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NONLINEAR BOUNDARY VALUE PROBLEMS WITH p-LAPLACIAN

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ABSTRACT. We study the second order nonlinear boundary value problem with p-Laplacian consisting of the equation $-[\phi(y')]' + q(t)\phi(y) = w(t)f(y)$ with $\phi(y) = |y|^{p-1}y$ for $p > 0$ on $[a, b]$ and a general separated boundary condition. By comparing it with a half-linear Sturm-Liouville problem we obtain conditions for the existence and nonexistence of nodal solutions of this problem. More specifically, let $\lambda_n, n = 0, 1, 2, \ldots,$ be the n-th eigenvalue of the corresponding half-linear Sturm-Liouville problem. Then the boundary value problem has a pair of solutions with exactly n zeros in (a, b) if λ_n is in the interior of the range of $f(y)/\phi(y)$; and does not have any solution with exactly n zeros in (a, b) if λ_n is outside the range. These conditions become necessary and sufficient when $f(y)/\phi(y)$ is monotone on $(-\infty, 0)$ and on $(0, \infty)$. We also study the changes of the number of different types of nodal solutions as the equation or the boundary condition changes. Our results are obtained based on the global existence and uniqueness of solutions of the corresponding initial value problems established earlier by the authors.

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1. INTRODUCTION

In this paper, we study the existence and nonexistence of nodal solutions of the boundary value problem (BVP) with p-Laplacian consisting of the equation

$$-[\phi(y')]' + q(t)\phi(y) = w(t)f(y) \quad \text{on } [a, b], \tag{1.1}$$

where $\phi(y) = |y|^{p-1}y$, $p > 0$, and the general separated boundary condition (BC)

$$\begin{aligned} a_{11}y(a) &- a_{12}y'(a) = 0, \\ a_{21}y(b) &- a_{22}y'(b) = 0, \end{aligned} \tag{1.2}$$

where $a_{ij} \in \mathbb{R}$.

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Throughout this paper, we make the following assumptions:
(H1) $q, w \in C^1[a,b]$ and $w > 0$ on $[a,b]$;
(H2) $f \in C(\mathbb{R})$ such that $yf(y) > 0$ for $y \neq 0$ and f is locally Lipschitz continuous on $\mathbb{R}\backslash\{0\}$;
(H3) there exist limits f_0 and f_∞ such that $0 \leq f_0, f_\infty \leq \infty$, where

$$f_0 = \lim_{y\to 0}\frac{f(y)}{\phi(y)} \quad \text{and} \quad f_\infty = \lim_{y\to\pm\infty}\frac{f(y)}{\phi(y)}.$$

Special cases of BVP (1.1), (1.2) with $p = 1$ have been investigated in numerous papers using various methods and techniques. Most existing results are about the existence of positive solutions and are for special cases of Eq. (1.1) such as $q \geq 0$ ($q \equiv 0$ for many of them) and for special BCs such as Dirichlet and Neumann BCs.

When $p = 1$, Erbe [3] initiated the idea of connecting BVP (1.1), (1.2) with the eigenvalues of its corresponding linear Sturm-Liouville problem (SLP). In that paper, using the fixed point index theory, the existence of positive solutions of BVP (1.1), (1.2) was established by comparing the values of $f(y)/y$, $y \in (0,\infty)$, with the smallest eigenvalue of the corresponding linear SLP. However, due to the limitation of the approach, results were given only for the case with a nonnegative q and a BC (1.2) satisfying certain conditions, and nothing was found for the existence of nodal solutions.

Recently, with $p = 1$ and $q \equiv 0$ and the Dirichlet BC, Naito and Tanaka [10] obtained results on the existence of nodal solutions to BVP (1.1), (1.2) with prescribed numbers of zeros in the interval by comparing the range of $f(y)/y$ with an eigenvalue of the corresponding SLP. This work has been extended by Kong [6] to the BVP (1.1), (1.2) with an arbitrary $q \in C^1$ and a general separated BC.

Later, Naito and Tanaka [11] extended their results in [10] to the BVP (1.1) with $q \equiv 0$ and the Dirichlet BC.

Motivated by the idea in [6] and [11], in this paper, we extend their results to the BVP (1.1), (1.2) with an arbitrary $q \in C^1$ and a general separated BC. However, the extension is not trivial due to the fact that Eq. (1.1) is not linear and cannot be transformed to a linear equation in any situation. Therefore, many tools used for the case when $p = 1$ cannot be applied to the general problems with p-Laplacian. For instance, the fundamental solution set, the classical Prüfer transformation and the Gronwall's inequality are among such tools. New properties of the energy function used in the proofs need to be derived since it contains a sign-changing function q which results in sign-changes of the energy function. In addition to studying the existence and nonexistence of nodal solutions of the BVP (1.1), (1.2), we also investigate the structural changes in the number and the types of nontrivial solutions as the equation and the BC change.

This paper is organized as follows. After this introduction, we introduce some supporting background knowledge for the statements and poofs of our main results in Section 2. Then we state our main results in Section 3, and derive several technical lemmas and use them to prove the main results in Section 4. Finally, we extend our main results to a generalized BVP in Section 5.

2. PRELIMINARIES

1. Generalized Trigonometric Functions

We first introduce some basic knowledge on generalized trigonometric functions introduced by Elbert [2]. These functions will be used to normalize BC (1.2) and to establish the generalized Prüfer transformation for equations with p-Laplacian which will be used in the proofs.

