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ON DEGREE OF APPROXIMATION OF A FUNCTION BELONGING TO $Lip(\xi(t), r)$ CLASS BY $(E, q)(C, 1)$ PRODUCT MEANS OF FOURIER SERIES

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ABSTRACT. In present paper, a new theorem on degree of approximation of a function belonging to $Lip(\xi(t), r)$ class by $(E, q)(C, 1)$ product summability means of Fourier series has been obtained.

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1. INTRODUCTION

Alexits [1], Sahney and Goel [11], Chandra [2],Qureshi and Neha [9], Leindler [5] and Rhoades [10] have determined the degree of approximation of a function belonging to $Lip\alpha$ class by Cesàro, Nörlund and generalised Nörlund single summability methods. Working in the same direction Sahney and Rao [12], Khan [4], Quershi [7, 8] have studied the degree of approximation of a function belonging to $Lip(\alpha, r)$ class by Nörlund and generalised Nörlund single summability methods. But nothing seems to have been done so far to obtain the degree of approximation of a function belonging to $Lip(\xi(t), r)$ class by $(E, q)(C, 1)$product summability method. The $Lip(\xi(t), r)$ is a generalization of $Lip\alpha$ class and $Lip(\alpha, r)$ class. Therefore, in present paper, a theorem on degree of approximation of a function belonging to $Lip(\xi(t), r)$ class by $(E, q)(C, 1)$ product summability means of Fourier series have been established.

2. DEFINITIONS AND NOTATIONS

Let $f(x)$ be periodic with period 2π and integrable in the sense of Lebesgue. The Fourier series of $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{2.1}$$

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with n^{th} partial sum $s_n(f;x)$. L_r-norm is defined by

$$\|f\|_r = \left\{\int_0^{2\pi} |f(x)|^r dx\right\}^{\frac{1}{r}}, r \geq 1 \tag{2.2}$$

and let the degree of approximation of a function be given by Zygmund [14].

$$E_n(f) = \min \|t_n - f\|_r \tag{2.3}$$

where $t_n(x)$ is some n^{th} degree trigonometric polynomial. A function $f \in Lip\alpha$ if

$$f(x+t) - f(x) = O(|t|^\alpha) \quad \text{for } 0 < \alpha \leq 1 \tag{2.4}$$

$f \in Lip(\alpha, r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(|t|^\alpha), \quad 0 < \alpha \leq 1 \text{ and } r \geq 1 \tag{2.5}$$

(definition 5.38 of Mc Fadden [6])

Given a positive increasing function $\xi(t)$ and an integer $r \geq 1$, $f \in Lip(\xi(t), r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(\xi(t)) \tag{2.6}$$

If $\xi(t) = t^\alpha$ then $Lip(\xi(t), r)$ class reduces to the class $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ class reduces to the class $Lip\alpha$.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of its n^{th} partial sum $\{s_n\}$. The $(C,1)$ transform is defined as the n^{th} partial sum of $(C,1)$ summability

$$\begin{aligned} t_n &= \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} \\ &= \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s \quad \text{as } n \to \infty \end{aligned} \tag{2.7}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number s by $(C,1)$ method. If

$$(E,q) = E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \to s \text{ as } n \to \infty \tag{2.8}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable (E,q) to the definite number s [3]. The (E,q) transform of the $(C,1)$ transform defines $(E,q)(C,1)$ transform and we denote it by $E_n^q C_n^1$. Thus if

$$E_n^q C_n^1 = \frac{1}{(1+q)^n}\sum_{k=0}^{n} \binom{n}{k} q^{n-k}\, C_k^1 \to s \tag{2.9}$$

where E_n^q denotes the (E,q) transform of s_n and C_n^1 denotes the $(C,1)$ transform of s_n. Then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(E,q)(C,1)$ means or summable $(E,q)(C,1)$ to a definite number s. We use the following notations:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$K_n(t) = \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \left\{ \begin{pmatrix} n \\ k \end{pmatrix} q^{n-k} \left(\frac{1}{1+k}\right) \sum_{\nu=0}^{k} \frac{\sin(\upsilon+\frac{1}{2})t}{\sin\frac{t}{2}} \right\}$$

