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MONOTONE TECHNIQUE FOR NONLINEAR DEGENERATE WEAKLY COUPLED SYSTEM OF PARABOLIC PROBLEMS

D.B.DHAIGUDE[1], B.R.SONTAKKE[2], AND CHANDRADEEPA D.DHAIGUDE[3]

[1,3]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad - 431 004 , INDIA
E-mail:dnyanraja@gmail.com
[2]Department of Mathematics, Pratishthan Mahavidyalaya, Paithan - 431 107 Dist.Aurangabad, INDIA
E-mail:brsontakke@rediffmail.com

ABSTRACT. The purpose of this paper is to develop monotone technique by introducing the notion of upper and lower solutions together with the associated monotone iterations for nonlinear weakly coupled time degenerate parabolic system with initial and boundary conditions. Under suitable initial iterations and for mixed quasimonotone boundary functions, two monotone sequences are constructed. It is shown that these two sequences converge monotonically from above and below respectively to maximal and minimal solutions of Dirichlet initial boundary value problem for nonlinear weakly coupled time degenerate parabolic system which leads to existence-comparison and uniqueness results for the solution of the Dirichlet initial boundary value problem for nonlinear weakly coupled time degenerate parabolic system.

1. INTRODUCTION

Monotone technique is one of the important and widely known method in the theory of applied nonlinear analysis. It is developed and extensively employed in the study of various aspects of both elliptic and parabolic boundary value problems, which arise in physical, chemical and biological phenomena.The method of upper and lower solutions is employed successfully in the study of existence-comparison and uniqueness of solutions of initial boundary value problem (IBVP) of a nonlinear partial differential equation.An excellent account of these results are given in the elegant books by Ladde, Lakshmikantham and Vatsala[4], Leung[5] and Pao[6]. Recently, the monotone technique is developed by Dhaigude, Dhaigude and Dhaigude [2], for nonlinear time degenerate parabolic IBVP.The qualitative properties such as existence-comparison and uniqueness of solution of time degenerate problem are studied. We extend this study by developing monotone technique for system of weakly

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coupled nonlinear time degenerate parabolic problem when the reaction functions are mixed quasimonotone.Such results for weakly coupled nonlinear uniformly parabolic problems have been established by Chandra, Dressel and Norman [1].

2.UPPER-LOWER SOLUTIONS

In this section, we state the following Dirichlet IBVP for weakly coupled nonlinear time degenerate parabolic system

$$\begin{aligned} d^{(1)}(x,t)(u_1)_t - D^{(1)}\nabla^2 u_1 &= f^{(1)}(x,t,u_1,u_2) \\ d^{(2)}(x,t)(u_2)_t - D^{(2)}\nabla^2 u_2 &= f^{(2)}(x,t,u_1,u_2) \end{aligned} \quad \text{in } D_T \tag{2.1}$$

with boundary conditions:

$$\begin{aligned} u_1(x,t) &= g^{(1)}(x,t) \\ u_2(x,t) &= g^{(2)}(x,t) \end{aligned} \quad \text{on } S_T \tag{2.2}$$

and initial conditions:

$$\begin{aligned} u_1(x,0) &= u_{1,0}(x) \\ u_2(x,0) &= u_{2,0}(x) \end{aligned} \quad \text{in } \Omega \tag{2.3}$$

Here Ω is a bounded domain in $R^n (n = 1, 2, \dots)$ with boundary $\partial\Omega$, $D_T := \Omega \times (O, T]$ is parabolic domain and $S_T := \delta\Omega \times (O, T], T > 0$ is parabolic boundary.

Suppose that the functions $d^{(1)}(x,t), d^{(2)}(x,t)$ are nonnegative in D_T.However we will not assume that $d^{(1)}(x,t)$ and $d^{(2)}(x,t)$ are bounded away from zero.Since we assume that $d^{(1)}(x,t) = 0$, $d^{(2)}(x,t) = 0$ for some $(x,t) \in D_T$ and hence the system is time degenerate.Further suppose that $D^{(1)} > 0, D^{(2)} > 0$ are constants in D_T.The functions $f^{(1)}(x,t,u_1,u_2), f^{(2)}(x,t,u_1,u_2)$ are in general nonlinear in u_1, u_2 and depend explicitly on (x,t).The functions $f^{(1)}(x,t,u_1,u_2), f^{(2)}(x,t,u_1,u_2)$, $g^{(1)}(x,t), g^{(2)}(x,t)$ and $u_{1,0}(x), u_{2,0}(x)$ are Hölder continuous in their respective domains.Suppose that the reaction functions $f^{(1)}(x,t,u_1,u_2)$ and $f^{(2)}(x,t,u_1,u_2)$ are mixed quasimonotone.

Definition 2.1. A C^1-function $(f^{(1)}, f^{(2)})$ is said to be mixed quasimonotone in $J \subset R^2$, if

$$\frac{\partial f^{(1)}}{\partial u_2} \leq 0, \ \frac{\partial f^{(2)}}{\partial u_1} \geq 0; \quad \text{(or vice versa)}$$

for $(u_1, u_2) \in J_1 \times J_2 = J$.