Let $S = S(\theta)$ be the unique solution of the half-linear differential equation

$$\frac{d}{d\theta}(\phi(\frac{dS}{d\theta})) + p\phi(S) = 0$$

satisfying the initial condition

$$S(0) = 0, \frac{dS(\theta)}{d\theta}|_{\theta=0} = 1.$$

Then $S = S(\theta)$ is called the generalized sine function and is periodic with the period $2\pi_p$, where

$$\pi_p = \frac{2\pi}{(p+1)} \Big/ \sin\frac{\pi}{p+1}\,.$$

$S(\theta)$ is an odd function having zeros at $\theta = k\pi_p$, $k \in \mathbb{Z}$, $S(\theta) > 0$ for $\theta \in (2k\pi_p, (2k+1)\pi_p)$, and $S(\theta) < 0$ for $\theta \in ((2k+1)\pi_p, (2k+2)\pi_p)$. The generalized cosine function $C(\theta)$ is defined by $C(\theta) = dS(\theta)/d\theta$ and is even and periodic with the period $2\pi_p$. For $k \in \mathbb{Z}$, $C(k+1/2)\pi_p = 0$, $C(\theta) > 0$ for $\theta \in ((2k-1/2)\pi_p, (2k+1/2)\pi_p)$, and $C(\theta) < 0$ for $\theta \in ((2k+1/2)\pi_p, (2k+3/2)\pi_p)$.
The functions $S(\theta)$ and $C(\theta)$ satisfy the relation

$$|S(\theta)|^{p+1} + |C(\theta)|^{p+1} = 1 \quad \text{for } \theta \in \mathbb{R}.$$

The generalized tangent function $T(\theta)$ is defined by

$$T(\theta) = \frac{S(\theta)}{C(\theta)} \quad \text{for } \theta \neq (k+1/2)\pi_p, \quad k \in \mathbb{Z}.$$

It is a periodic function of period π_p and satisfies

$$T'(\theta) = 1 + |T(\theta)|^{p+1} \quad \text{for } \theta \neq (k+1/2)\pi_p, \quad k \in \mathbb{Z}.$$

For $k \in \mathbb{Z}$, $T(\theta)$ is strictly increasing for $\theta \in ((k-1/2)\pi_p, (k+1/2)\pi_p)$ and $T(\theta) \to -\infty$ as $\theta \to (k-1/2)\pi_p^+$ and $T(\theta) \to \infty$ as $\theta \to (k+1/2)\pi_p^-$.

Based on the above, we can normalize BC (1.2) to the form

$$\begin{array}{lclcl} C(\alpha)y(a) & - & S(\alpha)y'(a) & = 0, & \alpha \in [0, \pi_p), \\ C(\beta)y(b) & - & S(\beta)y'(b) & = 0, & \beta \in (0, \pi_p]. \end{array} \tag{2.1}$$

2. Results on IVPs Associated with Eq. (1.1)

The criteria for the existence of nodal solutions of BVP (1.1), (1.2) will be derived using the shooting method which requires the continuous dependence of solutions on the initial conditions for initial value problems (IVPs) associated with Eq. (1.1). The proposition below given by the authors in [7] provides the desired result.

Consider the IVP consisting of Eq. (1.1) and the initial condition

$$y(t_0) = y_0, \quad y'(t_0) = z_0, \tag{2.2}$$

We have the following result:

Proposition 2.1. *For any* $t_0 \in [a, b]$ *and* $y_0, z_0 \in \mathbb{R}$, *IVP* (1.1), (2.2) *has a unique solution which exists on the whole interval* $[a, b]$. *Consequently, the unique solution depends on the initial condition continuously.*

3. MAIN RESULTS

To present our main results, we need to compare BVP (1.1), (1.2) with the half-linear SLP consisting of the equation

$$-[\phi(y')]' + q(t)\phi(y) = \lambda w(t)\phi(y) \quad \text{ on } [a, b] \tag{3.1}$$

and BC (1.2).

It has been shown, see Theorem 3.1 in [1], that SLP (3.1), (1.2) has a countably infinite number of real eigenvalues; they are bounded below and unbounded above, and they are all simple and can be ordered to satisfy

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \text{ with } \lambda_n \to \infty.$$

Moreover, any eigenfunction $y_n = y_n(t, \lambda_n)$ associated with λ_n has exactly n zeros in (a, b).

Note that $f(y) = (f(y)/\phi(y))\phi(y)$ for $y \neq 0$. By employing the shooting method together with the generalized Sturm comparison theorem, we obtain the theorems below.

Let λ_k be the first positive eigenvalue of SLP (3.1), (1.2), and define $\mathbb{N}_k = \{k, k+1, k+2, \ldots\}$.

The first theorem is concerned with the existence of certain types of nodal solutions of BVP (1.1), (1.2).

Theorem 3.1. *Assume there exists $n \in \mathbb{N}_k$ such that either $f_0 < \lambda_n < f_\infty$ or $f_\infty < \lambda_n < f_0$. Then BVP (1.1), (1.2) has two solutions $y_n^\pm(t)$ which have exactly n zeros in (a, b) and have opposite signs in a right-neighborhood of a.*

The second result is about the nonexistence of nodal solutions of BVP (1.1), (1.2).

Theorem 3.2. *(i) Assume $f(y)/\phi(y) < \lambda_n$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP (1.1), (1.2) has no solution with exactly i zeros in (a, b) for any $i \geq n$.*

(ii) Assume $f(y)/\phi(y) > \lambda_n$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP (1.1), (1.2) has no solution with exactly i zeros in (a, b) for any $i \leq n$.

(iii) Assume $f(y)/\phi(y) \neq \lambda_n$ for any $i \in \mathbb{N}_0$ and $y \neq 0$. Then BVP (1.1), (1.2) has no nontrivial solution.

