_k \in H(y_k)$ such that $\|u - v_k\| < \epsilon$. Hence, for each $y \in Y$, one has $H(y) \subset \bigcup_{i=1}^{n} \mathcal{B}(v_i, \epsilon)$, where $v_i \in H(y_i)$, $i = 1, 2, \ldots, n$. This further implies that $h(S) \subset H(S) \subset \bigcup_{i=1}^{n} \mathcal{B}(v_i, \epsilon)$, and so h is a compact operator on S. Now the mapping $h : S \to S$ satisfies all the assumptions of Schauder's fixed point theorem and hence h has a fixed point, that is, there is a point $u \in S$ such that $u = h(u)$. From (3.2) it follows that $u = h(u) \in A(h(u))\,Bu + C(h(u)) = Au\,Bu + Cu$ This completes the proof. □

An improvement of Theorem 3.4 under weaker hypothesis (c) thereof is given in the following multi-valued hybrid fixed point theorem.

Theorem 3.5. *Let S be a closed convex and bounded subset of the Banach algebra X and let $A, C : X \to \mathcal{P}_{bd,cl,cv}(X)$ and $B : S \to \mathcal{P}_{cp,cv}(X)$ be three multi-valued operators such that* (*a*) *A and C are multi-valued Lipschitz with the Lipschitz constants q_1 and q_2, respectively,*

(*b*) *B is l.s.c. and compact,*

(c) *$Ax\,By + Cx \in \mathcal{P}_{cl,cv}(X)$ and $x \in Ax\,By + Cx \Rightarrow x \in S$ for all $y \in S$, and*

(*d*) *$q_1 M + q_2 < 1$, where $M = \| \cup B(S)\| = \sup\{\|B(x)\| \mid x \in S\}$.*

Then the operator inclusion $x \in Ax\,Bx + Cx$ has a solution in S.

Proof. Let $y \in S$ be fixed and define the multi-valued operator $T_y : X \times S \to \mathcal{P}_{cl,cv}(X)$ by

$$T_y(x) = Ax\,By + Cx, \quad x \in X.$$

Then proceeding as in the proof of Theorem 3.4, it can be proved that T_y is a multi-valued contraction on X. Now an application of Theorem 2.6 yields that the fixed point set $\mathcal{F}_{T_y}$ of T_y is non-empty and closed in X. Thus we have

$$\mathcal{F}_{T_y} = \{u \in X \mid u \in AuBy + Cu\} \subset X$$

is nonempty and closed for each $y \in S$. From hypothesis (c), it follows that $\mathcal{F}_{T_y} \subset S$ for all $y \in S$.

Note that the function $T(x, y)$ satisfies all the conditions of Theorem 3.3 and hence an application of it yields that there is a continuous function $f : X \times S \to S$ satisfying

$$f(x, y) \in T(f(x, y), y) = A(f(x, y))By + C(f(x, y))$$

for each $y \in S$. Now define a multi-valued operator $H : S \to S$ by $H(y) = \mathcal{F}_{T_y}$. Consider the single-valued mapping $h : S \to X$ by

$$h(y) = f(y, y) \in A(f(y, y))\,Bx + C(f(y, y)) = A(h(y))\,By + C(h(y)).$$

Clearly h is continuous and maps S into itself. Obviously $h(y) \in H(y)$ for each $y \in S$. Again proceeding with the arguments as in the proof of Theorem 3.4, it can be shown that h is compact on S. Now the desired conclusion follows by an application of Schauder's fixed point principle to the mapping h on S. This completes the proof. □

The Kuratowskii and Hausdorff measures α and β of noncompactness of a bounded set S in a Banach space are the nonnegative real numbers $\alpha(S)$ and $\beta(S)$ defined by

$$\alpha(S) = \inf\Big\{r > 0 : S \subseteq \bigcup_{i=1}^{n} S_i,\ \mathrm{diam}\,(S_i) \le r\ \forall\, i\Big\}, \tag{3.6}$$

and

$$\beta(S) = \inf\Big\{r > 0 : S \subset \bigcup_{i=1}^{n} \mathcal{B}_i(x_i, r),\ \text{for some } x_i \in X\Big\}, \tag{3.7}$$

where, $\mathcal{B}_i(x_i, r) = \{x \in X \mid d(x, x_i) < r\}$.

Discussion of Kuratowskii and Hausdorff measures of noncompactness appear in Akhmerov *et al.* [16], Zeidler [5] and the references therein.

Remark 3.6. It is known that $\beta(S) \leq \alpha(S) \leq 2\beta(S)$ for every bounded subset S of the Banach space X.

Remark 3.7. (Deimling [17]) It is known that if $T : X \to \mathcal{P}_{cp}(X)$ is a multi-valued contraction with a contraction constant k, then $\beta(T(S)) \leq k\,\beta(S)$ for all $S \in \mathcal{P}_{cl,bd}(X)$. Similarly, if T is a single-valued contraction on X with a contraction constant k, then $\alpha(T(S)) \leq k\alpha(S)$ for $S \in \mathcal{P}_{bd}(X)$.

Definition 3.8. A mapping $T : X \to \mathcal{P}_{cl,bd}(X)$ is called condensing if for any bounded subset S of X, $T(S)$ is bounded and $\beta(T(S)) < \beta(S)$ for $\beta(S) > 0$.

Note that contractions and completely continuous mappings are condensing but the converse may not be true. The following fixed point theorem for condensing multi-valued mappings is well-known. See Hu and Papageorgiou [18] and the references therein.

Theorem 3.9. *Let S be a closed convex and bounded subset of a Banach space X and let $T : S \to \mathcal{P}_{cp,cv}(S)$ be a upper semi-continuous and β-condensing multi-valued operator. Then T has a fixed point point.*

We need the following result in the sequel.

Lemma 3.10 (Banas and Lecho [19])**.** *Let X be a Banach algebra. If $S_1, S_2 \in \mathcal{P}_{bd}(X)$, then*

$$\beta(S_1 \circ S_2) \leq \beta(S_1)\|S_2\|_{\mathcal{P}} + \beta(S_2)\|S_1\|_{\mathcal{P}}.$$

Now we apply Theorem 3.9 in conjunction with Lemma 3.10 to yield the following fixed point theorem.

Theorem 3.11. *Let S be a closed, convex and bounded subset of the Banach algebra X and let $A, B, C : S \to \mathcal{P}_{cp,cv}(X)$ be three multi-valued operators satisfying*

(a) *A and C are Lipschitz with the Lipschitz constants q_1 and q_2, respectively,*
(b) *B is compact and upper semi-continuous,*
(c) *$Ax\,Bx + Cx \in \mathcal{P}_{cv}(S)$ for each $x \in S$, and*
(d) *$Mq_1 + q_2 < 1$, where $M = \| \cup B(S)\|_{\mathcal{P}} = \sup\{\|B(x)\|_{\mathcal{P}} \mid x \in S\}$.*

Then the operator inclusion $x \in Ax\,Bx + Cx$ has a solution.

Proof. Define the mapping $T : S \to \mathcal{P}_p(S)$ by

$$Tx = Ax\,Bx + Cx, \quad x \in S. \tag{3.8}$$

We shall show that T satisfies all the conditions of Theorem 2.6 on S.

Step I: First we claim that T defines a multi-valued map $T : S \to \mathcal{P}_{cp,cv}(S)$. Obviously, Tx is a convex subset of S for each $x \in S$ in view of hypothesis (c). From Lemma 3.10 and Remark 3.2, it follows that

$$\begin{aligned} \beta(Tx) &\leq \beta(Ax \cdot Bx) + \beta(Cx) \\ &\leq \beta(Ax).\|B(x)\|_{\mathcal{P}} + \beta(Bx).\|A(x)\|_{\mathcal{P}} + \beta(Cx) \\ &\leq q_1\beta(\{x\})\,\|B(x)\|_{\mathcal{P}} + 0 + q_2\beta(\{x\}) \\ &= 0 \end{aligned}$$

for every $x \in S$ and the claim follows immediately.

Step II: Next, we shall show that the mapping T is an upper semi-continuous on S. Since S is a bounded set in X and A is a multi-valued Lipschitz operator, by hypothesis (a), there exists a constant $\delta > 0$ such that $\|Ax\|_{\mathcal{P}} \leq \delta$ for all $x \in S$.

