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STABILITY ANALYSIS IN TERMS OF TWO MEASURES FOR SETVALUED HYBRID INTEGRO-DIFFERENTIAL EQUATIONS OF MIXED TYPE

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ABSTRACT. We study some stability criteria in terms of two measures for setvalued perturbed hybrid integro-differential equations of mixed type with fixed moments of impulse. Stability properties of perturbed system are obtained via a comparison result which connects the solutions of perturbed system and the unperturbed one through the solutions of a comparison system.

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1. INTRODUCTION

The study of setvalued differential equations has recently gained much attention due to its applicability to multivalued differential inclusions and fuzzy differential equations, for instance, see [1-6] and the references therein. An other interesting feature of the setvalued differential equations is that the results obtained in this new framework become the corresponding results of ordinary differential equations as the Hukuhara derivative and the integral used in formulating the set differential equations reduce to the ordinary vector derivative and integral when the set under consideration is a single valued mapping.

In the perturbation theory of nonlinear differential systems, a flexible mechanism known as variation of Lyapunov second method, was introduced in [7]. This technique, which essentially connects the solutions of perturbed system and the unperturbed one through the solutions of a comparison system using a comparison principle, was extended to integral equations in [8]. The concept of stability in terms of two measures [9] which unifies a number of stability concepts such as Lyapunov stability, partial stability, conditional stability, etc. has become an important area of investigation in the qualitative analysis [10-15].

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Impulsive hybrid dynamical systems form a class of hybrid systems in which continuous time states are reset discontinuously when the discrete event states change. Recently, a number of research papers has dealt with dynamical systems with impulsive effect as a class of general hybrid systems [16-22]. In this paper, we develop the stability criteria in terms of two measures for setvalued perturbed hybrid integro-differential equations of mixed type with fixed moments of impulsive effect through the variation of Lyapunov second method.

2. PRELIMINARIES AND COMPARISON RESULT

Let $K_c(\mathbb{R}^n)$ denote the collection of nonempty, compact and convex subsets of $\mathbb{R}^n$. We define the Hausdorff metric as

$$D[X,Y] = \max[\sup_{y\in Y} d(y,X), \sup_{x\in X} d(x,Y)], \tag{1}$$

where $d(y,X) = \inf[d(y,x) : x \in X]$ and X, Y are bounded subsets of $\mathbb{R}^n$. Notice that $K_c(\mathbb{R}^n)$ with the metric is a complete metric space. Moreover, $K_c(\mathbb{R}^n)$ equipped with the natural algebraic operations of addition and nonnegative scalar multiplication becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space [6,23]. The Hausdorff metric (1) satisfies the following properties:

$$D[X+Z, Y+Z] = D[X,Y] \text{ and } D[X,Y] = D[Y,X], \tag{2}$$

$$D[\mu X, \mu Y] = \mu D[X,Y], \tag{3}$$

$$D[X,Y] \le D[X,Z] + D[Z,Y], \tag{4}$$

$\forall\ X, Y, Z \in K_c(\mathbb{R}^n)$ and $\mu \in \mathbb{R}_+$.

Definition 2.1. The set $Z \in K_c(\mathbb{R}^n)$ satisfying $X = Y + Z$ is known as the Hukuhara difference of the sets X and Y in $K_c(\mathbb{R}^n)$ and is denoted as $X - Y$.

Definition 2.2. For any interval $I \in R$, the mapping $F : I \to K_c(\mathbb{R}^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if there exists an element $D_H F(t_0) \in K_c(\mathbb{R}^n)$ such that the limits

$$\lim_{h\to 0^+} \frac{F(t_0+h) - F(t_0)}{h} \text{ and } \lim_{h\to 0^+} \frac{F(t_0) - F(t_0-h)}{h}, \tag{5}$$

exist in the topology of $K_c(\mathbb{R}^n)$ and each one is equal to $D_H F(t_0)$.