Definition 2.2. Two functions $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ and $\hat{u} = (\hat{u}_1, \hat{u}_2)$in $C(\overline{D}_T) \cap C^{2,1}(D_T)$ with the condition $(\tilde{u}_1, \tilde{u}_2) \geq (\hat{u}_1, \hat{u}_2)$are called ordered upper and lower solutions of the Dirichlet IBVP (2.1)-(2.3) if they satisfy the differential inequalities:

$$\left.\begin{aligned} d^{(1)}(x,t)(\tilde{u}_1)_t - D^{(1)}\nabla^2\tilde{u}_1 &\geq f^{(1)}(x,t,\tilde{u}_1,\hat{u}_2)\\ d^{(1)}(x,t)(\hat{u}_1)_t - D^{(1)}\nabla^2\hat{u}_1 &\leq f^{(1)}(x,t,\hat{u}_1,\tilde{u}_2)\\ d^{(2)}(x,t)(\tilde{u}_2)_t - D^{(2)}\nabla^2\tilde{u}_2 &\geq f^{(2)}(x,t,\tilde{u}_1,\tilde{u}_2)\\ d^{(2)}(x,t)(\hat{u}_2)_t - D^{(2)}\nabla^2\hat{u}_2 &\leq f^{(2)}(x,t,\hat{u}_1,\hat{u}_2) \end{aligned}\right\} \quad \text{in } D_T$$

boundary conditions:

$$\left.\begin{aligned} \tilde{u}_1(x,t) &\geq g^{(1)}(x,t)\\ \hat{u}_1(x,t) &\leq g^{(1)}(x,t)\\ \tilde{u}_2(x,t) &\geq g^{(2)}(x,t)\\ \hat{u}_2(x,t) &\leq g^{(2)}(x,t) \end{aligned}\right\} \quad \text{on } S_T$$

and initial conditions:

$$\left.\begin{aligned} \tilde{u}_1(x,0) &\geq u_{1,0}(x)\\ \hat{u}_1(x,0) &\leq u_{1,0}(x)\\ \tilde{u}_2(x,0) &\geq u_{2,0}(x)\\ \hat{u}_2(x,0) &\leq u_{2,0}(x) \end{aligned}\right\} \quad \text{in } \Omega$$

Definition 2.3. Let $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ and $\hat{u} = (\hat{u}_1, \hat{u}_2)$ be any two functions with $(\tilde{u}_1, \tilde{u}_2) \geq (\hat{u}_1, \hat{u}_2)$ then we define the sector

$$\langle \hat{u}, \tilde{u} \rangle = \{\, (u_1, u_2) \in C(\overline{D}_T) : (\hat{u}_1, \hat{u}_2) \leq (u_1, u_2) \leq (\tilde{u}_1, \tilde{u}_2) \,\}$$

Assume that $(f^{(1)}, f^{(2)})$ satisfies the one sided Lipschitz condition then there exists nonnegative constants $\underline{c}_1$ and $\underline{c}_2$ such that for every pair of $(u_1, u_2), (v_1, v_2)$ in the sector $\langle \hat{u}, \tilde{u} \rangle$,

$$\begin{aligned} f^{(1)}(x,t,u_1,u_2) - f^{(1)}(x,t,v_1,u_2) &\geq -\underline{c}_1(u_1 - v_1) \qquad \text{for} \quad \hat{u} \leq v_1 \leq u_1 \leq \tilde{u}\\ f^{(2)}(x,t,u_1,u_2) - f^{(2)}(x,t,u_1,v_2) &\geq -\underline{c}_2(u_2 - v_2) \qquad \text{for} \quad \hat{u} \leq v_2 \leq u_2 \leq \tilde{u} \end{aligned} \tag{2.4}$$

which ensure the existence of a solution of Dirichlet IBVP (2.1)-(2.3). Further assume that

$$\begin{aligned} F^{(1)}(x,t,u_1,u_2) &= \underline{c}_1 + f^{(1)}(x,t,u_1,u_2)\\ F^{(2)}(x,t,u_1,u_2) &= \underline{c}_2 + f^{(2)}(x,t,u_1,u_2) \end{aligned}$$

are Hölder continuous in $D_T \times \langle \hat{u}, \tilde{u} \rangle$.

Lemma 2.4. *Suppose (u_1, u_2) and (v_1, v_2) are any two functions in the sector $\langle \hat{u}, \tilde{u} \rangle$ such that $(u_1, u_2) \geq (v_1, v_2)$. Assume that $(f^{(1)}, f^{(2)})$ is mixed quasimonotone and satisfy Lipchitz conditions (2.4). Then*

$$\begin{aligned} F^{(1)}(x,t,u_1,u_2) &\geq F^{(1)}(x,t,v_1,v_2)\\ F^{(2)}(x,t,u_1,u_2) &\geq F^{(2)}(x,t,v_1,v_2) \end{aligned}$$

Proof: The Lipschitz conditions (2.4)and mixed quasimonotone property of $(f^{(1)}, f^{(2)})$, implies

$$F^{(1)}(x,t,u_1,v_2) - F^{(1)}(x,t,v_1,u_2) = [\underline{c}_1(u_1-v_1) + f^{(1)}(x,t,u_1,v_2) - f^{(1)}(x,t,v_1,v_2)] + [f^{(1)}(x,t,v_1,v_2) - f^{(1)}(x,t,v_1,u_2)] \geq 0.$$