Let $\{x_n\}$ be a sequence in S converging to the point $x^* \in S$ and let $\{y_n\}$ be sequence defined by $y_n \in Tx_n$ converging to the point y^*. It is enough to prove that $y^* \in Tx^*$. Now for any $x, y \in S$, we have

$$\begin{aligned} \rho(Tx,Ty) &\leq \rho(AxBx, AyBy) + d_H(Cx,Cy) \\ &\leq d_H(AxBx, AyBx) + \rho(AyBx, AyBy) + d_H(Cx,Cy) \\ &\leq d_H(Ax,Ay)d_H(0,Bx) + d_H(0,Ay)\,\rho(Bx,By) + d_H(Cx,Cy) \\ &\leq q_1\,\|x-y\|\|B(S)\|_{\mathcal{P}} + \delta\rho(Bx,By) + d_H(Cx,Cy) \\ &\leq Mq_1\,\|x-y\| + \delta\rho(Bx,By) + q_2\|x-y\| \\ &\leq (Mq_1+q_2)\,\|x-y\| + \delta\rho(Bx,By) \end{aligned} \tag{3.9}$$

Since B is u.s.c., it is H-upper semi-continuous and consequently

$$\rho(Bx_n, Bx^*) \to 0 \quad \text{whenever} \quad x_n \to x^*.$$

Therefore,

$$\rho(Tx_n, Tx^*) \leq (Mq_1+q_2)\|x_n - x^*\| + \delta\rho(Bx_n, Bx^*) \to 0 \text{ as } n \to \infty.$$

This shows that the multi-valued operator T is H-upper semi-continuous on S. Since the multi-valued map T is compact-valued, it is upper semi-continuous on S in view of Remark 2.1.

Step III: Finally, we show that that T is β-condensing on S. Let $G \subset S$ be arbitrary. Then G is bounded. Also $A(G)$ and $C(G)$ are bounded in view of Lemma 3.2. Since B compact, the set $B(G)$ is compact and hence bounded in X. Therefore, the set $A(G)\,B(G) + C(G)$ is bounded. As $T(G) \subset A(G)\,B(G) + C(G)$, we have that $T(G)$ is a bounded set in X for each $G \subset S$. Hence by Lemma 3.10,

$$\beta(T(G)) \leq \beta(A(G)\,B(G)) + \beta(C(G))$$

$$\begin{aligned} &\leq \beta(A(G))\|B(G)\|_{\mathcal{P}} + \beta(B(G))\|A(G)\|_{\mathcal{P}} + \beta(C(G)) \\ &\leq Mq_1\beta(G) + q_1\beta(G) \\ &\leq k\beta(G) \end{aligned}$$

for all $G \subset S$, where $k = Mq_1 + q_2 < 1$. This shows that T is β-condensing multi-valued mapping on S into itself. Now an application of Theorem 3.9 yields that T has a fixed point. This further implies that the operator inclusion $x \in Ax\,Bx + Cx$ has a solution. This completes the proof. □

A special case of Theorem 3.11, useful in applications to differential and integral inclusions is given in the following theorem.

Theorem 3.12. *Let S be a closed, convex and bounded subset of the Banach algebra X and let $A, C : S \to X$ and $B : S \to \mathcal{P}_{cp,cv}(X)$ be three multi-valued operators satisfying*

(a) *A and C are single-valued Lipschitz with the Lipschitz constant q_1 and q_2 respectively,*
(b) *B is compact and upper semi-continuous,*
(c) *$Ax\,Bx + Cx \subset S$ for each $x \in S$, and*
(d) *$Mq_1 + q_2 < 1/2$, where $M = \|B(S)\|_{\mathcal{P}} = \sup\{\|B(x)\|_{\mathcal{P}} \mid x \in S\}$.*

Then the operator inclusion $x \in Ax\,Bx + Cx$ has a solution.

Proof. It is clear in view of Remark 3.1 that every single-valued Lipschitz mapping with a Lipschitz constant k is multi-valued Lipschitz mapping with the Lipschitz constant $2k$ (see Hu and Papageorgiou [18] and the references therein). Again, $Ax\,Bx + Cx$ is a convex subset of X for each $x \in S$ in view of the fact that $a\,K + b$ is a convex subset of X for all $a, b \in \mathbb{R}$ and a convex set K in X. Again, the set $a\,K + b$ is closed if K is a closed subset of X. Now the desired conclusion follows by an application of Theorem 3.11. □

4. LERAY-SCHAUDER TYPE FIXED POINT PRINCIPLE

4.1. Nonlinear Alternatives. In this section we obtain multi-valued analogues of the following nonlinear alternative of Leray-Schauder type due to Dhage [15].

Theorem 4.1. *Let U and $\overline{U}$ be respectively open-bounded and closed-bounded subsets of a Banach algebra X such that $0 \in U$ and let $A, B, C : \overline{U} \to X$ be three operators satisfying*

(a) *A and C are Lipschitz with the Lipschitz constants q_1 and q_2, respectively,*
(b) *B is upper semi-continuous and compact, and*
(c) *$Mq_1 + q_2 < 1$, where $M = \|B(\overline{U})\|_{\mathcal{P}} = \sup\left\{\|B(x)\|_{\mathcal{P}} : x \in \overline{U}\right\}$.*

Then, either

(i) *the equation* $AxBx + Cx = x$ *has a solution in* $\overline{U}$*, or*
(ii) *there is an element* $u \in \partial U$ *such that* $u = \lambda[AuBu + Cu]$ *for some* $\lambda \in (0, 1)$*, where* ∂U *is the boundary of* U *in* X.

We need the following Leray-Schauder principle in the sequel.

Theorem 4.2. *Let* U *and* $\overline{U}$ *be respectively open-bounded and closed-bounded subsets of a Banach space* X *such that* $0 \in U$*. Let* $T : \overline{U} \to X$ *be a continuous and compact mapping. Then either*

(i) *the equation* $Tx = x$ *has a solution in* $\overline{U}$*, or*
(ii) *there is an element* $u \in \partial U$ *such that* $u = \lambda Tu$ *for some* $\lambda \in (0, 1)$*, where* ∂U *is the boundary of* U *in* X.

The underlined principle of Theorem 4.2 will be used to prove the following fixed point theorem.

Theorem 4.3. *Let* U *and* $\overline{U}$ *be respectively open-bounded and closed-bounded subsets of a Banach algebra* X *such that* $0 \in U$ *and let* $A, C : X \to \mathcal{P}_{cl,cv,bd}(X)$ *and* $B : \overline{U} \to \mathcal{P}_{cp,cv}(X)$ *be three operators satisfying*

(a) A *and* C *are multi-valued Lipschitz with the Lipschitz constants* q_1 *and* q_2*, respectively,*
(b) B *is l.s.c. and compact,*
(c) $Ax\,By + Cx \in \mathcal{P}_{cl,cv}(X)$ *for all* $x, y \in U$*, and*
(d) $Mq_1 + q_2 < 1$*, where* $M = \| \cup B(\overline{U})\|_{\mathcal{P}} = \sup\big\{\|B(x)\|_{\mathcal{P}} : x \in \overline{U}\big\}$.

Then either

(i) *the equation* $x \in Ax\,Bx + Cx$ *has a solution in* $\overline{U}$*, or*
(ii) *there is an element* $u \in \partial U$ *such that* $\mu u \in A(\mu u)\,Bu + C(\mu u)$ *for some* $\mu > 1$*, where* ∂U *is the boundary of* U *in* X.

Proof. Let $y \in \overline{U}$ be fixed and define the multi-valued operator $T_y : X \times \overline{U} \to \mathcal{P}_{cl,cv}(X)$ by

$$T_y(x) = AxBy + Cx, \quad x \in X.$$

Then proceeding as in the proof of Theorem 3.4, it can be proved that T_y is a multi-valued contraction on X. Now an application of Theorem 2.6 yields that the fixed point set $\mathcal{F}_{T_y}$ of T_y is non-empty and closed in X. Thus, we have

$$\mathcal{F}_{T_y} = \{u \in X \mid u \in AuBy + Cu\} \subset X$$

is nonempty and closed for each $y \in \overline{U}$.

Note that the function $T(x,y)$ satisfies all the conditions of Theorem 3.3 and hence an application of it yields that there is a continuous function $f : X \times \overline{U} \to \overline{U}$ satisfying

$$f(x,y) \in T(f(x,y),y) = A(f(x,y))By + C(f(x,y))$$

for each $y \in \overline{U}$. Now define a multi-valued operator $H : \overline{U} \to X$ by $H(y) = \mathcal{F}_{T_y}$. Consider the single-valued mapping $h : \overline{U} \to X$ defined by

$$h(y) = f(y,y) \in A(f(y,y))\, Bx + C(f(y,y)) = A(h(y))\, By + C(h(y)).$$

Clearly h is continuous and maps $\overline{U}$ into X. Obviously $h(y) \in H(y)$ for each $y \in \overline{U}$. Again, proceeding with arguments as in the proof of Theorem 3.4 shows that that h is compact on $\overline{U}$. Now an application of Theorem 4.2 yields that either

(i) the equation $x = hx$ has a solution in $\overline{U}$, or
(ii) there is an element $u \in \partial U$ such that $u = \lambda hu$ for some $\lambda \in (0,1)$, where ∂U is the boundary of U in X.

Furthermore, the definition of h implies that either

(i) the operator equation $x \in AxBx + Cx$ has a solution, or
(ii) there exists an $u \in \partial U$ such that $\mu u \in A(hu)Bu + C(hu) = A(\mu u)\, Bu + C(\mu u)$ for $\mu > 1$, where ∂U is the boundary of U in X.