By embedding $K_c(\mathbb{R}^n)$ as a complete cone in a corresponding Banach space and taking into account the result on differentiation of Bochner integral, it is found that if

$$F(t) = X_0 + \int_0^t \Phi(\eta)d\eta,\ X_0 \in K_c(\mathbb{R}^n), \tag{6}$$

where $\Phi : I \to K_c(\mathbb{R}^n)$ is integrable in the sense of Bochner, then $D_H F(t)$ exists and

$$D_H F(t) = \Phi(t) \text{ a.e. on } I. \tag{7}$$

Consider the following perturbed setvalued integro-differential equations of mixed type with fixed moments of impulse

$$\begin{cases} D_H U(t) = F(t, U(t), L_1 U(t), S_1 U(t)), & t \neq t_k, \\ U(t_k^+) = U(t_k) + I_k(U(t_k)), & k = 1, 2, 3, \dots, \\ U(t_0^+) = U_0, & t_0 \geq 0, \end{cases} \tag{8}$$

together with the unperturbed ones

$$\begin{cases} D_H V(t) = G(t, V(t), L_2 V(t), S_2 V(t)), & t \neq t_k, \\ V(t_k^+) = V(t_k) + J_k(V(t_k)), & k = 1, 2, 3, \dots, \\ V(t_0^+) = U_0, & t_0 \geq 0, \end{cases} \tag{9}$$

where $F, G : \mathbb{R}_+ \times K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$ are continuous on $(t_{k-1}, t_k] \times K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^n)$, with G smooth enough or containing the linear terms of system (8), L_i, S_i denote the integral in sense of Hukuhara [24-25] and are defined by $L_i U(t) = \int_{t_0}^{\gamma(t)} K_i(t, \eta, U(\eta)) d\eta,\ S_i U(t) = \int_{t_0}^{T} H_i(t, \xi, U(\xi)) d\xi,\ K_i,\ H_i : \mathbb{R}_+ \times \mathbb{R}_+ \times K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$ are continuous on $(t_{k-1}, t_k] \times (t_{k-1}, t_k] \times K_c(\mathbb{R}^n),\ i = 1, 2,\ \gamma \in C(\mathbb{R}_+, \mathbb{R}_+),\ I_k, J_k : K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$ and $\{t_k\}$ is a sequence of points such that $t_0 < t_1 < \cdots t_k < \cdots$ with $\lim_{k\to\infty} t_k = \infty$.

Letting ρ to be a positive real number, we define the following classes of functions:

$$K = \{\nu : [0, \rho) \to \mathbb{R}_+ \text{ is continuous, strictly increasing and } \nu(0) = 0\};$$

$$PC = \{\mu : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is continuous on } (t_{k-1}, t_k] \text{ and } \mu \to \mu(t_k^+) \text{ exists as } t \to t_k^+\};$$

$$PCK = \{\phi : R_+ \times [0, \rho) \to \mathbb{R}_+,\ \phi(., m) \in PC \text{ for each } m \in [0, \rho), \phi(t, .) \in K \text{ for each } t \in \mathbb{R}_+\};$$

$$\Gamma = \{h : \mathbb{R}_+ \times K_c(\mathbb{R}^n) \to \mathbb{R}_+,\ \inf_{U \in K_c(\mathbb{R}^n)} h(t, U) = 0,\ h(., U) \in PC, \text{ for each } U \in K_c(\mathbb{R}^n), \text{ and } h(t, .) \in C(K_c(\mathbb{R}^n), \mathbb{R}_+) \text{ for each } t \in \mathbb{R}_+\};$$

$$S(h, \rho) = \{(t, U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n) : h(t, U) < \rho, h \in \Gamma\};$$

$$S(\rho) = \{U \in K_c(\mathbb{R}^n) : (t, U) \in S(h, \rho) \text{ for each } t \in \mathbb{R}_+\}.$$

Definition 2.3. W is said to belong to a class W_0 if $W(t, U) \in PC$ for each $U \in S(\rho)$ and $W(t, U)$ is locally Lipschitzian in U.