Thus, $F^{(1)}(x,t,u_1,v_2) - F^{(1)}(x,t,v_1,u_2) \geq 0$. Similarly we can prove $F^{(2)}(x,t,u_1,u_2) - F^{(2)}(x,t,v_1,v_2) \geq 0$

Lemma 2.5 (Positivity Lemma. Dhaigude, Dhaigude and Dhaigude [2])**.** *Suppose that* $u \in C(\overline{D}_T) \cap C^{2,1}(D_T)$ *and satisfies the inequalities*

(i) $d(x,t)u_t - D(x,t)\nabla^2 u + c(x,t)u \geq 0$ *in* D_T

(ii) $\alpha(x)\frac{\partial u}{\partial \nu} + \beta(x)u \geq 0$ *on* S_T

(iii) $u(x,0) \geq 0$ *in* Ω

where $d(x,t) \geq 0, \quad D(x,t) > 0, \quad c(x,t) \geq 0$ *in* D_T

Then $u(x,t) \geq 0$ *in* $\overline{D}_T$.

3. MONOTONE ITERATIVE TECHNIQUE

In this section, we develop monotone method for time degenerate parabolic Dirichlet IBVP (2.1)-(2.3); by introducing the notion of upper and lower solutions. The operators L_1 and L_2 are

$$\left.\begin{aligned} L_1[u_1] &\equiv d^{(1)}(x,t)(u_1)_t - D^{(1)}\nabla^2 u_1 + \underline{c}_1 u_1 \\ L_2[u_2] &\equiv d^{(2)}(x,t)(u_2)_t - D^{(2)}\nabla^2 u_2 + \underline{c}_2 u_2 \end{aligned}\right. \quad \text{in } D_T$$

then the differential equations in (2.1) are equivalent to

$$\begin{aligned} L_1[u_1] &= \underline{c}_1 u_1 + f^{(1)}(x,t,u_1,u_2) \\ L_2[u_2] &= \underline{c}_2 u_2 + f^{(2)}(x,t,u_1,u_2) \end{aligned}$$

Monotone Iterative Process. The monotone iterative processes are given by

$$\begin{aligned} L_1[\overline{u}_1^{(k)}] &= \underline{c}_1 \overline{u}_1^{(k-1)} + f^{(1)}(x,t,\overline{u}_1^{(k-1)}, \underline{u}_2^{(k-1)}) \\ \overline{u}_1^{(k)}(x,t) &= g^{(1)}(x,t) \\ \overline{u}_1^{(k)}(x,0) &= u_{1,0}(x) \end{aligned} \tag{3.1}$$

where $k = 1, 2, \ldots$

$$\begin{aligned} L_1[\underline{u}_1^{(k)}] &= \underline{c}_1 \underline{u}_1^{(k-1)} + f^{(1)}(x,t,\underline{u}_1^{(k-1)}, \overline{u}_2^{(k-1)}) \\ \underline{u}_1^{(k)}(x,t) &= g^{(1)}(x,t) \\ \underline{u}_1^{(k)}(x,0) &= u_{1,0}(x) \end{aligned} \tag{3.2}$$

where $k = 1, 2, \ldots$

$$\begin{aligned} L_2[\overline{u}_2^{(k)}] &= \underline{c}_2\overline{u}_2^{(k-1)} + f^{(2)}(x,t,\overline{u}_1^{(k-1)},\overline{u}_2^{(k-1)}) \\ \overline{u}_2^{(k)}(x,t) &= g^{(2)}(x,t) \\ \overline{u}_2^{(k)}(x,0) &= u_{2,0}(x) \end{aligned} \tag{3.3}$$

where $k = 1, 2, \dots$

$$\begin{aligned} L_2[\underline{u}_2^{(k)}] &= \underline{c}_2\underline{u}_2^{(k-1)} + f^{(2)}(x,t,\underline{u}_1^{(k-1)},\underline{u}_2^{(k-1)}) \\ \underline{u}_2^{(k)}(x,t) &= g^{(2)}(x,t) \\ \underline{u}_2^{(k)}(x,0) &= u_{2,0}(x) \end{aligned} \tag{3.4}$$

where $k = 1, 2, \dots$

For $k = 1$, we start with $(\overline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\hat{u}_1, \hat{u}_2)$ as an initial iteration in the iteration process (3.3) and applying the existence theory for linear time degenerate parabolic initial boundary value problem Ippolito [3], we get $\overline{u}_2^{(1)}$.

Similarly, we start with $(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\tilde{u}_1, \tilde{u}_2)$ as an initial iteration in the iteration process (3.4) and applying the existence theory for linear time degenerate parabolic initial boundary value problem Ippolito [3], we get $\underline{u}_2^{(1)}$.

Now, we start with $(\overline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\hat{u}_1, \tilde{u}_2)$ as an initial iteration in the iteration process (3.1) and applying the existence theory for linear time degenerate parabolic initial boundary value problem Ippolito [3], we get $\overline{u}_1^{(1)}$.

Similarly, consider initial iterations as $(\underline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\tilde{u}_1, \hat{u}_2)$ in the iteration process (3.2) and applying the existence theory for linear time degenerate parabolic initial boundary value problem Ippolito [3], we get $\underline{u}_1^{(1)}$.
Thus for $k = 1$, we obtain first iterations $(\overline{u}_1^{(0)}, \overline{u}_2^{(0)})$ and $(\underline{u}_1^{(0)}, \underline{u}_2^{(0)})$.