The combination of Theorem 3.1 and 3.2 leads to the following:

Corollary 3.3. *(i) Assume $f_0 < f(y)/\phi(y) < f_\infty$ for all $y \neq 0$. Then for $n \in \mathbb{N}_k$, BVP (1.1), (1.2) has two solutions $y_n^\pm(t)$ which have exactly n zeros in (a, b) and have opposite signs in a right-neighborhood of a if and only if $f_0 < \lambda_n < f_\infty$.*

(ii) Assume $f_\infty < f(y)/\phi(y) < f_0$ for all $y \neq 0$. Then for $n \in \mathbb{N}_k$, BVP (1.1), (1.2) has two solutions $y_n^\pm(t)$ which have exactly n zeros in (a, b) and have opposite signs in a right-neighborhood of a if and only if $f_\infty < \lambda_n < f_0$.

The next theorem and corollary are for the existence of multiple and even an infinite number of solutions.

Theorem 3.4. *(i) Assume $f_0 < \lambda_m < \lambda_n < f_\infty$ for some $m, n \in \mathbb{N}_0$, then BVP (1.1), (1.2) has solutions $y_m^\pm$ and $y_n^\pm$ which have exactly m and n zeros in (a, b) and have opposite signs in a right-neighborhood of a, respectively, and satisfy $r_m(a) < r_n(a)$, where $r_i = (|y_i^\pm(t)|^{p+1} + |y_i^{\pm\prime}(t)|^{p+1})^{1/(p+1)}$ for $i = m, n$.*

(ii) Assume $f_\infty < \lambda_m < \lambda_n < f_0$ for some $m, n \in \mathbb{N}_0$, then BVP (1.1), (1.2) has solutions $y_m^\pm$ and $y_n^\pm$ which have exactly m and n zeros in (a, b) and have opposite signs in a right-neighborhood of a, respectively, and satisfy $r_m(a) > r_n(a)$, where $r_i = (|y_i^\pm(t)|^{p+1} + |y_i^{\pm\prime}(t)|^{p+1})^{1/(p+1)}$ for $i = m, n$.

Corollary 3.5. *Assume either (i) $f_0 = 0$ and $f_\infty = \infty$, or (ii) $f_\infty = 0$ and $f_0 = \infty$. Then BVP (1.1), (1.2) has an infinite number of solutions $\{y_n^\pm : n \in \mathbb{N}_k\}$ such that $y_n^\pm$ have exactly n zeros in (a, b) and have opposite signs in a right-neighborhood of a respectively for each $n \in \mathbb{N}_k$, and*

$$r_k(a) < r_{k+1}(a) < r_{k+2}(a) < \cdots \quad \text{if (i) holds}$$

and

$$r_k(a) > r_{k+1}(a) > r_{k+2}(a) > \cdots \quad \text{if (ii) holds,}$$

where $r_n = (|y_n^{\pm}(t)|^{p+1} + |y_n^{\pm\prime}(t)|^{p+1})^{1/(p+1)}$ *for* $n \in \mathbb{N}_k$.
Furthermore, BVP (1.1), (1.2) *has no solutions with exactly* m *zeros in* (a, b) *for* $m < k$ *if* $\lambda_{k-1} < 0$.

Now, we discuss the structural changes in the existence of nodal solutions to BVP (1.1), (1.2) as the problem changes, more specifically, as the interval $[a, b]$ shrinks, the BC angles α, β in BC (2.1) vary, and as the functions q, w grow in certain directions.

The first theorem is about changes of the number and the types of nontrivial solutions as the interval $[a, b]$ shrinks, more specifically, as $b \to a+$.

Theorem 3.6. *(i) Assume either* $f_0 < \infty$ *and* $f_\infty = \infty$*, or* $f_\infty < \infty$ *and* $f_0 = \infty$.

(a) For any $n \in \mathbb{N}_1$*, there exists* $b_n > a$ *such that for any* $b \in (a, b_n)$ *and for any* $i \geq n$*, BVP* (1.1), (1.2) *has two solutions which have exactly* i *zeros in* (a, b) *and have opposite signs in a right-neighborhood of* a.

(b) Let $\alpha \in [0, \pi_p)$, $\beta \in (0, \pi_p]$*. Then for* $\alpha < \beta$*, there exists* $b_0 > a$ *such that for any* $b \in (a, b_0)$*, BVP* (1.1), (1.2) *has a positive solution and a negative solution; and for* $\beta < \alpha$*, there exists* $b_0 > a$ *such that for any* $b \in (a, b_0)$*, BVP* (1.1), (1.2) *has no positive or negative solution.*

(ii) Assume $f_0 < \infty$ *and* $f_\infty < \infty$*. Let* $\alpha \in [0, \pi_p)$ *and* $\beta \in (0, \pi_p]$ *with* $\alpha \neq \beta$*. Then there exists* $b_* > a$ *such that for any* $b \in (a, b_*)$*, BVP* (1.1), (1.2) *has no nontrivial solution.*

The theorem below is about the nonexistence of positive and negative solutions and solutions with one zero in (a, b) for some values of α and β.

Theorem 3.7. *(i) For each* $\beta \in (0, \pi_p]$ *there exists* $\alpha_* \in [0, \pi_p)$ *such that BVP* (1.1), (1.2) *has no positive or negative solution for any* $\alpha \in (\alpha_*, \pi_p)$.

(ii) For each $\alpha \in [0, \pi_p)$*, there exists* $\beta_* \in (0, \pi_p]$ *such that BVP* (1.1), (1.2) *has no positive or negative solution for any* $\beta \in (0, \beta_*)$.

(iii) There exists $\alpha_* \in [0, \pi_p)$ *and* $\beta_* \in (0, \pi_p]$ *such that BVP* (1.1), (1.2) *has no positive or negative solution nor solution with one zero in* (a, b) *for any* $\alpha \in (\alpha_*, \pi_p)$ *and* $\beta \in (0, \beta_*)$.

We next present results on the structural changes as the functions q or w grow in a given direction.