This completes the proof. □

Corollary 4.4. *Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}}_r(0)$ denote respectively the open and closed balls centered at the origin 0 of radius r in a Banach algebra X and let $A, C : X \to \mathcal{P}_{cl,cv,bd}(X)$ and $B : \overline{\mathcal{B}}_r(0) \to \mathcal{P}_{cp,cv}(X)$ be three multi-valued operators such that*

(a) *A and C are multi-valued Lipschitz with the Lipschitz constants q_1 and q_2 respectively,*
(b) *B is l.s.c. and compact,*
(c) *$Ax\, By + Cx \in \mathcal{P}_{cl,cv,bd}(X)$ for all $x, y \in \mathcal{B}_r(0)$, and*
(d) *$Mq_1 + q_2 < 1$, where $M = \| \cup \overline{\mathcal{B}}_r(0)\|_{\mathcal{P}} = \sup\left\{\|B(x)\|_{\mathcal{P}} : x \in \overline{\mathcal{B}_r(0)}\right\}$.*

Then either

(i) *the operator inclusion $x \in Ax\, Bx + Cx$ has a solution in $\overline{\mathcal{B}}_r(0)$, or*
(ii) *there exists an $u \in X$ with $\|u\| = r$ satisfying $\mu u \in A(\mu u)Bu + C(\mu u)$ for some $\mu > 1$.*

A nonlinear alternative for condensing multi-valued mappings useful in the applications to differential and integral inclusions is the following variant of a fixed point theorem of O'Regan [7].

Theorem 4.5. *Let U and $\overline{U}$ be respectively the open-bounded and closed-bounded subsets of a Banach space X such that $0 \in U$ and let $T : \overline{U} \to \mathcal{P}_{cp,cv}(X)$ be an upper semi-continuous and condensing mapping such that $T(\overline{U})$ is bounded. Then either*

(i) *T has a fixed point, or*

(ii) *there exists an element $u \in \partial U$ such that $\lambda u \in Tu$ for some $\lambda > 1$, where ∂U is the boundary of U in X..*

As an application of Theorem 4.5 we obtain the following hybrid fixed point theorem.

Theorem 4.6. *Let U and $\overline{U}$ be respectively open-bounded and closed-bounded subsets of a Banach algebra X such that $0 \in U$ and let $A, B, C : \overline{U} \to \mathcal{P}_{cp,cv}(X)$ be three multi-valued operators satisfying*

(a) *A and C are multi-valued Lipschitz with the Lipschitz constants q_1 and q_2 respectively,*
(b) *B is upper semi-continuous and compact,*
(c) *$Ax\,Bx + Cx \in \mathcal{P}_{cv}(X)$ for all $x \in \overline{U}$, and*
(d) *$Mq_1 + q_2 < 1$, where $M = \|\bigcup \overline{U}\|_{\mathcal{P}} = \sup\big\{\|B(x)\|_{\mathcal{P}} : x \in \overline{U}\big\}$.*

Then either

(i) *the operator inclusion $x \in Ax\,Bx + Cx$ has a solution in $\overline{U}$, or*
(ii) *there exists an $u \in \partial U$ such that $\mu u \in Au\,Bu + Cu$ for some $\mu > 1$, where ∂U is the boundary of U in X.*

Proof. Define the multi-valued mapping $T : \overline{U} \to \mathcal{P}_{cp,cv}(X)$ by

$$Tx = Ax\,Bx + Cx,\ x \in \overline{U}. \tag{4.1}$$

It can be shown as in the proof of Theorem 3.11 that T is upper semi-continuous and β-condensing on $\overline{U}$. We just show that $T(\overline{U})$ is bounded. Since A and C are multi-valued Lipschitz, there are constants $\delta_1 > 0$ and $\delta_1 > 0$ such that $\|A(\overline{U})\|_{\mathcal{P}} \leq \delta_1$ and $\|C(\overline{U})\|_{\mathcal{P}} \leq \delta_2$. Again, the multi-valued map B is compact, so the set $B(\overline{U})$ is bounded and there is a constant $\delta_3 > 0$ such that $\|B(\overline{U})\|_{\mathcal{P}} \leq \delta_3$. As a result, we have

$$\|T(\overline{U})\|_{\mathcal{P}} \leq \|A(\overline{U})\|_{\mathcal{P}}\,\|B(\overline{U})\|_{\mathcal{P}} + \|C(\overline{U})\|_{\mathcal{P}} \leq \delta_1\delta_2 + \delta_3 = \delta.$$

Hence, $T(\overline{U})$ is bounded and now the desired result follows by an application of Theorem 3.5. □

Corollary 4.7. *Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}}_r(0)$ denote respectively open and closed balls centered at origin of radius r in a Banach algebra X and let $A, B, C : \overline{\mathcal{B}}_r(0) \to \mathcal{P}_{cp,cv}(X)$ be three multi-valued operators such that*

(*a*) *A and C are multi-valued Lipschitz with the Lipschitz constants* q_1 *and* q_2*, respectively,*
(*b*) *B is u.s.c. and compact,*
(*c*) $Ax\,Bx + Cx \in \mathcal{P}_{cv}(X)$ *for each* $x \in \overline{\mathcal{B}}_r(0)$*, and*
(*d*) $Mq_1 + q_2 < 1$*, where* $M = \|B(\overline{\mathcal{B}}_r(0))\|_{\mathcal{P}} = \sup\{\|B(x)\|_{\mathcal{P}} \mid x \in \overline{\mathcal{B}}_r(0)\}$*.*

Then either

(i) *the operator inclusion* $x \in Ax\,B + Cx$ *has a solution, or*
(ii) *there exists an* $u \in X$ *with* $\|u\| = r$ *such that* $\mu u \in Au\,Bu + Cu$ *for some* $\mu > 1$*.*

The following special case is useful in applications to differential and integral inclusions in Banach algebras for proving existence results under mixed conditions.

Theorem 4.8. *Let* $\mathcal{B}_r(0)$ *and* $\overline{\mathcal{B}}_r(0)$ *denote respectively the open and closed balls centered at the origin* 0 *of radius r in a Banach algebra X and let* $A, C : \overline{\mathcal{B}}_r(0) \to X$ *and* $B : \overline{\mathcal{B}}_r(0) \to \mathcal{P}_{cp,cv}(X)$ *be three operators such that*

(*a*) *A and C are single-valued Lipschitz with the Lipschitz constants* q_1 *and* q_2*, respectively,*
(*b*) *B is u.s.c. and compact, and*
(*c*) $Mq_1 + q_2 < 1/2$*, where* $M = \|B(\overline{\mathcal{B}}_r(0))\|_{\mathcal{P}} = \sup\{\|B(x)\|_{\mathcal{P}} \mid x \in \overline{\mathcal{B}}_r(0)\}$*.*

Then either

(i) *the operator inclusion* $x \in Ax\,B + Cx$ *has a solution, or*
(ii) *there exists an* $u \in X$ *with* $\|u\| = r$ *such that* $\mu u \in Au\,Bu + Cu$ *for some* $\mu > 1$*.*

Proof. The proof is similar to Theorem 3.12 and now the conclusion follows by an application of Corollary 4.7. □

4.2. Schaefer type multi-valued hybrid fixed point theory. It is common knowledge that Schaefer type fixed point theory for single as well as multi-valued mappings provides powerful tools in the theory of differential equations and inclusions for proving existence theorems under suitable conditions. The method is also known as an "**a priori bound method**" for differential equations and inclusions. Here, we prove some the multi-valued analogues of some hybrid fixed point theorems of Schaefer type due to the present author [10]. We need the following fixed point theorem in the sequel.

Theorem 4.9 ([5, page 245])**.** *Let X be a Banach space and let* $T : X \to X$ *be completely continuous. Then, either*

(i) *the operator equation* $x = Tx$ *has a solution, or*
(ii) *the set* $\mathcal{E} = \{u \in X \mid u = \lambda Tu,\ 0 < \lambda < 1\}$ *is unbounded.*

Theorem 4.9 will be used in the proof of following fixed point theorem.