For $k = 2$, we start with $(\overline{u}_1^{(1)}, \overline{u}_2^{(1)}) = (\hat{u}_1, \hat{u}_2)$ as an initial iteration in the iteration process (3.3) and applying the existence theory for linear time degenerate parabolic initial boundary value problem Ippolito [3], we get $\overline{u}_2^{(2)}$.

Similarly, we start with $(\underline{u}_1^{(1)}, \underline{u}_2^{(1)}) = (\tilde{u}_1, \tilde{u}_2)$ as an initial iteration in the iteration process (3.4) and applying the existence theory for linear time degenerate parabolic initial boundary value problem Ippolito [3], we get $\underline{u}_2^{(2)}$.

Now, we start with $(\overline{u}_1^{(1)}, \underline{u}_2^{(1)}) = (\hat{u}_1, \tilde{u}_2)$ as an initial iteration in the iteration process (3.1) and applying the existence theory for linear time degenerate parabolic initial boundary value problem Ippolito [3], we get $\overline{u}_1^{(2)}$.

Similarly, consider initial iterations as $(\underline{u}_1^{(1)}, \overline{u}_2^{(1)}) = (\tilde{u}_1, \hat{u}_2)$ in the iteration process (3.2)and applying the existence theory for linear time degenerate parabolic initial boundary value problem Ippolito [3], we get $\underline{u}_1^{(2)}$.
Thus for $k = 2$, we obtain second iterations $(\overline{u}_1^{(2)}, \overline{u}_2^{(2)})$ and $(\underline{u}_1^{(2)}, \underline{u}_2^{(2)})$.
Similarly for $k = 3, 4, \dots$ we obtain the sequence of these iterations as $\{\, \overline{u}_1^{(k)}, \overline{u}_2^{(k)} \,\}$ and $\{\, \underline{u}_1^{(k)}, \underline{u}_2^{(k)} \,\}, \dots$

Thus we observe that in this iteration process the equations in (3.1) - (3.4) are uncoupled but are inter related in the sense that the k^{th} iteration $(\overline{u}_1^{(k)}, \overline{u}_2^{(k)})$ or $(\underline{u}_1^{(k)}, \underline{u}_2^{(k)})$ depends on all four components in the $(k-1)^{th}$ iteration.

Now, we prove monotone property of these two sequences $\{\, \overline{u}_1^{(k)}, \overline{u}_2^{(k)} \,\}$ and $\{\, \underline{u}_1^{(k)}, \underline{u}_2^{(k)} \,\}$.

Lemma 3.1 (Monotone Property)**.** *Suppose that*

(i) $(\tilde{u}_1, \tilde{u}_2), (\hat{u}_1, \hat{u}_2)$ *are ordered upper and lower solutions of the Dirichlet IBVP* (2.1)-(2.3).

(ii) the reaction functions $f^{(1)}, f^{(2)}$ *are mixed quasimonotone*

(iii) the functions $f^{(1)}(x,t,u_1,u_2)$ *and* $f^{(2)}(x,t,u_1,u_2)$ *satisfy the onesided Lipschitz conditions in* u_1, u_2.

$$f^{(1)}(x,t,u_1,u_2) - f^{(1)}(x,t,v_1,u_2) \geq -\underline{c}_1(u_1 - v_1,) \quad for \quad \hat{u} \leq v_1 \leq u_1 \leq \tilde{u}$$

$$f^{(2)}(x,t,u_1,u_2) - f^{(2)}(x,t,u_1,v_2) \geq -\underline{c}_2(u_2 - v_2,) \quad for \quad \hat{u} \leq v_2 \leq u_2 \leq \tilde{u}$$

Then the sequences $\{\, \overline{u}_1^{(k)}, \overline{u}_2^{(k)} \,\}$ *and* $\{\, \underline{u}_1^{(k)}, \underline{u}_2^{(k)} \,\}$ *possess the monotone property*

$$\hat{u}_1 \leq \underline{u}_1^{(k)} \leq \underline{u}_1^{(k+1)} \leq \overline{u}_1^{(k+1)} \leq \overline{u}_1^{(k)} \leq \tilde{u}_1 \qquad in\ \overline{D}_T \tag{3.5}$$

$$\hat{u}_2 \leq \underline{u}_2^{(k)} \leq \underline{u}_2^{(k+1)} \leq \overline{u}_2^{(k+1)} \leq \overline{u}_2^{(k)} \leq \tilde{u}_2 \qquad in\ \overline{D}_T \tag{3.6}$$

for $k = 1, 2, \ldots$

Proof: Define

$$w_1 = \overline{u}_1^{(0)} - \overline{u}_1^{(1)} = \tilde{u}_1 - u_1^{(1)} \qquad (\overline{u}_1^{(0)} = \tilde{u}_1)$$

By definition 2.2, we have

$$\begin{aligned} d^{(1)}(x,t)(\tilde{u}_1)_t - D^{(1)}\nabla^2\tilde{u}_1 &\geq f^{(1)}(x,t,\tilde{u}_1,\hat{u}_2) \quad \text{in } D_T \\ \tilde{u}_1(x,t) &\geq g^{(1)}(x,t) \qquad \text{on } S_T \\ \tilde{u}_1(x,0) &\geq u_{1,0}(x) \qquad \text{in } \Omega \end{aligned}$$