Let $s \in \mathbb{R}$ and $h \in C^1[a, b]$ be such that $h > 0$ on $[a, b]$, and consider the equation

$$-[\phi(y')]' + [q(t) + sh(t)]\phi(y) = w(t)f(y) \quad \text{on } [a, b]. \tag{3.2}$$

Theorem 3.8. *(i) For any* $n \in \mathbb{N}_0$*, there exists* $s_n \leq 0$ *such that for any* $s < s_n$ *and for any* $i \leq n$*, BVP* (3.2), (1.2) *has no solution with exactly* i *zeros in* (a, b).

(ii) Assume either $f_0 < \infty$ *and* $f_\infty = \infty$*, or* $f_\infty < \infty$ *and* $f_0 = \infty$*. Then for any*

$n \in \mathbb{N}_0$, there exists $s_n \geq 0$ such that for any $s > s_n$ and for any $i \geq n$, BVP (3.2), (1.2) *has two solutions which have exactly i zeros in (a, b) and have opposite signs in a right-neighborhood of a.*

(iii) Assume $f_0 < \infty$ and $f_\infty < \infty$. Then there exists $s_* \geq 0$ such that for any $s > s_*$, BVP (3.2), (1.2) *has no nontrivial solutions.*

Let $s \geq 0$ and $h \in C^1[a, b]$ be such that $h > 0$ on $[a, b]$, and consider the equation

$$-[\phi(y')]' + q(t)\phi(y) = [w(t) + sh(t)]f(y) \quad \text{on } [a, b]. \tag{3.3}$$

Theorem 3.9. *Assume $f(y)/\phi(y) \geq f_* > 0$ for all $y \neq 0$. Then for any $n \in \mathbb{N}_0$, there exists $s_n \geq 0$ such that for any $s > s_n$ and for any $i \leq n$, BVP* (3.3), (1.2) *has no solution with exactly i zeros in (a, b).*

Finally, we comment that all the above results can be extended to the BVP consisting of the equation

$$-[\nu(t)\phi(y')]' + q(t)\phi(y) = w(t)f(y) \quad \text{on } [a, b] \tag{3.4}$$

and BC

$$\begin{array}{rcl} a_{11}y(a) - a_{12}(\nu^{1/p}y')(a) &=& 0, \\ a_{21}y(b) - a_{22}(\nu^{1/p}y')(b) &=& 0, \end{array} \tag{3.5}$$

where $a_{ij} \in \mathbb{R}$, $\nu \in C^1[a, b]$ such that $\nu > 0$ on $[a, b]$. This is because Eq. (3.4) can be transformed to an equation in the form of Eq. (1.1) with a change of independent variable. In fact, let $\tau(t) = \int_a^t 1/\nu^{1/p}(s)ds$, $t = t(\tau)$ be the inverse function, and $u(\tau) = y(t(\tau))$. Define $Q(\tau) = \nu^{1/p}(t(\tau))q(t(\tau))$ and $W(\tau) = \nu^{1/p}(t(\tau))w(t(\tau))$. Then Eq. (3.4) becomes

$$-\frac{d}{d\tau}(\phi(\frac{du}{d\tau})) + Q(\tau)\phi(u) = W(\tau)f(u) \quad \text{on } \left[0, \int_a^b 1/\nu^{1/p}(s)ds\right], \tag{3.6}$$

and BC (3.5) becomes

$$\begin{array}{rcl} a_{11}u(a) - a_{12}u'(a) &=& 0, \\ a_{21}u(b) - a_{22}u'(b) &=& 0, \end{array} \tag{3.7}$$

Therefore, all results for BVP (3.6), (3.7) can be transformed back to BVP (3.4), (3.5). We omit the details.

4. PROOFS OF THE MAIN RESULTS

To prove Theorem 3.1, we need to study the IVP consisting of Eq. (1.1) and the initial condition

$$y(a) = \rho S(\alpha), \quad y'(a) = \rho C(\alpha), \tag{4.1}$$

where $\rho > 0$ is a parameter. Let $y(t, \rho)$ be the solution of IVP (1.1), (4.1) and $\theta(t, \rho)$ be the Prüfer angle of $y(t, \rho)$. Then $\theta(\cdot, \rho)$ is a continuous function on $[a, b]$

such that $T(\theta(t,\rho)) = y(t,\rho)/y'(t,\rho)$ and $\theta(a,\rho) = \alpha$. Let $r(t,\rho) = (|y(t,\rho)|^{p+1} + |y'(t,\rho)|^{p+1})^{1/(p+1)}$. From Eq. (1.1) we have

$$\theta'(t,\rho) = |C(\theta(t,\rho))|^{p+1} + \frac{w(t)f(y(t,\rho))S(\theta(t,\rho))}{p[r(t,\rho)]^p} - \frac{q(t)|S(\theta(t,\rho))|^{p+1}}{p} \tag{4.2}$$

and

$$r'(t,\rho) = \left[\left(1+\frac{q(t)}{p}\right)\phi(y(t,\rho)) - \frac{w(t)f(y(t,\rho))}{p}\right]\frac{C(\theta(t,\rho))}{[r(t,\rho)]^{p-1}}. \tag{4.3}$$

Lemma 4.1. *Assume $f_0 < \lambda_n$ for some $n \in \mathbb{N}_k$. Then there exists ρ_* such that $\theta(b,\rho) < n\pi_p + \beta$ for all $\rho \in (0,\rho_*)$.*

Proof. $f_0 < \lambda_n < \infty$ implies that $f(y)/\phi(y)$ can be extended continuously to $y = 0$ and there exists $\delta > 0$ such that $f(y)/\phi(y) < \lambda_n$ for $|y| < \delta$. Since $y \equiv 0$ is a solution of Eq. (1.1), by the continuous dependence of solution on the initial condition, there exists $\rho_* > 0$ such that $|y(t,\rho)| < \delta$ for $\rho \in (0,\rho_*)$ and $t \in [a,b]$. Let $\tilde{w}(t) = w(t)f(y(t))/\phi(y(t))$. Then $\tilde{w}(t)$ is continuous on $[a,b]$, and $\tilde{w}(t) < \lambda_n w(t)$ on $[a,b]$. From (4.2) we have

$$\begin{aligned}\theta'(t,\rho) &= |C(\theta(t,\rho))|^{p+1} + [\tilde{w}(t) - q(t)]\frac{|S(\theta(t,\rho))|^{p+1}}{p} \\ &< |C(\theta(t,\rho))|^{p+1} + [\lambda_n w(t) - q(t)]\frac{|S(\theta(t,\rho))|^{p+1}}{p}.\end{aligned}$$