Theorem 4.10. *Let X be a Banach algebra and let $A, C : X \to \mathcal{P}_{bd,cl,cv}(X)$ and $B : X \to \mathcal{P}_{cp,cv}(X)$ be three multi-valued operators such that*

(a) *A and C are Lipschitz with the Lipschitz constants q_1 and q_2, respectively,*
(b) *B is l.s.c. and compact,*
(c) *$Ax\,By + Cx \in \mathcal{P}_{cl,cv}(X)$ for each $x, y \in X$, and*
(d) *$Mq_1 + q_2 < 1$, where $M = \|B(X)\|_{\mathcal{P}}$.*

Then either

(i) *the operator inclusion $x \in Ax\,Bx + Cx$ has a solution, or*
(ii) *the set $\mathcal{E} = \{u \in X \mid \mu u \in A(\mu u)\,Bu + C(\mu u),\ \mu > 1\}$ is unbounded.*

Proof. Let S be a bounded subset of the Banach algebra X and define the multi-valued mapping $T : X \times S \to \mathcal{P}_{cl,cv}(X)$ by (3.2). It can be shown as in the proof of Theorem 3.4 that the multi-valued map T satisfies all the conditions of Theorem 3.3. Hence there is a continuous function $f : X \times S \to X$ such that

$$f(x, y) \in A(f(x, y))By + C(f(x, y)).$$

Now define a function $h : S \to X$ by

$$h(y) = f(y, y) \in A(h(y))By + C(h(y)).$$

Again proceeding as in the proof of Theorem 3.4 we can show that the function h is completely continuous on S into X. As a result, an application of Theorem 4.8 yields the desired result. The proof of the theorem is complete. □

To prove our final Schaefer type fixed point result, we need the following generalization of the Schaefer principle.

Theorem 4.11 (Martelli [23]). *Let X be a Banach space and let $T : X \to \mathcal{P}_{cp,cv}(X)$ be an upper semi-continuous and β-condensing multi-valued operator. Then either*

(i) *the operator inclusion $x \in Tx$ has a solution, or*
(ii) *the set $\mathcal{E} = \{u \in X \mid \lambda u \in Tu,\ \lambda > 1\}$ is unbounded.*

An application of Theorem 4.11 yields the following fixed point theorem.

Theorem 4.12. *Let X be a Banach algebra and let $A, B, C : X \to \mathcal{P}_{cp,cv}(X)$ be three multi-valued operators satisfying*

(a) *A and C are Lipschitz with the Lipschitz constants q_1 and q_2 respectively,*
(b) *B is compact and upper semi-continuous,*
(c) *$Ax\,Bx + Cx$ is a convex subset of X for each $x \in X$, and*
(d) *$Mq_1 + q_2 < 1$, where $M = \|\bigcup B(X)\|_{\mathcal{P}}$.*

Then either

(i) *the operator inclusion* $x \in Ax\,Bx + Cx$ *has a solution, or*
(ii) *the set* $\mathcal{E} = \{u \in X \mid \mu u \in Au\,Bu + Cu,\ \mu > 1\}$ *is unbounded.*

Proof. Let S be a bounded subset of the Banach algebra X and define the multi-valued map $T : S \to \mathcal{P}_p(X)$ by (3.8). Now it can be shown as in the proof of Theorem 3.11 that T defines a upper semi-continuous multi-valued mapping $T : S \to \mathcal{P}_{cp,cv}(X)$ satisfying $\beta(T(S)) < \beta(S)$ for $\beta(S) > 0$. This is true for every bounded set S in X. Therefore, T is an upper semi-continuous and β-condensing map on X. Now the desired conclusion follows by an application of Theorem 4.11. □

An interesting hybrid fixed point principle of Schaefer type in a form applicable to the differential inclusions is the following.

Theorem 4.13. *Let X be a Banach algebra and let $A, C : X \to X$ be two single-valued and $B : X \to \mathcal{P}_{cp,cv}(X)$ be multi-valued operator, satisfying*

(a) *A and C are Lipschitz with the Lipschitz constants q_1 and q_2, respectively,*
(b) *B is compact and upper semi-continuous, and*
(c) *$Mq_1 + q_2 < 1/2$, where $M = \|\bigcup B(X)\|_{\mathcal{P}}$.*

Then either

(i) *the operator inclusion* $x \in Ax\,Bx + Cx$ *has a solution, or*
(ii) *the set* $\mathcal{E} = \{u \in X \mid \mu u \in Au\,Bu + Cu,\ \mu > 1\}$ *is unbounded.*

Proof. The proof is similar to Theorem 3.12 and now the conclusion follows by an application of Theorem 4.12. □

Remark 4.14. If $C \equiv 0$ in Theorems 3.2, 3.3, 3.5, 4.3, 4.5 and 4.10, then they reduce to the multi-valued hybrid fixed point theorems of Dhage [20, 1, 2]. Therefore, the hybrid fixed point theorems of this paper are new to the literature in the multi-valued case of nonlinear analysis.

In the following section, we apply the abstract results of this section to first order differential inclusions with initial and periodic boundary conditions for proving the existence results under generalized Lipschitz and Carathéodory conditions.

5. DIFFERENTIAL INCLUSIONS

5.1. Multi-valued initial value problems. First we consider the quadratic initial value problems (in short IVP) for first order ordinary differential inclusions. Given a closed and bounded interval $J = [0, a]$ in $\mathbb{R}$ for some $a \in \mathbb{R}$, $a > 0$, consider the IVP

$$\left.\begin{aligned} \left(\frac{x(t) - k(t, x(t))}{f(t, x(t))}\right)' \in G(t, x(t)) \quad \text{a.e.} \quad t \in J, \\ x(0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{5.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R} - \{0\}$, $k : J \times \mathbb{R} \to \mathbb{R}$ and $G : J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$.

By a solution to IVP (5.1) we mean a function $x \in AC(J, \mathbb{R})$ that satisfies

(i) the function $t \mapsto \left(\dfrac{x(t) - k(t, x(t))}{f(t, x(t))}\right)$ is differentiable, and

(ii) $\left(\dfrac{x(t) - k(t, x(t))}{f(t, x(t))}\right)' = v(t),\ t \in J$ for some $v \in L^1(J, \mathbb{R})$ such that $v(t) \in G(t, x(t))$ a.e. $t \in J$ satisfying $x(0) = x_0$,

where $AC(J, \mathbb{R})$ is the space of all absolutely continuous real-valued functions on J.

The IVP (5.1) is new in the theory of differential inclusions and special cases of it have been discussed in the literature extensively. For example, if $f(t, x) = 1$, then the IVP (5.1) reduces to IVP

$$\left.\begin{aligned} & x'(t) \in G(t, x(t)) \quad \text{a.e.} \quad t \in J, \\ & x(0) = x_0 \in \mathbb{R}. \end{aligned}\right\} \tag{5.2}$$

There is a considerable work available in the literature for the IVP (5.2). See Aubin and Cellina [22], Deimling [17] and Hu and Papageorgiou [18], etc. Similarly, in the special case when $G(t, x) = \{g(t, x)\}$, we obtain the differential equation

$$\left.\begin{aligned} & \left(\frac{x(t) - k(t, x(t))}{f(t, x(t))}\right)' = g(t, x(t)) \quad \text{a.e.} \quad t \in J, \\ & x(0) = x_0 \in \mathbb{R}. \end{aligned}\right\} \tag{5.3}$$

The differential equation (5.3) has been studied recently in Dhage [10, 11] for the existence of solutions. Therefore, it of interest to discuss the the IVP (5.3) for various aspects of its solution under suitable conditions. In this section, we shall prove the existence of solutions for the IVP (5.1) under mixed generalized Lipschitz and Carathéodory conditions.

Define a norm $\|\cdot\|$ and a multiplication " $\cdot$ " in the Banach algebra $C(J, \mathbb{R})$ of continuous and real-valued functions on J by

$$\|x\| = \sup_{t \in J} |x(t)|$$

and

$$(x \cdot y)(t) = (xy)(t) = x(t)y(t),\ t \in J$$

for all $x, y \in C(J, \mathbb{R})$. Then, $C(J, \mathbb{R})$ is a Banach algebra with respect to the above norm and multiplication in it.

We need the following definitions in the sequel.

Definition 5.1. A multi-valued mapping $F : J \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be measurable if for any $y \in X$, the function $t \mapsto d(y, F(t)) = inf\{|y - x| : x \in F(t)\}$ is measurable.

Definition 5.2. A multi-valued function $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is called Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in E$, and
(ii) $x \mapsto F(t,x)$ is an upper semi-continuous almost everywhere for $t \in J$.

Again a Carathéodory multi-valued function F is called L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(J,\mathbb{R})$ such that

$$\|F(t,x)\|_{\mathcal{P}} \leq h(t) \quad \text{ a.e. } \; t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq r$.

Further, a Carathéodory multi-valued function F is called L^1_X-Carathéodory if

(iv) there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$\|F(t,x)\|_{\mathcal{P}} \leq h(t) \quad \text{ a.e. } \; t \in J$$

for all $x \in \mathbb{R}$, and the function h is called a growth function of F on $J \times \mathbb{R}$.

Set

$$S^1_F(x) = \{v \in L^1(J,\mathbb{R}) \mid v(t) \in F(t,x(t)) \text{ a.e. } t \in J\}.$$

Then we have the following lemmas due to Lasota and Opial [26].