We have

$$\begin{aligned} L_1[w_1] &= d^{(1)}(x,t)(w_1)_t - D^{(1)}\nabla^2 w_1 + \underline{c}_1 w_1 \\ &= [d^{(1)}(x,t)](\tilde{u}_1)_t - D^{(1)}\nabla^2\tilde{u}_1 + \underline{c}_1\tilde{u}_1] \\ &\qquad - [d^{(1)}(x,t)(\overline{u}_1^{(1)})_t - D^{(1)}(x,t)\nabla^2\overline{u}_1^{(1)} + \underline{c}_1\overline{u}_1^{(1)}] \end{aligned}$$

$$\begin{aligned} L_1[w_1] &= [d^{(1)}(x,t)(\tilde{u}_1)_t - D^{(1)}\nabla^2\tilde{u}_1 + \underline{c}_1\tilde{u}_1] - [\underline{c}_1\overline{u}_1^{(0)} + f^{(1)}(x,t,\overline{u}_1^{(0)},\underline{u}_2^{(0)})] \\ &\qquad \text{(By iterative process)} \\ &= d^{(1)}(x,t)(\tilde{u}_1)_t - D^{(1)}\nabla^2\tilde{u}_1 + \underline{c}_1\tilde{u}_1 - [\underline{c}_1\tilde{u}_1 - f^{(1)}(x,t,\tilde{u}_1,\hat{u}_2)] \\ &\qquad \text{(By using } \overline{u}_1^{(0)} = \tilde{u}_1 \text{ and } \underline{u}_2^{(0)} = \hat{u}_2) \\ &= d^{(1)}(x,t)(\tilde{u}_1)_t - D^{(1)}\nabla^2\tilde{u}_1 - f^{(1)}(x,t,\tilde{u}_1,\hat{u}_2) \geq 0 \end{aligned}$$

$$L_1[w_1] \equiv d^{(1)}(x,t)(w_1^{(0)})_t - D^{(1)}\nabla^2 w_1^{(0)} + \underline{c}_1 w_1^{(0)} \geq 0 \qquad \text{in } D_T.$$

Also

$$w_1(x,t) = \tilde{u}_1(x,t) - \overline{u}_1^{(1)}(x,t) \geq \tilde{u}_1(x,t) - g^{(1)}(x,t) \geq 0 \qquad \text{on } S_T$$

and

$$w_1^{(}x,0) = \tilde{u}_1(x,0) - \overline{u}_1^{(1)}(x,0) \geq \tilde{u}_1(x,0) - u_{1,0}(x) \geq 0 \qquad \text{in } \Omega$$

Now applying the Lemma 2.5, we get

$$w_1^{(}x,t) \geq 0 \quad \text{in } \overline{D}_T$$

This implies that

$$\overline{u}_1^{(1)} \leq \overline{u}_1^{(0)} \qquad \text{in } \overline{D}_T \tag{3.7}$$

Define

$$w_1 = \underline{u}_1^{(1)} - \underline{u}_1^{(0)} = \underline{u}_1^{(1)} - \hat{u}_1 \qquad (\underline{u}_1^{(0)} = \hat{u}_1)$$

We get on similar lines

$$\underline{u}_1^{(0)} \leq \underline{u}_1^{(1)} \quad \text{in } \overline{D}_T \tag{3.8}$$

Now we define,

$$w_1^{(1)} = \overline{u}_1^{(1)} - \underline{u}_1^{(1)}$$

We have

$$\begin{aligned} L_1[w_1^{(1)}] &= d^{(1)}(x,t)(w_1^{(1)})_t - D^{(1)}\nabla^2 w_1^{(1)} + \underline{c}_1 w_1^{(1)} \\ &= [d^{(1)}(x,t)(\overline{u}_1^{(1)})_t - D^{(1)}\nabla^2 \overline{u}_1^{(1)} + \underline{c}_1 \overline{u}_1^{(1)}] \\ &\qquad - [d^{(1)}(x,t)(\underline{u}_1^{(1)})_t - D^{(1)}\nabla^2 \underline{u}_1^{(1)} + \underline{c}_1 \underline{u}_1^{(1)}] \end{aligned}$$

$$\begin{aligned} L_1[w_1^{(1)}] &= [\underline{c}_1(x,t)\overline{u}_1^{(0)} + f^{(1)}(x,t,\overline{u}_1^{(0)},\underline{u}_2^{(0)})] - [\underline{c}_1(x,t)\underline{u}_1^{(0)} + f^{(1)}(x,t,\underline{u}_1^{(0)},\overline{u}_2^{(0)})] \\ &\qquad \text{(By iterative scheme)} \\ &= [\underline{c}_1\tilde{u}_1 + f^{(1)}(x,t,\tilde{u}_1,\hat{u}_2)] - [\underline{c}_1\hat{u}_1 + f^{(1)}(x,t,\hat{u}_1,\tilde{u}_2)] \\ &\qquad \text{(By using } \overline{u}_1^{(0)} = \tilde{u}_1,\ \underline{u}_1^{(0)} = \hat{u}_1,\ \overline{u}_2^{(0)} = \tilde{u}_2 \text{ and } \underline{u}_2^{(0)} = \hat{u}_2) \\ &= F^{(1)}(x,t,\tilde{u}_1,\hat{u}_2) - F^{(1)}(x,t,\hat{u}_1,\tilde{u}_2) \geq 0 \text{ (By Lemma 2.4)} \end{aligned}$$

$$L_1[w_1^{(1)}] \equiv d^{(1)}(x,t)(w_1^{(1)})_t - D^{(1)}\nabla^2 w_1^{(1)} + \underline{c}_1 w_1^{(1)} \geq 0 \qquad \text{in } D_T.$$