Definition 2.4. Let $W \in W_0$ and $V(t, \eta, U)$ be any solution of (9). Then for any fixed $t > t_0,\ (\eta, U) \in (t_{k-1}, t_k) \times S(\rho),\ t_0 \leq \eta < t$, we define

$$\begin{aligned} & D^+ W(\eta, V(t, \eta, U)) \\ = & \lim sup_{h \to 0^+} \frac{1}{h} [W(\eta + h, V(t, \eta + h, U + hF(\eta, U, L_1 U, S_1 U))) - W(\eta, V(t, \eta, U))], \end{aligned}$$

where $V(t, \eta, U)$ is any solution of (9) such that $V(\eta, \eta, U) = U$.

Definition 2.5. Let $h,\ h_0 \in \Gamma$. We say that

(i) h_0 is finer than h if there exists a $\overline{\lambda} > 0$ and a function $\phi \in PCK$ such that

$$h_0(t,U) < \overline{\lambda} \text{ implies } h(t,U) \leq \phi(t, h_0(t,U));$$

(ii) h_0 is uniformly finer than h if (i) holds for $\phi \in K$.

Definition 2.6. Let $h,\ h_0 \in \Gamma$ and $W \in W_0$. Then $W(t,U)$ is said to be

(i) $h-$ positive definite if there exists a $\lambda > 0$ and a function $b \in K$ such that

$$h(t,U) < \lambda \text{ implies } b(h(t,U)) \leq W(t,U);$$

(ii) weakly h_0-decrescent if there exists a $\lambda_1 > 0$ and a function $a \in PCK$ such that

$$h_0(t,U) < \lambda_1 \text{ implies } W(t,U) \leq a(t, h_0(t,U));$$

(iii) h_0-decrescent if (ii) holds with $a \in K$.

Definition 2.7. Let $h,\ h_0 \in \Gamma$ and $U((t) = U(t,t_0,U_0)$ be any solution of (8), then the system (8) is said to be

(I) $(h_0,h)-$stable if for each $\epsilon > 0$, there exists a $\delta = \delta(t_0,\epsilon) > 0$ such that

$$h_0(t_0,U_0) < \delta \text{ implies } h(t,U(t))) < \epsilon,\ t \geq t_0;$$

(II) (h_0,h)-uniformly stable if (I) holds with δ independent of t_0;

(III) (h_0,h)-attractive if there exists a $\delta = \delta(t_0) > 0$ and for each $\epsilon > 0$, there exists $T = T(t_0,\epsilon) > 0$ such that

$$h_0(t_0,U_0) < \delta_0 \text{ implies } h(t,U(t)) < \epsilon,\ t \geq t_0 + T;$$

(IV) (h_0,h)-uniformly attractive if (III) holds with δ and T independent of t_0;

(V) (h_0,h)-asymptotically stable if it is (h_0,h)-stable and (h_0,h)-attractive;

(VI) (h_0,h)-uniformly asymptotically stable if it is (h_0,h)-uniformly stable and (h_0,h)-uniformly attractive.

Now, we prove a comparison result which is needed for the sequel.

Lemma 2.8. *Assume that*

($\mathbf{A_1}$) *the solution $V(t) = V(t,t_0,U_0)$ of (9) existing for all $t \geq t_0$ is unique, continuous with respect to the initial values, locally Lipschitzian in U_0 and $V(t_0) = U_0$;*

($\mathbf{A_2}$) *$W \in PC[\mathbb{R}_+ \times K_c(\mathbb{R}^n)]$ satisfies $|W(t,X) - W(t,Y)| \leq ND[X,Y]$, where N is the local Lipschitz constant, $X, Y \in K_c(\mathbb{R}^n),\ t \in \mathbb{R}_+$;*