Lemma 5.3. *Let E be a Banach space. If $dim(E) < \infty$ and $F : J \times E \to \mathcal{P}_{cp}(E)$ is L^1-Carathéodory, then $S^1_F(x) \neq \emptyset$ for each $x \in E$.*

Lemma 5.4. *Let E be a Banach space, F a Carathéodory multi-valued operator with $S^1_F \neq \emptyset$, and let $\mathcal{L} : L^1(J,E) \to C(J,E)$ be a continuous linear mapping. Then the composite operator*

$$\mathcal{L} \circ S^1_F : C(J,E) \to \mathcal{P}_{bd,cl}(C(J,E))$$

is a closed graph operator on $C(J,E) \times C(J,E)$.

We consider the following hypotheses in the sequel.

(H_0) The function $x \mapsto \dfrac{x - k(0,x)}{f(0,x)}$ is increasing in $\mathbb{R}$.

(H_1) The function $f : J \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous and there exists a bounded function $\ell_1 : J \to \mathbb{R}$ with bound $\|\ell_1\|$ satisfying

$$|f(t,x) - f(t,y)| \leq \ell_1(t)|x-y| \quad \text{ a.e. } \quad t \in J$$

for all $x, y \in \mathbb{R}$.

(H_2) The function $k : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a bounded function $\ell_2 : J \to \mathbb{R}$ with bound $\|\ell_2\|$ satisfying

$$|k(t,x) - k(t,y)| \leq \ell_2(t)|x-y| \quad \text{ a.e. } \quad t \in J$$

for all $x, y \in \mathbb{R}$.

(H_3) The multi-valued operator $G : J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is $L^1_{\mathbb{R}}$-Carathéodory with growth function h.

Theorem 5.5. *Assume that the hypotheses* (H_0)*–*(H_3) *hold. If*

$$\|\ell_1\| \left(\left| \frac{x_0 - k(0, x_0)}{f(0, x_0)} \right| + \|h\|_{L^1} \right) + \|\ell_2\| < 1/2, \tag{5.4}$$

then the IVP (5.1) *has a solution on* J.

Proof. Let $X = C(J, \mathbb{R})$. Define three mappings A, B and C on X by

$$Ax(t) = f(t, x(t)), \tag{5.5}$$

$$Bx = \left\{ u \in X \,\middle|\, u(t) = \frac{x_0}{f(0, x_0)} + \int_0^t v(s)\, ds, \quad v \in S^1_G(x) \right\} \tag{5.6}$$

and

$$Cx(t) = k(t, x(t)) \tag{5.7}$$

for all $t \in J$. Then the IVP (5.1) is equivalent to the operator inclusion

$$x(t) \in Ax(t)Bx(t) + Cx(t), \; t \in J. \tag{5.8}$$

We will show that the multi-valued operators A, B and C satisfy all the conditions of Theorem 4.13. Clearly the operator B is well defined since $S^1_G(x) \neq \emptyset$ for each $x \in X$.

Step I: We first show that the operators A and C define single-valued operators $A, C : X \to X$ and $B : X \to \mathcal{P}_{cp,cv}(X)$. The claim concerning A and C is obvious, because the functions f and k are continuous on $J \times \mathbb{R}$. We only prove the claim for the multi-valued operator B on X. First, we show that B has compact values on X. Observe that the operator B is equivalent to the composition $\mathcal{K} \circ S^1_G$ of two operators on $L^1(J, \mathbb{R})$, where $\mathcal{K} : L^1(J, \mathbb{R}) \to X$ is the continuous operator defined by

$$\mathcal{K}v(t) = \frac{x_0 - k(0, x_0)}{f(0, x_0)} + \int_0^t v(s)\, ds. \tag{5.9}$$

To show B has compact values, it then suffices to prove that the composition operator $\mathcal{K} \circ S^1_G$ has compact values on X. Let $x \in X$ be arbitrary and let $\{v_n\}$ be a sequence in $S^1_G(x)$. Then, by the definition of S^1_G, $v_n(t) \in G(t, x(t))$ a.e. for $t \in J$. Since $G(t, x(t))$ is compact, there is a convergent subsequence of $v_n(t)$ (for simplicity call it $v_n(t)$ itself) that converges in measure to some $v(t)$, where $v(t) \in G(t, x(t))$ a.e. for $t \in J$. From the continuity of $\mathcal{L}$, it follows that $\mathcal{K}v_n(t) \to \mathcal{K}v(t)$ pointwise on J as $n \to \infty$. In order to show that the convergence is uniform, we need to show that $\{\mathcal{K}v_n\}$ is an equi-continuous sequence. Let t, $\tau \in J$; then

$$\begin{aligned} |\mathcal{K}v_n(t) - \mathcal{K}v_n(\tau)| &\leq \left| \int_0^t v_n(s)\, ds - \int_0^\tau v_n(s)\, ds \right| \\ &\leq \left| \int_t^\tau |v_n(s)|\, ds \right|. \end{aligned} \tag{5.10}$$

Since $v_n \in L^1(J,\mathbb{R})$, the right hand side of (5.10) tends to 0 as $t \to \tau$. Hence, the sequence $\{\mathcal{K}v_n\}$ is equi-continuous, and an application of the Ascoli theorem implies that there is a uniformly convergent subsequence. We then have $\mathcal{K}v_{n_j} \to \mathcal{K}v \in (\mathcal{K}\circ S^1_G)(x)$ as $j \to \infty$, and so $(\mathcal{K}\circ S^1_G)(x)$ is a compact set for all $x \in X$. Therefore, B is a compact-valued multi-valued operator on X.

Again let $u_1, u_2 \in Bx$. Then there are $v_1, v_2 \in S^1_G(x)$ such that

$$u_1(t) = \frac{x_0 - k(0,x_0)}{f(0,x_0)} + \int_0^t v_1(s)\,ds, \quad t \in J,$$

and

$$u_2(t) = \frac{x_0 - k(0,x_0)}{f(0,x_0)} + \int_0^t v_2(s)\,ds, \quad t \in J.$$

Now for any $\gamma \in [0,1]$,

$$\begin{aligned}\gamma u_1(t) + (1-\gamma)u_2(t) &= \gamma\left(\frac{x_0 - k(0,x_0)}{f(0,x_0)} + \int_0^t v_1(s)\,ds\right)\\ &\quad + (1-\gamma)\left(\frac{x_0 - k(0,x_0)}{f(0,x_0)} + \int_0^t v_2(s)\,ds\right)\\ &= \frac{x_0 - k(0,x_0)}{f(0,x_0)} + \int_0^t [\gamma v_1(s) + (1-\gamma)v_2(s)]\,ds\\ &= \frac{x_0 - k(0,x_0)}{f(0,x_0)} + \int_0^t v(s)\,ds\end{aligned}$$

where $v(t) = \gamma v_1(t) + (1-\gamma)v_2(s) \in G(t,x)$ for all $t \in J$. Hence $\gamma u_1 + (1-\gamma)u_2 \in Bx$ and consequently Bx is convex for each $x \in X$. As a result B defines a multi-valued operator $B : X \to \mathcal{P}_{cp,cv}(X)$.

Step II: Next we show A and C are single-valued Lipschitz operators on X. Let $x, y \in S$. Then

$$\|Ax - Ay\| = \sup_{t\in J} |f(t,x(t)) - f(t,y(t))| \le \sup_{t\in J} \ell_1(t)|x(t) - y(t)| \le \|\ell_1\|\|x - y\|,$$

which shows that A is a multi-valued Lipschitz operator on X with the Lipschitz constant $\|\ell_1\|$. Similarly, it can be proved that C is again a Lipschitz operator on X with the Lipschitz constant $\|\ell_2\|$.

Step III: Next we show that B is completely continuous on X. Let S be a bounded subset of X. Then there is a constant $r > 0$ such that $\|x\| \le r$ for all $x \in S$. First we prove that B is compact operator on S. To do this, it is enough to prove that $B(S)$ is a uniformly bounded and equi-continuous set in X. To see this, let $u \in B(S)$ be arbitrary. Then there is a $v \in S^1_G(x)$ such that

$$u(t) = \frac{x_0 - k(0,x_0)}{f(0,x_0)} + \int_0^t v(s)\,ds$$

for some $x \in S$. Hence, by (H_3),

$$\begin{aligned}|u(t)| &\leq \Big|\frac{x_0 - k(0,x_0)}{f(0,x_0)}\Big| + \int_0^t |v(s)|\,ds\\ &\leq \Big|\frac{x_0 - k(0,x_0)}{f(0,x_0)}\Big| + \int_0^t \|G(s,x(s))\|_{\mathcal{P}}\,ds\\ &\leq \Big|\frac{x_0 - k(0,x_0)}{f(0,x_0)}\Big| + \int_0^t h(s)ds\\ &= \Big|\frac{x_0 - k(0,x_0)}{f(0,x_0)}\Big| + \|h\|_{L^1}\end{aligned}$$

for all $x \in S$ and so $B(S)$ is a uniformly bounded set in X. Again proceeding with the arguments as in Step I, we see that that

$$|u(t) - u(\tau)| \leq |p(t) - p(\tau)|$$

where $p(t) = \displaystyle\int_0^t h(s)ds$.