Also

$$w_1^{(1)}(x,t) = \overline{u}_1^{(1)}(x,t) - \underline{u}_1^{(1)}(x,t) \geq g^{(1)}(x,t) - g^{(1)}(x,t) = 0 \qquad \text{on } S_T$$

and

$$w_1^{(1)}(x,0) = \overline{u}_1^{(1)}(x,0) - \underline{u}_1^{(1)}(x,0) \geq u_{1,0}(x) - u_{1,0}(x) = 0 \qquad \text{in } \Omega$$

Applying the Lemma 2.5, we get

$$w_1^{(1)}(x,t) \geq 0 \quad \text{in } \overline{D}_T$$

This implies that

$$\underline{u}_1^{(1)} \leq \overline{u}_1^{(1)} \qquad \text{in } \overline{D}_T \tag{3.9}$$

Thus from (3.7), (3.8), (3.9), we get

$$\underline{u}_1^{(0)} \leq \underline{u}_1^{(1)} \leq \overline{u}_1^{(1)} \leq \overline{u}_1^{(0)} \quad \text{in } \overline{D}_T \tag{3.10}$$

Thus result is true for $k = 1$. We assume,the result is true for k

$$\underline{u}_1^{(k-1)} \leq \underline{u}_1^{(k)} \leq \overline{u}_1^{(k)} \leq \overline{u}_1^{(k-1)}$$

and prove it for $k + 1$

$$\underline{u}_1^{(k)} \leq \underline{u}_1^{(k+1)} \leq \overline{u}_1^{(k+1)} \leq \overline{u}_1^{(k)} \quad \text{in } \overline{D}_T$$

Define a function

$$w_1^{(k)} = \overline{u}_1^{(k)} - \overline{u}_1^{(k+1)}$$

We have

$$\begin{aligned} L_1[w_1^{(k)}] &= d^{(1)}(x,t)(w_1^{(k)})_t - D^{(1)}\nabla^2 w_1^{(k)} + \underline{c}_1 w_1^{(k)} \\ &= [d^{(1)}(x,t)(\overline{u}_1^{(k)})_t - D^{(1)}(x,t)\nabla^2 \overline{u}_1^{(k)} + \underline{c}_1 \overline{u}_1^{(k)}] \\ &\qquad - [d^{(1)}(x,t)(\overline{u}_2^{(k+1)})_t - D^{(1)}(x,t)\nabla^2 \overline{u}_1^{(k+1)} + \underline{c}_1 \overline{u}_1^{(k+1)}] \end{aligned}$$

$$\begin{aligned} L_1[w_1^{(k)}] &= [\underline{c}_1 \overline{u}_1^{(k-1)} + f^{(1)}(x,t,\overline{u}_1^{(k-1)}, \underline{u}_2^{(k-1)})] - [\underline{c}_1 \overline{u}_1^{(k)} + f^{(1)}(x,t,\overline{u}_1^{(k)}, \underline{u}_2^{(k)})] \\ &\qquad \text{(By iterative process)} \\ &= F^{(1)}(x,t,\overline{u}_1^{(k-1)}, \underline{u}_2^{(k-1)}) - F^{(1)}(x,t,\overline{u}_1^{(k)}, \underline{u}_2^{(k)}) \geq 0 \ \text{(By Lemma 2.4)} \\ L_1[w_1^{(k)}] &\equiv d^{(1)}(x,t)(w_1^{(k)})_t - D^{(1)}(x,t)\nabla^2 w_1^{(k)} + \underline{c}_1 w_1^{(k)} \geq 0 \qquad \text{in } D_T \end{aligned}$$

Also

$$w_1^{(k)}(x,t) = \overline{u}_1^{(k)}(x,t) - \overline{u}_1^{(k+1)}(x,t) \geq g^{(1)}(x,t) - g^{(1)}(x,t) = 0 \qquad \text{on } S_T$$

and

$$w_1^{(k)}(x,0) = \overline{u}_1^{(k)}(x,0) - \overline{u}_1^{(k+1)}(x,0) \geq u_{1,0}(x) - u_{1,0}(x) = 0 \qquad \text{in } \Omega$$