3. MAIN THEOREM

Theorem 3.1. *If a function $f, 2\pi$-periodic, means of its Fourier series is given by*

$$\| E_n^q C_n^1 - f \|_r = O\left[(n+1)^{\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right] \tag{3.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t|\phi(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \tag{3.2}$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left\{(n+1)^{\delta}\right\} \tag{3.3}$$

where δ is an arbitrary number such that $s(1-\delta)-1>0, \frac{1}{r}+\frac{1}{s}=1,\ 1\le r\le\infty$, conditions (3.2) and (3.3) hold uniformly in x and $E_n^q C_n^1$ is $(E,1)(C,1)$ means of the series (2.1).

4. LEMMAS

For the proof of our theorem, following lemmas are required.

Lemma 4.1. $|K_n(t)| = O(n+1)$, *for* $0 \le t \le \frac{1}{n+1}$.

Proof. For $0 \le t \le \frac{1}{n+1}$, $\sin nt \le n \sin t$

$$\begin{aligned}
|K_n(t)| &= \frac{1}{2\pi\ (1+q)^n}\left|\sum_{k=0}^{n}\left[\begin{pmatrix} n \\ k \end{pmatrix} q^{n-k}\left(\frac{1}{1+k}\right)\sum_{\upsilon=0}^{k}\frac{\sin(\upsilon+\frac{1}{2})t}{\sin\frac{t}{2}}\right]\right| \\
&\le \frac{1}{2\pi\ (1+q)^n}\left|\sum_{k=0}^{n}\left[\begin{pmatrix} n \\ k \end{pmatrix} q^{n-k}\left(\frac{1}{1+k}\right)\sum_{\upsilon=0}^{k}\frac{(2\upsilon+1)\sin\frac{t}{2}}{\sin\frac{t}{2}}\right]\right| \\
&\le \frac{1}{2\pi\ (1+q)^n}\left|\sum_{k=0}^{n}\left[\begin{pmatrix} n \\ k \end{pmatrix} q^{n-k}\ (k+1)\right]\right| \\
&= O\left[\frac{(n+1)}{(1+q)^n}\sum_{k=0}^{n}\left\{\begin{pmatrix} n \\ k \end{pmatrix} q^{n-k}\right\}\right]
\end{aligned}$$

$$= O(n+1) \qquad\qquad \text{since } \sum_{k=0}^{n} \binom{n}{k} q^{n-k} = (1+q)^n$$

□

Lemma 4.2. $|K_n(t)| = O\left(\frac{1}{t}\right)$, *for* $\frac{1}{n+1} \leq t \leq \pi$.

Proof. For $\frac{1}{n+1} \leq t \leq \pi$, by applying Jordan's lemma $\sin\frac{t}{2} \geq \frac{t}{\pi}$ and $\sin nt \leq 1$

$$\begin{aligned}
|K_n(t)| &= \frac{1}{2\pi\ (1+q)^n}\left|\sum_{k=0}^{n}\left[\binom{n}{k} q^{n-k}\left(\frac{1}{1+k}\right)\sum_{\upsilon=0}^{k}\frac{\sin(\upsilon+\frac{1}{2})t}{\sin\frac{t}{2}}\right]\right| \\
&\leq \frac{1}{2\pi\ (1+q)^n}\left|\sum_{k=0}^{n}\left[\binom{n}{k} q^{n-k}\left(\frac{1}{1+k}\right)\sum_{\upsilon=0}^{k}\frac{1}{t/\pi}\right]\right| \\
&= \frac{1}{2(1+q)^n\ t}\left|\sum_{k=0}^{n}\left[\binom{n}{k} q^{n-k}\left(\frac{1}{1+k}\right)\sum_{\upsilon=0}^{k}(1)\right]\right| \\
&= \frac{1}{2(1+q)^n\ t}\left|\sum_{k=0}^{n}\left[\binom{n}{k} q^{n-k}\right]\right| \\
&= O\left(\frac{1}{t}\right) \qquad\qquad \text{since } \sum_{k=0}^{n}\binom{n}{k} q^{n-k} = (1+q)^n
\end{aligned}$$