Let $y_n(t,\rho)$ be the solution of IVP (3.1), (4.1) with $\lambda = \lambda_n$ and $\theta_n(t,\rho)$ its Prüfer angle. Then $y_n(t,\rho)$ is the eigenfunction of the SLP (3.1), (1.2) and

$$\theta_n'(t,\rho) = |C(\theta_n(t,\rho))|^{p+1} + [\lambda_n w(t) - q(t)]\frac{|S(\theta_n(t,\rho))|^{p+1}}{p}. \tag{4.4}$$

Therefore, $\theta_n(b,\rho) = n\pi_p + \beta$. By the theory of differential inequalities, we obtain that $\theta(b,\rho) < \theta_n(b,\rho) = n\pi_p + \beta$. □

Lemma 4.2. *Assume $f_0 > \lambda_n$ for some $n \in \mathbb{N}_k$. Then there exists ρ_* such that $\theta(b,\rho) > n\pi_p + \beta$ for all $\rho \in (0,\rho_*)$.*

Proof. $f_0 > \lambda_n$ implies that there exists $\delta > 0$ such that $f(y)/\phi(y) > \lambda_n$ for $0 < |y| < \delta$. For the same reason as in the proof of Lemma 4.1, there exists $\rho_* > 0$ such that $|y(t,\rho)| < \delta$ for $\rho \in (0,\rho_*)$ and $t \in [a,b]$, and hence $f(y(t,\rho))/\phi(y(t,\rho)) > \lambda_n$ for $\rho \in (0,\rho_*)$ and $t \in [a,b]$ whenever $y(t,\rho) \neq 0$. Then from (4.2) we have

$$\theta'(t,\rho) > |C(\theta(t,\rho))|^{p+1} + [\lambda_n w(t) - q(t)]\frac{|S(\theta(t,\rho))|^{p+1}}{p}.$$

By the theory of differential inequalities we obtain that for the $\theta_n(t,\rho)$ defined in the proof of Lemma 4.1, $\theta(b,\rho) > \theta_n(b,\rho) = n\pi_p + \beta$. □

We need the following lemma to prove results parallel to Lemmas 4.1 and 4.2 where f_0 is replaced by f_∞.

Lemma 4.3. *For $M>0$ and $\rho>0$ define $I_{M,\rho}=\{t\in[a,b]:|y(t,\rho)|<M\}$. Then for any $M,\tilde{P}>0$, there exists $\rho^*>0$ such that $|y'(t,\rho)|>\tilde{P}$ for $\rho>\rho^*$ and $t\in I_{M,\rho}$.*

Proof. (I) First we consider the case when $f_\infty=\infty$. Define an energy function $[E(y)](t,\rho)$ for the solution $y(t,\rho)$ by

$$[E(y)](t,\rho)=\frac{p}{p+1}|y'(t,\rho)|^{p+1}-\frac{1}{p+1}q(t)|y(t,\rho)|^{p+1}+w(t)F(y(t,\rho)).$$

Then we have

$$\begin{aligned}[E(y)]'(t,\rho) &\geq \frac{k+1}{p+1}q(t)|y(t,\rho)|^{p+1}-\frac{1}{p+1}[(k+1)q(t)+q'(t)]|y(t,\rho)|^{p+1}\\ &\quad -kw(t)F(y(t,\rho)),\end{aligned}$$

where $k=\max\{|w'(t)|/w(t):t\in[a,b]\}$. Since $w>0$ is continuous and q, q' are bounded on $[a,b]$, we can find a constant $h>0$ such that for $t\in[a,b]$

$$\frac{h}{p+1}[(k+1)q(t)-q'(t)]\leq w(t). \tag{4.5}$$

Since $f_\infty=\infty$, we have $|y|^{p+1}=o(F(y))$ as $|y|\to\infty$. This means that there exists $M>0$ such that

$$|y|^{p+1}\leq hF(y) \text{ for } |y|\geq M. \tag{4.6}$$

Define $I_1=\{t\in[a,b],|y(t,\rho)|<M\}$ and $I_2=\{t\in[a,b],|y(t,\rho)|\geq M\}$. Then by Eq. (4.5) and (4.6), there exists $N>0$ such that

$$[E(y)]'(t,\rho) \geq \begin{cases} -N, & t\in I_1;\\ -(k+1)[E(y)](t,\rho), & t\in I_2.\end{cases} \tag{4.7}$$

We claim that

$$[E(y)](t,\rho)\geq\left([E(y)](a,\rho)+\frac{N}{k+1}\right)e^{-(k+1)(\operatorname{sgn}[E(y)])(t-a)}-\frac{N}{k+1}e^{2(k+1)(b-a)}. \tag{4.8}$$

From (4.7), we have

$$\begin{aligned}[E(y)]'(t,\rho) &\geq -(k+1)|[E(y)](t,\rho)|-N\\ &= -(k+1)(\operatorname{sgn}[E(y)](t,\rho))[E(y)](t,\rho)-N.\end{aligned}$$

Hence,

$$[E(y)]'(t,\rho)+(k+1)(\operatorname{sgn}[E(y)](t,\rho))[E(y)](t,\rho)\geq -N. \tag{4.9}$$