Notice that p is a continuous function on J, so it is uniformly continuous on J. As a result, we have that

$$|u(t) - u(\tau)| \to 0 \quad \text{as } t \to \tau.$$

This shows that $B(S)$ is an equi-continuous set in X.

Next we show that B is a upper semi-continuous multi-valued mapping on X. Let $\{x_n\}$ be a sequence in X such that $x_n \to x_*$. Let $\{y_n\}$ be a sequence such that $y_n \in Bx_n$ and $y_n \to y_*$. We shall show that $y_* \in Bx_*$. Since $y_n \in Bx_n$, there exists a $v_n \in S^1_G(x_n)$ such that

$$y_n(t) = \frac{x_0 - k(0,x_0)}{f(0,x_0)} + \int_0^t v_n(s)\,ds, \quad t \in J.$$

We must prove that there is a $v_* \in S^1_G(x_*)$ such that

$$y_*(t) = \frac{x_0 - k(0,x_0)}{f(0,x_0)} + \int_0^t v_*(s)\,ds, \quad t \in J.$$

Consider the continuous linear operator $\mathcal{L} : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by

$$\mathcal{L}v(t) = \int_0^t v(s)\,ds, \quad t \in J.$$

Now

$$\left\|\left(y_n - \frac{x_0 - k(0,x_0)}{f(0,x_0)}\right) - \left(y_* - \frac{x_0 - k(0,x_0)}{f(0,x_0)}\right)\right\| \to 0 \ \text{ as } n \to 0.$$

From Lemma 5.4, it follows that $\mathcal{L} \circ S^1_G$ is a closed graph operator. Also from the definition of $\mathcal{L}$ we have

$$y_n(t) - \frac{x_0 - k(0,x_0)}{f(0,x_0)} \in \mathcal{L} \circ S^1_G(x_n).$$

Since $y_n \to y_*$, there is a point $v_* \in S^1_G(x_*)$ such that

$$y_*(t) = \frac{x_0 - k(0,x_0)}{f(0,x_0)} + \int_0^t v_*(s)\,ds, \quad t \in J.$$

This shows that B is a upper semi-continuous operator on X. Thus, B is an upper semi-continuous and compact and hence is completely continuous multi-valued operator on X.

Step IV: Finally, we show that the conclusion (ii) of Theorem 4.13 does not hold. Let u be any solution of the IVP (5.1) such that $\mu u \in Au\,Bu + Cu$ for some $\mu > 1$. Then there is a $v \in S^1_G(u)$ such that

$$u(t) = \lambda k(t,u(t)) + \lambda\big[f(t,u(t))\big]\left(\frac{x_0}{f(0,x_0)} + \int_0^t v(s)\,ds\right)$$

for all $t \in J$, where $\lambda = \frac{1}{\mu} < 1$. Therefore, we have

$$\begin{aligned}
|u(t)| &\le |k(t,u(t))| + \Big|\big[f(t,u(t))\big]\Big|\left|\left(\frac{x_0}{f(0,x_0)} + \int_0^t v(s)\,ds\right)\right| \\
&\le |k(t,u(t)) - k(t,0)| + k(t,0)| \\
&\quad + \big[|f(t,u(t)) - f(t,0)| + f(t,0)|\big]\left(\left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t |v(s)|\,ds\right) \\
&\le \ell_2(t)|u(t)| + K + \ell_1(t)|u(t)|\left(\left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t |v(s)|\,ds\right) \\
&\quad + F\left(\left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t |v(s)|\,ds\right) \\
&\le \|\ell_2\|\|u\| + K + \|\ell_1\|\|u\|\left(\left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t h(s)\,ds\right) \\
&\quad + F\left(\left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t h(s)\,ds\right) \\
&\le \|\ell_1\|\left(\left|\frac{x_0}{f(0,x_0)}\right| + \|h\|_{L^1}\right)\|u\| + \|\ell_2\|\|u\| \\
&\quad + K + F\left(\left|\frac{x_0}{f(0,x_0)}\right| + \|h\|_{L^1}\right)
\end{aligned}$$

for all $t \in J$, where $F = \sup_{t\in J}|f(t,0)|$ and $K = \sup_{t\in J}|k(t,0)|$. Taking the supremum over t in the above inequality, we obtain a constant $M > 0$ such that

$$\|u\| \le M = \frac{K + F\left(\left|\frac{x_0}{f(0,x_0)}\right| + \|h\|_{L^1}\right)}{1 - \|\ell_1\|\left(\left|\frac{x_0}{f(0,x_0)}\right| + \|h\|_{L^1}\right) - \|\ell_2\|}$$

which is a contradiction since $\|\ell_1\|\left(\left|\frac{x_0}{f(0,x_0)}\right| + \|h\|_{L^1}\right) + \|\ell_2\| < 1/2$. As a result, the conclusion (ii) of Theorem 4.13 does not hold. Hence, the conclusion (i) holds and consequently the problem (5.1) has a solution u on J. This completes the proof. □

5.2. Multi-valued periodic boundary value problems of first order. Given a closed and bounded interval $J = [0, T]$ in $\mathbb{R}$, consider the periodic boundary value problem (in short PBVP) for the first order ordinary differential inclusion,

$$\left.\begin{aligned} \left(\frac{x(t) - k(t, x(t))}{f(t, x(t))}\right)' \in G(t, x(t)) \quad \text{a.e.} \quad t \in J, \\ x(0) = x(T), \end{aligned}\right\} \tag{5.11}$$

where $f : J \times \mathbb{R} \to \mathbb{R} - \{0\}$, $k : J \times \mathbb{R} \to \mathbb{R}-$ and $G : J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$. By a *solution* of the PBVP (5.11) we mean a function $x \in AC(J, \mathbb{R})$ satisfying

(i) the function $t \mapsto \left(\frac{x(t) - k(t, x(t))}{f(t, x(t))}\right)$ is absolutely continuous, and

(ii) there exists a function $v \in L^1(J, \mathbb{R})$ such that

$$v(t) \in G(t, x(t)) \quad \text{a.e. } t \in J$$

satisfying

$$\left(\frac{x(t) - k(t, x(t))}{f(t, x(t))}\right)' = v(t), \quad x(0) = x(T),$$

where $AC(J, \mathbb{R})$ is the space of absolutely continuous real-valued functions on J.

The PBVP (5.11) is new to the theory of differential inclusions and none of the special cases in the form of differential inclusion involving the product of two functions has been discussed in the literature.

The following useful lemma is obvious and the details may be found in Nieto [27].

Lemma 5.6. *For any $h \in L^1(J, \mathbb{R}^+)$ and $\sigma \in L^1(J, \mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{aligned} x' + h(t)x(t) = \sigma(t) \ \ a. \ e. \ t \in J \\ x(0) = x(T), \end{aligned}\right\} \tag{5.12}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T g_h(t, s)\sigma(s)\, ds \tag{5.13}$$

where

$$g_h(t, s) = \begin{cases} \dfrac{e^{H(s)-H(t)}}{1 - e^{-H(T)}}, & 0 \le s \le t \le T, \\ \dfrac{e^{H(s)-H(t)-H(T)}}{1 - e^{-H(T)}}, & 0 \le t < s \le T, \end{cases} \tag{5.14}$$

where $H(t) = \int_0^t h(s)\, ds$.

Notice that the Green's function g_h is nonnegative on $J \times J$ and the number

$$M_h := \max\{|g_h(t, s)| : t, s \in [0, T]\}$$

exists for all $h \in L^1(J, \mathbb{R}^+)$. Note also that $H(t) > 0$ for all $t > 0$ provided that h is not the identically zero function.

We will use the following hypotheses in the sequel.

(H_4) The functions $t \mapsto f(t,x)$ and $t \mapsto k(t,x)$ are periodic of period T for all $x \in \mathbb{R}$.

(H_5) The function $x \mapsto \dfrac{x - k(0,x)}{f(0,x)}$ is injective in $\mathbb{R}$.

Note that hypotheses (H_4) through (H_5) are common in the literature on the theory of nonlinear differential equations. Actually, the function $f : J \times \mathbb{R} \to \mathbb{R}$ defined by $f(t,x) = \alpha + \beta x$ for some $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta x \neq 0$ satisfies the hypotheses (H_4)–(H_5). Now consider the following PBVP with some perturbations,

$$\left.\begin{aligned} \Big(\frac{x(t) - k(t,x(t))}{f(t,x(t))}\Big)' + h(t)\Big(\frac{x(t) - k(t,x(t))}{f(t,x(t))}\Big) \in G_h(t,x(t)) \quad \text{a.e.} \quad t \in J, \\ x(0) = x(T), \end{aligned}\right\} \tag{5.15}$$

where $h \in L^1(J, \mathbb{R}^+)$ is bounded and the multi-valued function $G_h : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is defined by

$$G_h(t,x) = G(t,x) + h(t)\Big(\frac{x - k(t,x)}{f(t,x)}\Big). \tag{5.16}$$

Remark 5.7. Note that the PBVP (5.11) is equivalent to the PBVP (5.15) and a solution to the PBVP (5.11) is the solution for the PBVP (5.15) on J and vice versa.