□

5. PROOF OF THE THEOREM

Following Titchmarsh [13] and using Riemann-Lebesgue theorem, $s_n(f;x)$ of the series (2.1) is given by

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$

Therefore using (2.1), the $(C,1)$ transform C_n^1 of $s_n(f;x)$ is given by

$$C_n^1 - f(x) = \frac{1}{2\pi(n+1)}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}\frac{\sin(k+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$

Now denoting $(E,q)(C,1)$ transform of $s_n(f;x)$ by $E_n^qC_n^1$, we write

$$\begin{aligned}
E_n^qC_n^1 - f(x) &= \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\left[\binom{n}{k} q^{n-k}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\ \left(\frac{1}{k+1}\right)\left\{\sum_{\upsilon=0}^{k}\sin\left(\upsilon+\frac{1}{2}\right)t\right\}dt\ \right] \\
&= \int_0^{\pi}\phi(t)\ K_n(t)\ dt \\
&= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)K_n(t)\ dt \\
&= I_1 + I_2\ (\text{say}) \qquad (5.1)
\end{aligned}$$

We consider,

$$|I_1| \leq \int_0^{\frac{1}{n+1}} |\phi(t)||K_n(t)|dt$$

Using Hölder's inequality and the fact that $\phi(t) \in Lip(\xi(t), r)$,

$$|I_1 \leq \left[\int_0^{\frac{1}{n+1}} \left\{\frac{t|\phi(t)|}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)|K_n(t)|}{t}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)|K_n(t)|}{t}\right\}^s dt\right]^{\frac{1}{s}} \text{ by (3.2)}$$

$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{(n+1)\ \xi(t)}{t}\right\}^s dt\right]^{\frac{1}{s}} \text{ by Lemma 1}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals

$$I_1 = O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_\in^{\frac{1}{n+1}} \frac{dt}{t^s}\right]^{\frac{1}{s}} \text{ for some } 0 <\in< \frac{1}{n+1}$$

$$= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-s+1}}{-s+1}\right\}_\epsilon^{\frac{1}{n+1}}\right]^{\frac{1}{s}}$$

$$= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{1-\frac{1}{s}}\right\}$$

$$= O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} \quad \text{since } \frac{1}{r}+\frac{1}{s}=1 \tag{5.2}$$

Now we consider,

$$|I_2| \leq \int_{\frac{1}{n+1}}^{\pi} |\phi\,(t)\,|\,|K_n\,(t)|\,dt$$

Using Hölder's inequality,

$$|I_2| \leq \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta}\,|\phi\,(t)\,|}{\xi\,(t)}\right\}^r dt\,\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi\,(t)\,|K_n\,(t)|}{t^{-\delta}}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^\delta\right\}\left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi\,(t)\,|K_n\,(t)|}{t^{-\delta}}\right\}^s dt\right]^{\frac{1}{s}} \text{ by (3.3)}$$

$$= O\left\{(n+1)^\delta\right\}\left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi\,(t)}{t^{1-\delta}}\right\}^s dt\right] \text{ by Lemma 2}$$

Now putting $t = \frac{1}{y}$,

$$I_2 = O\left\{(n+1)^\delta\right\}\left[\int_{\frac{1}{\pi}}^{n+1} \left\{\frac{\xi\left(\frac{1}{y}\right)}{(y)^{\delta-1}}\right\} \frac{dy}{y^2}\,\right]^{\frac{1}{s}}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$\begin{aligned}
I_2 &= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\eta}^{n+1}\frac{dy}{y^{s(\delta-1)+2}}\right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi}\le\eta\le n+1\\
&= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{1}^{n+1}\frac{dy}{y^{s(\delta-1)+2}}\right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi}\le 1\le n+1\\
&= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{y^{s(1-\delta)-1}}{s(1-\delta)-1}\right\}_{1}^{n+1}\right]^{\frac{1}{s}}\\
&= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(1-\delta)-\frac{1}{s}}\right]\\
&= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{1-\frac{1}{s}}\right]\\
&= O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} \text{ since } \frac{1}{r}+\frac{1}{s}=1
\end{aligned} \tag{5.3}$$