($\mathbf{A_3}$) *for* $(\eta, U) \in S(h, \rho),\ t_0 \le \eta < t,\ W \in W_0$ *satisfies the inequality*

$$\begin{cases} D^+W(\eta, V(t, \eta, U)) \le g_1(\eta, W(\eta, V(t, \eta, U))), & t \ne t_k, \\ W(t_k^+, V(t, t_k^+, U(t_k^+))) \le \psi_k(W(t_k, V(t, t_k, U(t_k))), & k = 1, 2, \ldots, \\ W(t_0^+, V(t, t_0^+, U_0)) \le x_0, \end{cases}$$

where $g_1(t, .) \in PC$ *for each value of the second variable and* $\psi_k(.)$ *are nondecreasing functions for all* $= 1, 2, \ldots;$

($\mathbf{A_4}$) *the maximal solution* $r(t) = r(t, t_0, x_0)$ *of the following scalar impulsive differential equation exists on* $[t_0, \infty)$

$$\begin{cases} x' = g_1(t, x), & t \ne t_k, \\ x(t_k^+) = \psi_k(x(t_k)),\ k = 1, 2, \ldots, \\ x(t_0^+) = x_0 \ge 0. \end{cases} \tag{10}$$

Then $W(t, U(t, t_0, U_0)) \le r(t, t_0, x_0)$.

Proof. Let $U(t) = U(t, t_0, U_0)$ be any solutions of (8) with $(t_0, U_0) \in S(h, \rho)$. We set $m(\eta) = W(\eta, V(t, \eta, U(\eta)),\ \eta \in [t_0, t]$ and $\lim_{\eta \to t-0} m(\eta) = m(t)$. For small $h > 0$, we consider

$$\begin{aligned} m(\eta + h) - m(\eta) &= W(\eta + h, V(t, \eta + h, U(\eta + h))) - W(\eta, V(t, \eta, U(\eta)) \\ &= W(\eta + h, V(t, \eta + h, U(\eta + h))) - W(\eta + h, V(t, \eta + h, U(\eta) \\ &\quad + hF(\eta, U(\eta), L_1U(\eta), S_1U(\eta)))) + W(\eta + h, V(t, \eta + h, U(\eta) \\ &\quad + hF(\eta, U(\eta), L_1U(\eta), S_1U(\eta)))) - W(\eta, V(t, \eta, U(\eta))) \\ &\le ND[V(t, \eta + h, U(\eta + h)), V(t, \eta + h, U(\eta) + hF(\eta, U(\eta), L_1U(\eta), S_1U(\eta)))] \\ &\quad + W(\eta + h, V(t, \eta + h, U(\eta) + hF(\eta, U(\eta), L_1U(\eta), S_1U(\eta)))) - W(\eta, V(t, \eta, U(\eta))) \end{aligned}$$

where we have used the condition (A_2). Thus,

$$\begin{aligned} D^+m(t) = \limsup_{h \to 0^+} \frac{1}{h}[m(t + h) - m(t)] &\le D^+W(\eta, V(t, \eta, U(\eta)) \\ &\quad + N^2 \limsup_{h \to 0^+} \frac{1}{h} D[U(\eta + h), U(\eta) + hF(\eta, U(\eta), L_1U(\eta), S_1U(\eta))]. \end{aligned}$$

Letting $U(\eta + h) = U(\eta) + Z(\eta)$, where $Z(\eta)$ is the Hukuhara difference of $U(\eta + h)$ and $U(\eta)$ for small $h > 0$ and is assumed to exist. Hence, employing the properties of $D[., .]$, it follows that

$$\begin{aligned} &D[U(\eta + h), U(\eta) + hF(\eta, U(\eta), L_1U(\eta), S_1U(\eta))] \\ &\quad = D[U(\eta) + Z(\eta), U(\eta) + hF(\eta, U(\eta), L_1U(\eta), S_1U(\eta))] \\ &\quad = D[Z(\eta), hF(\eta, U(\eta), L_1U(\eta), S_1U(\eta))] \\ &\quad = D[U(\eta + h) - U(\eta), hF(\eta, U(\eta), L_1U(\eta), S_1U(\eta))]. \end{aligned}$$