In the following, we denote $\operatorname{sgn}[E(y)](t,\rho)$ by $\operatorname{sgn}[E(y)]$ for convenience. By (4.9), we have

$$\left([E(y)](t,\rho)e^{(k+1)(\operatorname{sgn}[E(y)])(t-a)}\right)'\geq -Ne^{(k+1)(\operatorname{sgn}[E(y)])(t-a)}\geq -Ne^{(k+1)(t-a)}$$

a.e. on $[a,b]$. Integrating both sides of the above inequality from a to t, we have

$$[E(y)](t,\rho)e^{(k+1)(\operatorname{sgn}[E(y)])(t-a)}-[E(y)](a,\rho)\geq -\frac{N}{k+1}\left(e^{(k+1)(t-a)}-1\right).$$

Hence

$$[E(y)](t,\rho) \geq \left([E(y)](a,\rho) + \frac{N}{k+1}\right) e^{-(k+1)(\operatorname{sgn}[E(y)])(t-a)} - \frac{N}{k+1} e^{(k+1)(1-\operatorname{sgn}[E(y)])(t-a)}.$$

Therefore, (4.8) holds. Note that

$$[E(y)](a,\rho) = \rho^{p+1}\left[\frac{p}{p+1}|C(\alpha)|^{p+1} - \frac{1}{p+1}q(a)|S(\alpha)|^{p+1} + \frac{w(a)F(\rho S(\alpha))}{\rho^{p+1}}\right].$$

Then when $\alpha = 0$, we have

$$\lim_{\rho\to\infty}[E(y)](a,\rho) = \lim_{\rho\to\infty}\frac{p}{p+1}\rho^{p+1} = \infty.$$

When $\alpha \in (0,\pi)$, since

$$\lim_{\rho\to\infty}\frac{F(\rho S(\alpha))}{\rho^{p+1}} = \lim_{y\to\infty}\frac{F(y)}{|y|^{p+1}}|S(\alpha)|^{p+1} = \infty,$$

we also have

$$\lim_{\rho\to\infty}[E(y)](a,\rho) = \infty.$$

Therefore, from (4.8)

$$\lim_{\rho\to\infty}[E(y)](t,\rho) = \infty \text{ uniformly for } t \in [a,b]. \tag{4.10}$$

Since $\left|-\frac{1}{p+1}q(t)|y(t,\rho)|^{p+1} + w(t)F(y(t,\rho))\right|$ is uniformly bounded by some $K_1 > 0$ for all $\rho > 0$ and $t \in I_{M,\rho}$, from (4.10) we may choose ρ^* so large that

$$[E(y)](t,\rho) > \frac{p}{p+1}\tilde{P}^{p+1} + K_1 \text{ for } \rho > \rho^* \text{ and } t \in [a,b].$$

Then for $\rho > \rho^*$ and $t \in I_{M,\rho}$

$$K_1 + \frac{p}{p+1}[y'(t,\rho)]^{p+1} \geq [E(y)](t,\rho) > \frac{p}{p+1}\tilde{P}^{p+1} + K_1.$$

Thus, we have $|y'(t,\rho)| > \tilde{P}$.

(II) Then we consider the case when $f_\infty < \infty$. For $\rho > 0$, define $r(t,\rho)$ as before. From (4.3), there exists $K_2 > 0$ such that for $\rho > 0$ and $t \in I_{M,\rho}$

$$r'(t,\rho) \geq -K_2 r(t,\rho)^{1-p}. \tag{4.11}$$

Since $f_\infty < \infty$, there exists $K_3 > 0$ such that $|f(y)/\phi(y)| \leq K_3$ for $|y| \geq M$, and by (4.3) again, we have

$$r'(t,\rho) = r(t,\rho)\phi(S(\theta(t,\rho)))C(\theta(t,\rho))\left[1 + \frac{q(t)}{p} - \frac{w(t)}{p}\frac{f(y(t,\rho))}{\phi(y(t,\rho))}\right]. \tag{4.12}$$

Then from (4.12) we see that for $|y(t,\rho)| \geq M$,

$$r'(t,\rho) \geq -r(t,\rho)\left[\left|1 + \frac{q(t)}{p}\right| + K_3\frac{w(t)}{p}\right] \geq -K_4 r(t,\rho), \tag{4.13}$$

where $K_4 = \max\{|1 + q(t)/p| + K_3 w(t)/p : t \in [a, b]\}$. Combining (4.11) and (4.13), we have that for $\rho > 0$ and $t \in [a, b]$

$$r'(t, \rho) \geq -K_2 r(t, \rho)^{1-p} - K_4 r(t, \rho). \tag{4.14}$$

Solving the above inequality we obtain that

$$\begin{aligned} r(t,\rho)^p &\geq r(a,\rho)^p e^{pK_4(a-t)} - \frac{K_2}{K_4} + \frac{K_2}{K_4} e^{pK_4(a-t)} \\ &= \rho^p e^{pK_4(a-t)} - \frac{K_2}{K_4} + \frac{K_2}{K_4} e^{pK_4(a-t)} \to \infty \quad \text{as } \rho \to \infty \end{aligned} \tag{4.15}$$

uniformly on $[a, b]$. Therefore, for any $\tilde{P} > 0$, there exists $\rho^* > 0$ such that for $\rho > \rho^*$ and $t \in I_{M,\rho}$

$$\left(M^p + [y'(t, \rho)]^p\right)^{1/p} \geq r(t, \rho) > \left(M^p + \tilde{P}^p\right)^{1/p}.$$

Hence, $|y'(t, \rho)| > \tilde{P}$. □

Lemma 4.4. *Assume $f_\infty > \lambda_n$ for some $n \in \mathbb{N}_k$. Then there exists ρ^* such that $\theta(b, \rho) > n\pi_p + \beta$ for all $\rho \in (\rho^*, \infty)$.*