Remark 5.8. If the function f is continuous on $J \times \mathbb{R}$ and the hypothesis (B_1) holds, then the multi-valued function G_h defined by (5.16) is Carathéodory on $J \times \mathbb{R}$. Similarly, if G has convex values on $J \times \mathbb{R}$, then G_h has also convex values on $J \times \mathbb{R}$.

Lemma 5.9. *Assume that hypotheses (H_4) and (H_5) hold. Then for any bounded $h \in L^1(J, \mathbb{R}^+)$, x is a solution to the differential inclusion* (5.15) *if and only if it is a solution of the integral equation*

$$x(t) \in k(t,x(t)) + \big[f(t,x(t))\big]\left(\int_0^T g_h(t,s) G_h(s,x(s))\right) \tag{5.17}$$

where the Green's function $g_h(t,s)$ is defined by (5.16).

Proof. Let $y(t) = \dfrac{x(t) - k(t,x(t))}{f(t,x(t))}$. Since $f(t,x)$ and $k(t,x)$ are periodic in t of period T for all $x \in \mathbb{R}$, we have

$$y(0) = \frac{x(0) - f(0,x(0))}{f(0,x(0))} = \frac{x(T) - k(T,x(T))}{f(T,x(T))} = y(T).$$

Now an application of Lemma 5.3 yields that the solution to differential equation (5.16) is the solution to integral equation (5.17). Conversely, suppose that x is any solution to the integral equation (5.17), then

$$\frac{x(0) - k(0,x(0))}{f(0,x(0))} = y(0) = y(T) = \frac{x(T) - k(0,x(T))}{f(0,x(T))}.$$

Since the function $x \mapsto \dfrac{x - k(0, x(0))}{f(0, x)}$ is injective, one has $x(0) = x(T)$ and so, x is a solution to PBVP (5.15). The proof of the lemma is complete. □

We make use of the following hypotheses in the sequel.

(H_6) The multi-valued operator $G : J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is Carathéodory.

(H_7) There exists a function $\gamma \in L^1(J, \mathbb{R}^+)$ with $\gamma(t) > 0$ a.e. $t \in J$ and a continuous and nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|G_h(t, x)\|_{\mathcal{P}} \leq \gamma(t)\psi(|x|) \quad \text{a.e. } t \in J$$

for each $x \in \mathbb{R}$.

Theorem 5.10. *Assume that the hypotheses (H_1)–(H_2) and (H_4)–(H_7) hold. Further if there exists a real number $r > 0$ such that*

$$r > \frac{K + F(M_h\|\gamma\|_{L^1}\psi(r))}{1 - \|\ell_1\|\,[M_h\|\gamma\|_{L^1}\psi(r)] - \|\ell_2\|} \tag{5.18}$$

where, $\|q_1\|M_h\|\gamma\|_{L^1}\psi(r) + \|\ell_2\| < 1/2$, $F = \sup_{t\in J}|f(t,0)|$ and $K = \sup_{t\in J}|k(t,0)|$, then the PBVP (5.11) has a solution on J.

Proof. Let $X = C(J, \mathbb{R})$ and define an open ball $\mathcal{B}_r(0)$ in X centered at origin of radius r, where the real number r satisfies the inequality (5.18). Now consider three mappings $A, C : \overline{\mathcal{B}}_r(0) \to X$ and $B : \overline{\mathcal{B}}_r(0) \to \mathcal{P}_p(\mathbb{R})$ defined by

$$Ax(t) = f(t, x(t)), \tag{5.19}$$

$$Bx = \left\{ u \in X \,\middle|\, u(t) = \int_0^T g_h(t,s)v(s)\,ds, \quad v \in S^1_{G_h}(x) \right\} \tag{5.20}$$

and

$$Cx(t) = k(t, x(t)), \tag{5.21}$$

for all $t \in J$. Then the PBVP (5.11) is equivalent to the operator inclusion

$$x(t) \in Ax(t)\,Bx(t) + Cx(t), \quad t \in J. \tag{5.22}$$

We shall show that the multi-valued operators A, B and C satisfy all the conditions of Theorem 4.8. Clearly the operator B is well defined since $S^1_{G_h}(x) \neq \emptyset$ for each $x \in \overline{\mathcal{B}}_r(0)$.

Step I: We first show that the operators A and B define, respectively, single-valued and multi-valued operators $A, C : \overline{\mathcal{B}}_r(0) \to X$ and $B : \overline{\mathcal{B}}_r(0) \to \mathcal{P}_{cp,cv}(X)$. The claim for A and C is obvious, because the functions f and k are continuous on $J \times \mathbb{R}$. We only prove the claim for the operator B. It is shown as in the Step III below that the multi-valued operator B has compact values on $\overline{\mathcal{B}}_r(0)$.

Again, let $u_1, u_2 \in Ax$. Then there are $v_1, v_2 \in S^1_{G_h}(x)$ such that

$$u_1(t) = \int_0^T g_h(t,s)v_1(s)\,ds, \; t \in J,$$

and

$$u_2(t) = \int_0^T g_h(t,s)v_2(s)\,ds,\ \ t \in J.$$

Now for any $\gamma \in [0,1]$,

$$\begin{aligned}\lambda u_1(t) + (1-\lambda)u_2(t) &= \lambda\left(\int_0^T g_h(t,s)v_1(s)\,ds\right) + (1-\lambda)\left(\int_0^T g_h(t,s)v_2(s)\,ds\right)\\ &= \int_0^T [\lambda g_h(t,s)v_1(s) + (1-\lambda)g_h(t,s)v_2(s)]\,ds\\ &= \int_0^T g_h(t,s)v(s)\,ds,\end{aligned}$$

where $v(t) = \lambda v_1(t) + (1-\lambda)v_2(s) \in G_h(t, x(t))$ for all $t \in J$. Hence, $\lambda u_1 + (1-\lambda)u_2 \in Bx$ and consequently Bx is convex for each $x \in X$. As a result, B defines a multi-valued operator $B : \overline{\mathcal{B}}_r(0) \to \mathcal{P}_{cp,cv}(X)$.

Step II: Here, it can be shown as in the Step II of the proof of Theorem 5.5 that the single-valued operators A and C are Lipschitz with the Lipschitz constants $\|\ell_1\|$ and $\|\ell_2\|$, respectively.

Step III: Next, we show that B is completely continuous on $\overline{\mathcal{B}}_r(0)$. First we prove that $B(\overline{\mathcal{B}}_r(0))$ is a totally bounded subset of X. To do this, it is enough to prove that $B(\overline{\mathcal{B}}_r(0))$ is a uniformly bounded and equi-continuous set in X. To see this, let $u \in B(\overline{\mathcal{B}}_r(0))$ be arbitrary. Then there is a $v \in S^1_{G_h}(x)$ such that

$$u(t) = \int_0^T g_h(t,s)v(s)\,ds$$

for some $x \in \overline{\mathcal{B}}_r(0)$. Hence,

$$\begin{aligned}|u(t)| &\le \int_0^T g_h(t,s)|v(s)|\,ds\\ &\le \int_0^T g_h(t,s)\big\|G_h(s,x(s)))\big\|_{\mathcal{P}}\,ds\\ &\le \int_0^T g_h(t,s)\gamma(s)\psi(r)\,ds\\ &= M_h\|\gamma\|_{L^1}\psi(r)\end{aligned}$$

for all $t \in J$ and so $B(\overline{\mathcal{B}}_r(0))$ is a uniformly bounded set in X.

Next, we show that $B(\overline{\mathcal{B}}_r(0))$ is an equi-continuous set. To finish, it is enough to show that u' is bounded on $[0,T]$. Now for any $t \in [0,T]$, one has

$$\begin{aligned}|u'(t)| &= \Big|\int_0^T \frac{\partial}{\partial t} g_h(t,s)v(s)\,ds\Big|\\ &= \Big|\int_0^T (-h(s))g_h(t,s)v(s)\,ds\Big|\\ &\le H\,M_h\|\gamma\|_{L^1}\psi(r)\end{aligned}$$

$$= c,$$

where $H = \max_{t \in J} h(t)$. Hence, for any $t, \tau \in [0, T]$ one has

$$|Bx(t) - Bx(\tau)| \leq c\, |t - \tau| \to 0 \quad \text{as} \quad t \to \tau.$$

This shows that $B(\overline{\mathcal{B}}_r(0))$ is a equi-continuous set in X. Hence, $B(\overline{\mathcal{B}}_r(0))$ is compact by Arzelá-Ascoli theorem. Thus, we have $B : \overline{\mathcal{B}}_r(0) \to \mathcal{P}_{cp,cv}(X)$ is totally bounded.