Consequently, we find that

$$\frac{1}{h}D[U(\eta+h),U(\eta)+hF(\eta,U(\eta),L_1U(\eta),S_1U(\eta))]$$
$$= D[\frac{U(\eta+h)-U(\eta)}{h},F(\eta,U(\eta),L_1U(\eta),S_1U(\eta))],$$

which, in view of the fact that $U(t)$ is a solution of (8), yields

$$\limsup_{h\to 0^+}\frac{1}{h}D[U(\eta+h),U(\eta)+hF(\eta,U(\eta),L_1U(\eta),S_1U(\eta))]$$
$$= \limsup_{h\to 0^+} D[\frac{U(\eta+h)-U(\eta)}{h},F(\eta,U(\eta),L_1U(\eta),S_1U(\eta))]$$
$$= D[U'_H(\eta),F(\eta,U(\eta),L_1U(\eta),S_1U(\eta))] = 0.$$

Hence, we have

$$D^+m(\eta) \le g(\eta,m(\eta)),\ t \ne t_k.$$

Also

$$m(t_k^+) = \psi_k(m(t_k)),\ k = 1,2,\ldots,$$

$$m(t_0) \le x_0.$$

Now, from reference [11], it follows that $m(\eta) \le r(\eta,t_0,x_0)$, $\eta \in [t_0,t]$, that is, $W(\eta,V(t,\eta,U(\eta)) \le r(\eta,t_0,x_0)$, $\eta \in [t_0,t]$. Since $V(t,t,U(t)) = U(t)$, therefore we have

$$W(t,U(t,t_0,U_0)) = W(t,V(t,t,U(t))) \le r(t,t_0,x_0).$$

This proves the assertion of the theorem. □

3. STABILITY CRITERIA FOE SETVALUED HYBRID INTEGRO-DIFFERENTIAL EQUATIONS

Theorem 3.1. *Assume that*

($\mathbf{B_1}$) *the solution $V(t) = V(t,t_0,U_0)$ of (9) existing for all $t \ge t_0$ is unique, continuous with respect to the initial values, locally Lipschitzian in U_0 and $V(t_0) = U_0$.*

($\mathbf{B_2}$) *$K_i(t,s,0) = 0$ so that $G(t,0,0) = G(t,0) = 0$, $g_1(t,0) = 0$ and $J_k(0) = 0, \psi_k(0) = 0$, $k = 1,2,\ldots$;*

($\mathbf{B_3}$) *$h_0, h \in \Gamma$ such that $h_0(t,0) = 0$ for $t \in \mathbb{R}_+$ and h_0 is finer than h;*

($\mathbf{B_4}$) *$W \in W_0$ be such that $W(t,U)$ is $h-$positive definite and weakly h_0- decrescent for $(t,U) \in S(h,\rho)$, and satisfies the inequality*

$$\begin{cases} D^+W(\eta,V(t,\eta,U)) \le g_1(\eta,W(\eta,V(t,\eta,U))),\ \eta \ne t_k, \\ (\eta,U) \in S(h,\rho), \eta \in [t_0,t), \\ W(t_k^+,V(t,t_k^+,U(t_k^+))) \le \psi_k(W(t_k,V(t,t_k,U(t_k)))),\ k = 1,2,\ldots; \end{cases}$$

$(\mathbf{B_5})$ *there exists a* $\rho_0 \in (0, \rho]$ *such that*

$$h(t_k, U(t_k)) < \rho_0 \text{ implies that } h(t_k^+, U(t_k^+)) < \rho, \ k = 1, 2, \ldots.$$

Then the stability of the trivial solution of (9) and the asymptotical stability of the trivial solution of (10) imply the $(h_0, h)-$ *asymptotical stability of (8).*