Proof. Assume the contrary. Then there exists a sequence $\rho_l \to \infty$ such that $\theta(b, \rho_l) \leq n\pi_p + \beta$. This implies that $y(t, \rho_l)$ has at most n zeros in (a, b). Choose $\lambda > 0$ satisfying $\lambda_n < \lambda < f_\infty$ and take $M > 0$ so large that $f(y)/\phi(y) \geq \lambda$ for $|y| \geq M$. Then for each $\rho = \rho_l$, $I_{M,\rho} \cap (a, b)$ is an open set and hence is a union of disjoint maximal open intervals in (a, b), i.e.,

$$I_{M,\rho} \cap (a, b) = \bigcup_{i=1}^{j} (a_i, b_i), \tag{4.16}$$

where $a \leq a_i < b_i \leq b$ and (a_i, b_i) is a maximal subinterval of $I_{M,\rho} \cap (a, b)$. If $a < a_i$ and $b_i < b$, by Lemma 4.3, for $\rho = \rho_l$ sufficiently large, $y(t, \rho)$ is monotone on $[a_i, b_i]$, and hence

$$|y(a_i, \rho)| = |y(b_i, \rho)| = M \text{ and } y(a_i, \rho)y(b_i, \rho) < 0.$$

This implies that (a_i, b_i) contains exactly one zero of $y(t, \rho)$, so $j \leq n+2$ for j defined in (4.16). By Lemma 4.3 again, for any $\tilde{P} > 0$ there exists $\rho(\tilde{P}) > 0$ such that if $\rho = \rho_l > \rho(\tilde{P})$, then $y'(t, \rho)$ has the same sign and $|y'(t, \rho)| > \tilde{P}$ in (a_i, b_i) for each $i = 1, \ldots, j$. Thus

$$2M \geq |y(b_i, \rho) - y(a_i, \rho)| = \int_{a_i}^{b_i} |y'(t, \rho)| dt \geq \tilde{P}(b_i - a_i)$$

which leads to $b_i - a_i \leq 2M/\tilde{P}$. Consequently,

$$\text{mess} I_{M,\rho} \leq \frac{2M}{\tilde{P}} j \leq \frac{2M(n + 2)}{\tilde{P}},$$

where "mess" means the Lebesgue measure. Therefore,

$$\lim_{\rho_l \to \infty} \text{mess} I_{M,\rho_l} = 0. \tag{4.17}$$

For each $\rho = \rho_l$, let $\psi(t,\rho)$ be the Prüfer angle of the solution of the IVP (3.1), (4.1) and $\psi_n(t,\rho)$ the Prüfer angle of the solution of the IVP (3.1), (4.1) with λ replaced by λ_n. Then $\psi_n(t,\rho)$ is an eigenfunction of SLP (3.1), (2.1), and hence $\psi_n(b,\rho) = n\pi_p + \beta$. By the generalized Sturm comparison theorem, $\psi(b,\rho) = n\pi_p + \beta + \epsilon$ for some $\epsilon > 0$. Define

$$g(t,\rho) \ = \begin{cases} \dfrac{f(y(t,\rho))}{\phi(r(t,\rho))}, & |y(t,\rho))| < M; \\ \lambda\phi(S(\theta)), & |y(t,\rho)| \geq M. \end{cases} \tag{4.18}$$

Note that when $|y(t,\rho)| \geq M$,

$$\begin{aligned} & \frac{f(y(t,\rho))}{\phi(r(t,\rho))} S(\theta(t,\rho)) - g(t,\rho) S(\theta(t,\rho)) \\ = \ & \frac{f(y(t,\rho))|S(\theta(t,\rho))|^{p+1}}{\phi(y(t,\rho))} - \lambda |S(\theta(t,\rho))|^{p+1} \\ = \ & \left(\frac{f(y(t,\rho))}{\phi(y(t,\rho))} - \lambda \right) |S(\theta(t,\rho))|^{p+1} \\ \geq \ & 0. \end{aligned}$$

We have

$$\frac{f(y(t,\rho))}{\phi(r(t,\rho))} S(\theta(t,\rho)) > g(t,\rho) S(\theta(t,\rho)).$$

Therefore, by (4.2)

$$\theta'(t,\rho) \geq |C(\theta(t,\rho))|^{p+1} + \frac{w(t)g(t,\rho)S(\theta(t,\rho))}{p} - q(t)\frac{|S(\theta(t,\rho))|^{p+1}}{p}. \tag{4.19}$$

Let $\vartheta(t,\rho)$ be the solution of the equation

$$\vartheta'(t,\rho) = |C(\vartheta(t,\rho))|^{p+1} + \frac{w(t)}{p} g(t,\rho) S(\vartheta(t,\rho)) - q(t) \frac{|S(\vartheta(t,\rho))|^{p+1}}{p} := F(t,\rho,\vartheta) \tag{4.20}$$

satisfying $\vartheta(a,\rho) = \alpha$. Since

$$\psi'(t,\rho) = |C(\psi(t,\rho))|^{p+1} + [\lambda w(t) - q(t)] \frac{|S(\psi(t,\rho))|^{p+1}}{p} := G(t,\rho,\psi), \tag{4.21}$$

and $\vartheta(a,\rho) = \psi(a,\rho)$, combining (4.20) and (4.21) we have that for $\rho = \rho_l$ and $t \in [a,b]$

$$\begin{aligned} & \vartheta(t,\rho) - \psi(t,\rho) \\ = \ & \int_a^t [F(s,\rho,\vartheta) - G(s,\rho,\psi)] ds \\ = \ & \int_a^t ([F(s,\rho,\vartheta) - G(s,\rho,\vartheta)] + [G(s,\rho,\vartheta) - G(s,\rho,\psi)]) ds \\ = \ & \int_a^t \frac{w(s)}{p} [g(s,\rho) - \lambda\phi(S(\vartheta))] S(\vartheta) ds \\ & + \int_a^t \frac{\partial}{\partial\psi} G(s,\rho,\xi)[\vartheta(s,\rho) - \psi(s,\rho)] ds \end{aligned} \tag{4.22}$$