Applying the Lemma 2.5, we get

$$w_1^{(k)}(x,t) \geq 0 \qquad \text{in } \overline{D}_T$$

This implies that
$$\overline{u}_1^{(k+1)} \leq \overline{u}_1^{(k)} \tag{3.11}$$
Define a function
$$w_1^{(k)} = \underline{u}_1^{(k+1)} - \underline{u}_1^{(k)}$$
On similar lines we get
$$\underline{u}_1^{(k)} \leq \underline{u}_1^{(k+1)} \quad \text{in } \overline{D}_T. \tag{3.12}$$
Define
$$w_1^{(k+1)} = \overline{u}_1^{(k+1)} - \underline{u}_1^{(k+1)}$$
On similar lines we get
$$\underline{u}_1^{(k+1)} \leq \overline{u}_1^{(k+1)} \quad \text{in } \overline{D}_T \tag{3.13}$$
Thus, we get
$$\underline{u}_1^{(k)} \leq \underline{u}_1^{(k+1)} \leq \overline{u}_1^{(k+1)} \leq \overline{u}_1^{(k)} \quad \text{in } \overline{D}_T$$
From the principle of induction, we get,
$$\hat{u}_1 \leq \underline{u}_1^{(k)} \leq \underline{u}_1^{(k+1)} \leq \overline{u}_1^{(k+1)} \leq \overline{u}_1^{(k)} \leq \tilde{u}_1 \qquad \text{in } \overline{D}_T$$
for $k = 1, 2, \ldots$ On similar lines we get monotone property (3.6),
$$\hat{u}_2 \leq \underline{u}_2^{(k)} \leq \underline{u}_2^{(k+1)} \leq \overline{u}_2^{(k+1)} \leq \overline{u}_2^{(k)} \leq \tilde{u}_2 \qquad \text{in } \overline{D}_T$$
for $k = 1, 2, \ldots$

4. APPLICATIONS

In this section, we prove existence - comparison and uniqueness of solution of time degenerate parabolic Dirichlet IBVP (2.1)-(2.3) when the functions $f^{(1)}, f^{(2)}$ are mixed quasimonotone.

Theorem 4.1 (Existence-Comparison). *Suppose that*

(i) $(\tilde{u}_1, \tilde{u}_2), (\hat{u}_1, \hat{u}_2)$ are the ordered upper and lower solutions of degenerate Dirichlit IBVP (2.1)-(2.3)

(ii) the reaction functions $f^{(1)}, f^{(2)}$ are mixed quasimonotone

(iii) the functions $f^{(1)}(x, t, u_1, u_2)$ and $f^{(2)}(x, t, u_1, u_2)$satisfy the one sided Lipschitz condition in u_1, u_2

$$f^{(1)}(x, t, u_1, u_2) - f^{(1)}(x, t, v_1, u_2) \geq -\underline{c}_1(u_1 - v_1) \quad \text{for} \quad \hat{u} \leq v_1 \leq u_1 \leq \tilde{u} \tag{4.1}$$
$$f^{(2)}(x, t, u_1, u_2) - f^{(2)}(x, t, u_1, v_2) \geq -\underline{c}_2(u_2 - v_2) \quad \text{for} \quad \hat{u} \leq v_2 \leq u_2 \leq \tilde{u} \tag{4.2}$$

Then the sequences $\{\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\}$ and $\{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\}$ converges monotonically to their respective maximal solution $(\overline{u}_1, \overline{u}_2)$ and minimal solution $(\underline{u}_1, \underline{u}_2)$ of degenerate Dirichlet IBVP (2.1)-(2.3) *and satisfy the relations*
$$\hat{u}_1 \leq \underline{u}_1^{(k)} \leq \underline{u}_1^{(k+1)} \leq \underline{u}_1 \leq \overline{u}_1 \leq \overline{u}_1^{(k+1)} \leq \overline{u}_1^{(k)} \leq \tilde{u}_1 \quad \textit{in } \overline{D}_T \tag{4.3}$$

and

$$\hat{u}_2 \leq \underline{u}_2^{(k)} \leq \underline{u}_2^{(k+1)} \leq \underline{u}_2 \leq \overline{u}_2 \leq \overline{u}_2^{(k+1)} \leq \overline{u}_2^{(k)} \leq \tilde{u}_2 \quad \text{in } \overline{D}_T \tag{4.4}$$

where $k = 1, 2, \ldots$

Proof: From Lemma 3.1, we conclude that the sequence $\{\,\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\,\}$ is monotone nonincreasing and bounded from below hence it is convergent to some limit function. Also the sequence $\{\,\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\,\}$ is monotone nondecreasing and is bounded from above hence it is convergent to some limit function. So

$$\lim_{k\to\infty} \overline{u}_1^{(k)}(x,t) = \overline{u}_1(x,t); \lim_{k\to\infty} \overline{u}_2^{(k)}(x,t) = \overline{u}_2(x,t)$$

and

$$\lim_{k\to\infty} \underline{u}_1^{(k)}(x,t) = \underline{u}_1(x,t); \lim_{k\to\infty} \underline{u}_2^{(k)}(x,t) = \underline{u}_2(x,t)$$

exist and called maximal and minimal solutions respectively of the degenerate parabolic Dirichlet initial boundary value problem (2.1)-(2.3) and they satisfy the monotone property

$$\hat{u}_1 \leq \underline{u}_1^{(k)} \leq \underline{u}_1^{(k+1)} \leq \underline{u}_1 \leq \overline{u}_1 \leq \overline{u}_1^{(k+1)} \leq \overline{u}_1^{(k)} \leq \tilde{u}_1 \quad \text{in } \overline{D}_T$$

and

$$\hat{u}_2 \leq \underline{u}_2^{(k)} \leq \underline{u}_2^{(k+1)} \leq \underline{u}_2 \leq \overline{u}_2 \leq \overline{u}_2^{(k+1)} \leq \overline{u}_2^{(k)} \leq \tilde{u}_2 \quad \text{in } \overline{D}_T$$