Proof. Let $U(t) = U(t, t_0, U_0), \ V(t) = V(t, t_0, U_0)$ and $x(t) = x(t, t_0, x_0)$ be any solutions of (8), (9) and (10) respectively. Since $W(t, U)$ is $h-$positive definite on $S(h, \rho)$, there exists $b \in K$ such that

$$h(t, U) < \rho \text{ implies } b(h(t, U)) \le W(t, U). \tag{11}$$

Also $W(t, U)$ is weakly h_0-decrescent and h_0 is finer than h, so there exists a $\lambda_0 > 0$ and $a \in PCK, \ \phi \in PCK$ such that

$$h(t, U) \le \phi(t, h_0(t, U)) \text{ implies } W(t, U) \le a(t, h_0(t, U)), \tag{12}$$

when $h_0(t, U) < \lambda_0$ and $\phi(t_0^+, \lambda_0) < \rho$. Since the trivial solution of (10) is stable, therefore, for given $b(\epsilon) > 0$, we can find a $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that

$$0 \le x_0 < \delta_1 \text{ implies that } \ x(t, t_0, x_0) < b(\epsilon), \ t \ge t_0, \tag{13}$$

where $0 < \epsilon < \rho_0$ and $t_0 \in \mathbb{R}_+$. Also, the trivial solution of (9) is stable, so there exists a $\delta_2 = \delta_2(t_0, \epsilon) > 0$ corresponding to δ_1 such that

$$\|U_0\| < \delta_2 \text{ implies } \|V(t)\| < a^{-1}(t_0, \delta_1),$$

while, from (B_3), we have

$$h_0(t_0^+, U_0) < \delta_2 \text{ implies } h_0(t_0^+, V(t)) < a^{-1}(t_0, \delta_1). \tag{14}$$

Select $\delta = \delta(t_0, \epsilon) > 0$ satisfying $\delta < \min\{\lambda_0, \delta_2\}$. Now if $h_0(t_0^+, U_0) < \delta$, then it follows from (11)-(14) that

$$b(h(t_0^+, U_0)) \le W(t_0^+, U_0) \le a(t_0^+, h_0(t_0^+, U_0)) < a(t_0^+, \delta_2) \le \delta_1 \le b(\epsilon),$$

which implies that $h(t_0^+, U_0)) < \epsilon$ when $h_0(t_0^+, U_0)) < \delta$.
We assert that

$$h(t, U(t)) < \epsilon \text{ whenever } h_0(t_0^+, U_0)) < \delta. \tag{15}$$

For the sake of contradiction, let us assume that (15) is false and there exists $t^* > t_0$ such that $h(t^*, U(t^*)) \ge \epsilon$. For $h \in \Gamma$, there are two cases: (i) $t_0 < t^* \le t_1$ (ii) $t_k < t^* \le t_{k+1}$ for some $k = 1, 2, \ldots$.
(i) Without loss of generality, let $t^* = \inf\{t : h(t, U(t)) \ge \epsilon\}$ and $h(t^*, U(t^*)) = \epsilon$.

Using Lemma 2.8 and (11)-(12) together with the fact that $r(t,t_0,x_1) \le r(t,t_0,x_2)$ if $x_1 \le x_2$ (which follows from Lemma 2.8), we obtain

$$W(t^*,U(t^*)) \le r(t^*,t_0,W(t_0^+,V(t^*,t_0,U_0))) \le r(t^*,t_0,a(t_0,h(t_0^+,V(t^*,t_0,U_0))) \le r(t^*,t_0,\delta_1) < b(\epsilon). (16)$$

On the other hand, it follows from (11) that

$$W(t^*,U(t^*)) \ge b(h(t^*,U(t^*))) = b(\epsilon),$$

which contradicts (15).