Next we show that B is a upper semi-continuous multi-valued mapping on X. Let $\{x_n\}$ be a sequence in X such that $x_n \to x_*$. Let $\{y_n\}$ be a sequence such that $y_n \in Bx_n$ and $y_n \to y_*$. We shall show that $y_* \in Bx_*$. Since $y_n \in Bx_n$, there exists a $v_n \in S^1_{G_h}(x_n)$ such that

$$y_n(t) = \int_0^T g_h(t,s) v_n(s)\, ds, \;\; t \in J.$$

We must prove that there is a $v_* \in S^1_{G_h}(x_*)$ such that

$$y_*(t) = \int_0^T g_h(t,s) v_*(s)\, ds, \;\; t \in J.$$

Consider the continuous linear operator $\mathcal{L} : L^1(J, \mathbb{R}) \to C(J, \mathbb{R})$ defined by

$$\mathcal{L}v(t) = \int_0^T g_h(t,s) v(s)\, ds, \;\; t \in J.$$

Now we have $\|y_n - y_*\| \to 0$ as $n \to 0$. From Lemma 5.4, it follows that $\mathcal{L} \circ S^1_{G_h}$ is a closed graph operator. Also, from the definition of $\mathcal{L}$, we have $y_n \in (\mathcal{L} \circ S^1_{G_h})(x_n)$. Since $y_n \to y_*$, there is a point $v_* \in S^1_{G_h}(x_*)$ such that

$$y_*(t) = \int_0^T g_h(t,s) v_*(s) ds, \;\; t \in J.$$

This shows that B is a completely continuous operator on $\overline{\mathcal{B}}_r(0)$. Thus, B is an upper semi-continuous and compact operator on $\overline{\mathcal{B}}_r(0)$.

Step IV: Finally, from condition (5.18) it follows that

$$Mq_1 + q_2 = \|\ell_1\| M_h \|\gamma\|_{L^1} \psi(r) + \|\ell_2\| < 1/2.$$

Thus all the conditions of Theorem 4.8 are satisfied and hence a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $u \in X$ be such that $\|u\| = r$ and assume that the conclusion (ii) holds.. Then, for any $\mu = \frac{1}{\lambda} > 1$, for some $\lambda \in (0, 1)$ one has

$$\mu u(t) \in k(t, u(t)) + [f(t, u(t))] \left(\int_0^T g_h(t,s) G_h(t, u(t))\, ds \right)$$

for all $t \in J$. Therefore, there is a $v \in S^1_{G_h}(u)$ such that

$$u(t) = \lambda k(t, u(t)) + \lambda \big[f(t, u(t)) \big] \Big(\int_0^T g_h(t,s) v(s)\, ds \Big)$$

for all $t \in J$. Hence, for any $t \in J$, we have

$$\begin{aligned}
|u(t)| &\leq |k(t,u(t))| + \Big|\lambda\big[f(t,u(t))\big]\Big| \, \Big|\int_0^T g_h(t,s)v(s)\,ds\Big| \\
&\leq |k(t,u(t))| + \Big|\big[f(t,u(t))\big]\Big| \, \Big(\int_0^T g_h(t,s)|v(s)|\,ds\Big) \\
&\leq |k(t,u(t)) - k(t,0)| + |k(t,0)| + \Big[|f(t,u(t)) - f(t,0)| + |f(t,0)|\Big] \\
&\qquad \times \Big(\int_0^T g_h(t,s)\big\|G_h(s,x(s)))\big\|_{\mathcal{P}}\,ds\Big) \\
&\leq |q_2(t)|\,|u(t)| + K + |q_1(t)|\,|u(t)|\Big(\int_0^T g_h(t,s)\gamma(s)\psi(r)\,ds\Big) \\
&\qquad + F\Big(\int_0^T g_h(t,s)\gamma(s)\psi(r)\,ds\Big) \\
&\leq \|\ell_2\|\,\|u\| + K + \|\ell_1\|\,\|u\|\,(M_h\|\gamma\|_{L^1}\psi(r)) + F\,(M_h\|\gamma\|_{L^1}\psi(r)) \\
&\leq \frac{K + FM_h\|\gamma\|_{L^1}\psi(r)}{1 - \|\ell_1\|\,[M_h\|\gamma\|_{L^1}\psi(r)] - \|\ell_2\|}. \qquad (5.23)
\end{aligned}$$

Taking the supremum over t in the above inequality (5.23), we obtain

$$\|u\| \leq \frac{K + FM_h\|\gamma\|_{L^1}\psi(r)}{1 - \|\ell_1\|\,[M_h\|\gamma\|_{L^1}\psi(r)] - \|\ell_2\|}$$

or,

$$r \leq \frac{K + FM_h\|\gamma\|_{L^1}\psi(r)}{1 - \|\ell_1\|\,[M_h\|\gamma\|_{L^1}\psi(r)] - \|\ell_2\|}$$

which is a contradiction to (5.18). Hence, the conclusion (ii) of Theorem 4.8 does not hold. Therefore, the operator inclusion $x \in AxBx + Cx$, and consequently the PBVP (5.11), has a solution in $\overline{\mathcal{B}}_r(0)$ defined on J. This completes the proof. □

ACKNOWLEDGMENTS

The author is very much thankful to **Prof. John R. Graef** (USA) for giving some useful suggestions for the improvement of this paper.

REFERENCES

[1] B. C. Dhage, *Multi-valued operators and fixed point theorems in Banach algebras*, Disc. Math. Differential Inclusions, Control & Optimization **24** (2004), 97–122.

[2] B. C. Dhage, *Multi-valued operators and fixed point theorems in Banach algebras II*, Computers & Mathematics with Applications **48** (2004), 1461–1478.

[3] B. C. Dhage, *Multi-valued mappings and fixed points I*, Nonlinear Functional Anal. & Appl. **10** (2005), 359–378.

[4] M. A. Krasnoselskii, *Topological Methods in The Theory of Nonlinear Integral Equations*, Pergamon Press 1964.

[5] E.Zeidler, *Nonlinear Functional Analysis and Its Applications : Part I,* Springer Verlag, 1985.

[6] T. A. Burton, *A fixed point theorem of Krasnoselskii*, Appl. Math. Lett. **11** (1998), 85–88.

[7] D. O'Regan, *New fixed point results for* 1*-set contractive set-valued maps*, Computers Math. Appl. **35** (1998), 27–34.

[8] B. C. Dhage, *On a fixed point theorem in Banach algebras with applications*, Appl. Math. Lett. **18** (2005), 273–280.

[9] B. C. Dhage, *A fixed point theorem in Banach algebras with applications to functional integral equations*, Kyungpook Math. Journal **44** (2004), 145–155.

[10] B. C. Dhage, *Some nonlinear alternatives in Banach algebras with applications I*, Nonlinear Studies **12** (2005), 271–283.

[11] B. C. Dhage, *Some nonlinear alternatives in Banach algebras with applications II*, Kyungpook Math. Journal **45** (2005), 281–292.

[12] B. C. Dhage, *Local fixed point theory involving three operators in Banach algebras*, Topological Methods in Nonlinear Anal. **24** (2004), 3773–86.

[13] A. Granas and J, Dugundji, *Fixed Point Theory*, Springer Verlag, 2003.

[14] A. Petruşel, *Multivalued operators and fixed points*, P. U. M. A. **9** (1998), 165–170.

[15] B. C. Dhage, *Periodic boundary value problems of first order Carathéodory and discontinuous differential equations*, Nonlinear Funct. Anal. & Appl. **13** (2008), 323–352.

[16] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodhina and B. N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, Birkhauser Verlag 1992.

[17] K. Deimling, *Multivalued Differential Equations*, W. de Gruyter, 1992.

[18] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis , Vol. I: Theory*, Kluwer Academic Publishers Dordrechet, 1997.

[19] J. Banas and M. Lecho, *Fixed points of the product of operators in Banach algebras*, PanAmerican Math. Journal **12** (2002), 101–109.

[20] B. C. Dhage, *Multi-valued operators and fixed point theorems in Banach algebras I*, Taiwanese J. Math. **10** (2006), 1025–1045.

[21] H. Covitz, S. B. Nadler jr., *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math. **8** (1970), 5–11.

[22] J. Aubin and A. Cellina, *Differential Inclusions*, Springer Verlag 1984.

[23] M. Martelli, *A Rothe's type theore for noncompact acyclic-valued maps*, Bull. Un. Mat. Ital. **4** (1975), 70–76.

[24] L. Rybinski, *An application of the continuous selection theorem to the study of the fixed points of multivalued mappings*, J. Math. Anal. Appl. **153** (1990), 391–396.

[25] T. C. Lim, *On fixed point stability for set-valued contraction mappings with applications to generalized differential equations*, J. Math. Anal. Appl. **110** (1985), 436–441.

[26] A. Lasota, and Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations* , Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781–786.

[27] J. J. Nieto, *Periodic boundary value problems for first order impulsive ODEs*, Nonlinear Anal. **51** (2002), 1223–1232.