Combining (5.1), (5.2) and (5.3),

$$\left|E_n^q C_n^1 - f(x)\right| = O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} \tag{5.4}$$

Now, using L_r-norm, we get

$$\begin{aligned}
\left\|E_n^q C_n^1 - f(x)\right\|_r &= \left\{\int_0^{2\pi}\left|E_n^q C_n^1 - f(x)\right|^r dx\right\}^{\frac{1}{r}}\\
&= O\left[\left\{\int_0^{2\pi}\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}}\right]\\
&= O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\int_0^{2\pi}dx\right\}^{\frac{1}{r}}\right]\\
&= O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}
\end{aligned}$$

This completes the proof of the theorem.

6. APPLICATIONS

Following corollaries can be derived from our main theorems.

Corollary 6.1. *If $\xi(t) = t^{\alpha}$, $0 < \alpha \le 1$, then the class $Lip(\xi(t), r)$, $r \ge 1$, reduces to the class $Lip(\alpha, r)$ and the degree of approximation of a function $f \in Lip(\alpha, r)$, $\frac{1}{r} < \alpha < 1$, is given by*

$$\left|E_n^q C_n^1 - f\right| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right)$$

Proof. We have

$$\left\|E_n^q C_n^1 - f\right\|_r = O\left\{\int_0^{2\pi} \left|E_n^q C_n^1 - f\right|^r dx\right\}^{\frac{1}{r}}$$

or

$$\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} = O\left\{\int_0^{2\pi} \left|E_n^q C_n^1 - f\right|^r dx\right\}^{\frac{1}{r}}$$

or

$$O(1) = O\left\{\int_0^{2\pi} \left|E_n^q C_n^1 - f\right|^r dx\right\}^{\frac{1}{r}} \; O\left\{\frac{1}{O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}}\right\}$$

Hence

$$\left|E_n^q C_n^1 - f\right| = O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}$$

for if not the right-hand side will be $O(1)$, therefore

$$\begin{aligned}\left|E_n^q C_n^1 - f\right| &= O\left\{\left(\frac{1}{n+1}\right)^{\alpha}(n+1)^{\frac{1}{r}}\right\}\\ &= O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right)\end{aligned}$$

□

Corollary 6.2. *If $r \to \infty$ in corollary 1, then the class Lip (α, r) reduces to the class $f \in Lip\alpha$ and the degree of approximation of a function $f \in Lip\alpha$, $0 < \alpha < 1$, is given by*

$$\| \; E_n^q C_n^1 - f \; \|_\infty = O\left\{\frac{1}{(n+1)^{\alpha}}\right\}$$

Corollary 6.3. *If $\xi(t) = t^{\alpha}$, $0 < \alpha \le 1$, then the class Lip $(\xi(t), r)$, $r \ge 1$, reduces to the class Lip (α, r) and if q=1 then (E, q) summability reduces to $(E, 1)$ summability and the degree of approximation of a function $f \in$ Lip (α, r), $\frac{1}{r} < \alpha < 1$, is given by*

$$\left|(EC)_n^1 - f\right| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right)$$

Corollary 6.4. *If $r \to \infty$ in corollary 3, then the class Lip (α, r) reduces to the class $f \in Lip\alpha$ and the degree of approximation of a function $f \in Lip\alpha$, $0 < \alpha < 1$, is given by*

$$\left\|(EC)_n^1 - f\right\|_\infty = O\left\{\frac{1}{(n+1)^{\alpha}}\right\}$$

Remark 6.5. Independent proofs of above corollaries 1 and 3 can be obtained along the same lines of our theorem.

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