(ii) In view of the impulse effect, we have

$$h(t^*,U(t^*)) \ge \epsilon \text{ and } h(t,U(t)) < \epsilon,\ t \in [t_0,t_k].$$

Since $0 < \epsilon < \rho_0$, it follows from assumption (B_5) that

$$h(t_k^+,U(t_k^+)) = h(t_k^+,U(t_k)+I_k(U(t_k))) < \rho.$$

Consequently, there exists a $t^{**} \in (t_k,t^*]$ such that

$$\epsilon \le h(t^{**},U(t^{**})) < \rho \text{ and } h(t,U(t)) < \rho,\ t \in [t_0,t_1). \tag{17}$$

Now, by virtue of Lemma 2.8 and (11)-(12), we obtain

$$W(t^{**},U(t^{**})) \le r(t^{**},t_0,W(t_0^+,V(t^{**},t_0,U_0))) \le r(t^{**},t_0,a(t_0,h(t_0^+,V(t^{**},t_0,U_0))) \le r(t^{**},t_0,\delta_1) < b(\epsilon),$$

whereas (11) and (17) yields

$$W(t^{**},U(t^{**})) \ge b(h(t^{**},U(t^{**}))) \ge b(\epsilon),$$

which is again a contradiction. Thus our assertion is true and the $(h_0,h)-$ stability of the system (8) is proved.

Next it is assumed that the trivial solution of (10) is asymptotically stable. In view of $(h_0,h)-$ stability of the system (8), we set $\epsilon = \rho_0$ and $\delta = \delta_3 = \delta_3(t_0,\rho_0) > 0$ in (15) and obtain

$$h(t,U(t)) < \rho_0 < \rho \text{ whenever } h_0(t_0^+,U_0)) < \delta_3,\ t \ge t_0.$$

In order to prove the $(h_0,h)-$ attractive of system (8), let the trivial solution of (10) be attractive, that is, for $t_0 \in R_+$, there exists a $\delta_0^* = \delta_0^*(t_0) > 0$ such that

$$x_0 < \delta_0^* \text{ implies that } \lim_{t\to\infty} x(t,t_0,x_0) = 0.$$

Now, for this δ_0^*, there is a $\delta_1^* = \delta_1^*(t_0,\delta_0^*) > 0$ such that

$$h_0(t_0^+,U_0) < \delta_1^* \text{ implies that } h_0(t_0^+,V(t)) < a^{-1}(t_0,\delta_0^*).$$

Taking $\delta_0 = \delta_0(t_0)$ (independent of ϵ) such that $0 < \delta_0 < \min\{\delta^*, \delta_0^*, \delta_1^*\}$ and applying the earlier arguments, we find that

$$b(h(t, U(t))) \leq W(t, U(t)) \leq r(t, t_0, W(t_0^+, V(t, t_0, U_0))) \leq r(t, t_0, \delta_0^*) \to 0,$$

as $t \to \infty$ when $h_0(t_0^+, U_0)) < \delta_0$. This implies that $\lim_{t\to\infty} h(t, U(t)) = 0$ when $h_0(t_0^+, U_0)) < \delta_0$, that is, system (8) is $(h_0, h)-$ attractive. Hence system (8) is $(h_0, h)-$asymptotically stable. □

Theorem 3.2. *Assume that all the assumptions of Theorem 3.1 hold except* (B_3) *and* (B_4) *which are modified as*

($\mathbf{B_3^*}$) h_0 *is uniformly finer than* h *instead of finer in* (B_3)*;*
($\mathbf{B_4^*}$) W *is* h_0- *decrescent instead of weakly* h_0- *decrescent in* (B_4).