Theorem 4.2 (Uniqueness)**.** *Suppose that*

(i) $(\tilde{u}_1, \tilde{u}_2), (\hat{u}_1, \hat{u}_2)$ *are the ordered upper and lower solutions of degenerate parabolic Dirichlet IBVP* (2.1)-(2.3)

(ii) the reaction functions $f^{(1)}, f^{(2)}$ *are mixed quasimonotone*

(iii) the functions $f^{(1)}(x,t,u_1,u_2)$ *and* $f^{(2)}(x,t,u_1,u_2)$ *satisfy the Lipschitz conditions*

$$\begin{aligned} -\underline{c}_1(u_1 - v_1) &\leq f^{(1)}(x,t,u_1,u_2) - f^{(1)}(x,t,v_1,u_2) \\ &\leq \overline{c}_1(u_1 - v_1) \quad \textit{for} \quad \hat{u} \leq v_1 \leq u_1 \leq \tilde{u} \end{aligned} \tag{4.5}$$

$$\begin{aligned} -\underline{c}_2(u_2 - v_2) &\leq f^{(2)}(x,t,u_1,u_2) - f^{(2)}(x,t,u_1,v_2) \\ &\leq \overline{c}_2(u_2 - v_2) \quad \textit{for} \quad \hat{u} \leq v_2 \leq u_2 \leq \tilde{u} \end{aligned} \tag{4.6}$$

Then the degenerate parabolic Dirichlet IBVP (2.1)-(2.3) *has unique solution.*

Proof: We know that $(\overline{u}_1, \overline{u}_2)$ and $(\underline{u}_1, \underline{u}_2)$ are maximal and minimal solutions respectively of the nonlinear weakly coupled time degenerate parabolic Dirichlet IBVP (2.1)-(2.3). To prove uniqueness,it is sufficient to show that

$$\overline{u}_1(x,t) \geq \underline{u}_1(x,t) \;\; \textit{and} \;\; \overline{u}_2(x,t) \geq \underline{u}_2(x,t)$$

First we consider $w_1(x,t) = \underline{u}_1 x, t - \overline{u}_1(x,t)$

$$\begin{aligned} d^{(1)}(x,t)(w_1)_t - D^{(1)}\nabla^2 w_1 &= f^{(1)}(x,t,\underline{u}_1,\underline{u}_2) - f^{(1)}(x,t,\overline{u}_1,\overline{u}_2) \\ &= [f^{(1)}(x,t,\underline{u}_1,\underline{u}_2) - f^{(1)}(x,t,\overline{u}_1,\underline{u}_2)] \\ &\quad + [f^{(1)}(x,t,\overline{u}_1,\underline{u}_2) - f^{(1)}(x,t,\overline{u}_1,\overline{u}_2)] \\ &\geq -\underline{c}_1(\underline{u}_1 - \overline{u}_1) + f^{(1)}(x,t,\overline{u}_1,\underline{u}_2) - f^{(1)}(x,t,\overline{u}_1,\overline{u}_2) \\ &\geq -\underline{c}_1 w_1 + f^{(1)}(x,t,\overline{u}_1,\underline{u}_2) - f^{(1)}(x,t,\overline{u}_1,\overline{u}_2) \end{aligned}$$

$$d^{(1)}(x,t)(w_1)_t - D^{(1)}\nabla^2 w_1 + \underline{c}_1 w_1 \geq f^{(1)}(x,t,\overline{u}_1,\underline{u}_2) - f^{(1)}(x,t,\overline{u}_1,\overline{u}_2) \geq 0$$

$$d^{(1)}(x,t)(w_1)_t - D^{(1)}\nabla^2 w_1 + \underline{c}_1 w_1 \geq 0 \qquad \text{in } D_T$$

also

$$w_1(x,t) = \underline{u}_1(x,t) - \overline{u}_1(x,t) \geq g^{(1)}(x,t) - g^{(1)}(x,t) = 0 \qquad \text{on } S_T$$

and

$$w_1(x,0) = \underline{u}_1(x,0) - \overline{u}_1(x,0) \geq u_{1,0}(x) - u_{1,0}(x) = 0 \qquad \text{in } \Omega$$

By using Lemma 2.5, we get

$$\begin{aligned} w_1(x,t) &\geq 0 \quad \text{in } \overline{D}_T \\ \underline{u}_1(x,t) &\geq \overline{u}_1(x,t) \quad \text{in } \overline{D}_T \\ \underline{u}_1(x,t) &\equiv \overline{u}_1(x,t) \quad \text{in } \overline{D}_T \end{aligned}$$

Similarly, we can show that,

$$\underline{u}_2(x,t) \geq \overline{u}_2(x,t) \quad \text{in } \overline{D}_T$$

Thus, $(\overline{u}_1, \overline{u}_2) \equiv (\underline{u}_1, \underline{u}_2)$. This shows that the nonlinear time degenerate parabolic Dirichlet IBVP (2.1)-(2.3) has unique solution.

Remark 4.3. If we replace linear Dirichlet boundary condition in IBVP (2.1)-(2.3) by nonlinear Dirichlet boundary condition then monotone property,existence and uniqueness results in this paper can be proved by assuming corresponding assumptions.

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