Then the uniform stability of the trivial solution of (9) and the uniformly asymptotical stability of the trivial solution of (10) imply the $(h_0, h)-$ *uniformly asymptotical stability of (8).*

Proof. From (B_3^*) and (B_4^*), it follows that there exists a $\lambda_0 > 0$ and $a,\ \phi \in K$ such that

$$h(t, U) \leq \phi(h_0(t, U)) \text{ implies } W(t, U) \leq a(h_0(t, U)), \tag{18}$$

when $h_0(t, U) < \lambda_0$ with $\phi(\lambda_0) < \rho$. The trivial solution of (10) is uniformly stable, therefore, for given $b(\epsilon) > 0$, we can find a $\delta_1 = \delta_1(\epsilon) > 0$ independent of t_0 such that

$$0 \leq x_0 < \delta_1 \text{ implies } \ x(t, t_0, x_0) < b(\epsilon),\ t \geq t_0, \tag{19}$$

where $0 < \epsilon < \rho_0$ and $t_0 \in \mathbb{R}_+$. From the hypothesis that the trivial solution of (9) is uniformly stable, for the above δ_1, there exists a $\delta_2 = \delta_2(\epsilon) > 0$ independent of t_0 such that

$$\|U_0\| < \delta_2 \text{ implies that } \ \|V(t)\| < a^{-1}(\delta_1).$$

On the other hand, from (B_3^*), we have

$$h_0(t_0^+, U_0) < \delta_2 \text{ implies that } \ h_0(t_0^+, V(t)) < a^{-1}(\delta_1). \tag{20}$$

Now, applying the arguments similar to the ones used in the proof of Theorem 3.1, we conclude that

$$h_0(t_0^+, U_0) < \delta \text{ implies } \ h(t_0^+, U(t)) < \epsilon,\ t \geq t_0,$$

where δ is independent of t_0 and satisfies $0 < \delta = \delta(\epsilon) < \min\{\lambda_0, \delta_2\}$. Thus, the system (8) is $(h_0, h)-$ uniformly stable.
Next, from the hypothesis that the trivial solution of (10) is uniformly asymptotically

stable, we can find a $\delta_0^* > 0$ independent of t_0 and any ϵ satisfying $0 < \epsilon < \rho_0$ such that there exists a $\tau = \tau(\epsilon)$ so that

$$0 < x_0 < \delta_0^* \text{ implies } \ x(t, t_0, x_0) < b(\epsilon),\ t \geq t_0 + \tau(\epsilon),\ t_0 \in R_+. \tag{21}$$

In view of that fact that (9) is uniformly stable, there is a δ_1^* independent of t_0 corresponding to δ_0^* such that

$$h_0(t_0^+, U_0) < \delta_1^* \text{ implies that } \ h_0(t_0^+, V(t)) < a^{-1}(\delta_0^*).$$

Since uniformly asymptotically stability of (10) implies its asymptotically stability, so system (8) is $(h_0, h)-$ uniformly stable. For $\epsilon = \rho_0$, there exists a $\delta^* = \delta^*(\rho_0)$ such that

$$h_0(t_0^+, U_0) < \delta^* \text{ implies } \ h(t, U(t)) < \rho_0 < \rho,\ t \geq t_0.$$

Choosing δ_0 such that $0 < \delta_0 < \min\{\delta^*, \delta_0^*, \delta_1^*\}$ and using the arguments employed in Theorem 3.1, we find that $h(t, U(t))) \leq \epsilon,\ t \geq t_0 + \tau$, when $h_0(t_0^+, U_0)) < \delta_0$, where δ_0 and τ are independent of t_0. This implies that system (8) is $(h_0, h)-$ uniformly attractive. Hence system (8) is $(h_0, h)-$ uniformly asymptotically stable. $\square$

Remarks The $(h_0, h)-$ equatability of (8) can be established on the same pattern if we require $\delta = \delta(t_0, \epsilon)$ in Definition 2.7 to be a continuous function in t_0 for each ϵ. Setting $L_1U \equiv 0 \equiv L_2U,\ S_1U \equiv 0 \equiv S_2U$ in (8) and (9), our results reduce to the ones for setvalued perturbed hybrid differential equations with fixed moments of impulse. Moreover, if the solution $U(t)$ is a single valued mapping, and Hukuhara derivative and integral used here reduce to the ordinary vector derivative and integral, then the results of reference [15] appear as a special case of our results for $S_1U \equiv 0 \equiv S_2U$ in (8) and (9).

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