where $\xi(s,\rho)$ is between $\vartheta(s,\rho)$ and $\psi(s,\rho)$. Since $g(t,\rho) = \lambda\phi(S(\vartheta))$ for $t \notin I_{M,\rho}$ and $g(t,\rho)$ is continuous on $I_{M,\rho}$, by (4.17)

$$\left| \int_a^t \frac{w(s)}{p} [g(s,\rho) - \lambda\phi(S(\vartheta))] S(\vartheta) ds \right|$$
$$\le \int_{I_{M,\rho}} \frac{w(s)}{p} |g(s,\rho) - \lambda\phi(S(\vartheta))| ds \to 0 \text{ as } \rho = \rho_l \to \infty.$$

Thus, for any $\delta > 0$, we can choose ρ^* so large that for $\rho = \rho_l \in (\rho^*, \infty)$

$$\left| \int_a^t \frac{w(s)}{p} [g(s,\rho) - \lambda\phi(S(\vartheta))] S(\vartheta) ds \right| < \delta.$$

Since $|\frac{\partial}{\partial\psi} G(t,\rho,\psi)|$ is uniformly bounded by some $K > 0$ for all $t \in [a,b]$, $\rho \in (\rho^*, \infty)$, and $\psi \in C^1([a,b] \times (\rho^*, \infty))$. Then from (4.22)

$$|\vartheta(t,\rho) - \psi(t,\rho)| < \delta + \int_a^t K |\vartheta(s,\rho) - \psi(s,\rho)| ds.$$

By the Gronwall inequality,

$$|\vartheta(s,\rho) - \psi(s,\rho)| < \delta e^{K(t-a)} \le \delta e^{K(b-a)} < \epsilon$$

if $\delta < \epsilon e^{-K(t-a)}$. Therefore,

$$\vartheta(t,\rho) > \psi(t,\rho) - \epsilon \text{ on } [a,b].$$

Comparing (4.19) and (4.20) we find that $\theta(t,\rho) \ge \vartheta(t,\rho)$ for $\rho = \rho_l \in (\rho^*, \infty)$ and $t \in [a,b]$. In particular,

$$\theta(b,\rho) \ge \vartheta(b,\rho) > \psi(b,\rho) - \epsilon = n\pi_p + \beta.$$

We have reached a contradiction. □

Lemma 4.5. *Assume $f_\infty < \lambda_n$ for some $n \in \mathbb{N}_k$. Then there exists ρ^* such that $\theta(b,\rho) < n\pi_p + \beta$ for all $\rho \in (\rho^*, \infty)$.*

Proof. We choose $\lambda > 0$ satisfying $f_\infty < \lambda < \lambda_n$ and take $M > 0$ so large that $f(y)/\phi(y) \le \lambda$ for $|y| \ge M$.

For any $\rho > 0$, let $\psi(t,\rho)$ be the Prüfer angle of the solution of IVP (3.1), (4.1) and $\psi_n(t,\rho)$ the Prüfer angle of the solution of IVP (3.1), (4.1) with λ replaced by λ_n. Then $\psi_n(t,\rho)$ is an eigenfunction of SLP (3.1), (2.1), and hence $\psi_n(b,\rho) = n\pi_p + \beta$. By the generalized Sturm comparison theorem, $\psi(b,\rho) = n\pi_p + \beta - \epsilon$ for some $\epsilon > 0$. Define $g(t,\rho)$ by (4.18). From (4.2)

$$\theta'(t,\rho) \le |C(\theta(t,\rho))|^{p+1} + \frac{w(t)g(t,\rho)S(\theta(t,\rho))}{p} - q(t)\frac{|S(\theta(t,\rho))|^{p+1}}{p}. \tag{4.23}$$

Since in this case (4.15) holds, there exists $\rho^* > 0$ such that $g(t,\rho)$ is uniformly bounded for $\rho > \rho^*$ and $t \in [a,b]$. Hence, (4.23) implies that $\theta(t,\rho)$ is uniformly bounded for $\rho > \rho^*$ and $t \in [a,b]$. As a result, the number of zeros of $y(t,\rho)$ in (a,b)

is uniformly bounded for $\rho > \rho^*$ and $t \in [a,b]$. By the same argument as in the proof of Lemma 4.4, we have

$$\lim_{\rho\to\infty} \text{mess} I_{M,\rho} = 0.$$

Then, with a similar discussion as in the proof of Lemma 4.4, we can show that

$$\theta(b,\rho) \le \vartheta(b,\rho) < \psi(b,\rho) + \epsilon = n\pi_p + \beta.$$

We omit the details. □

We now prove the main theorems.

Proof of Theorem 3.1. Assume $f_0 < \lambda_n < f_\infty$. Let $y(t,\rho)$ be the solution of IVP (1.1), (4.1) and $\theta(t,\rho)$ the generalized Prüfer angle. By Lemma 4.1, there exists $\rho_* > 0$ such that $\theta(b,\rho) < n\pi_p + \beta$ for all $\rho \in (0,\rho_*)$. By Lemma 4.4, there exists $\rho^* > \rho_*$ such that $\theta(b,\rho) > n\pi_p + \beta$ for all $\rho \in (\rho^*,\infty)$. Since $\theta(b,\rho)$ is continuous in ρ on $(0,\infty)$, there exists $\rho_n \in [\rho_*,\rho^*]$ such that $\theta(b,\rho) = n\pi_p + \beta$. This implies that $y(t,\rho_n)$ is a solution of BVP (1.1), (1.2) and $y(t,\rho_n)$ has exactly n zeros in (a,b). Another solution is obtained in the same way by replacing ρ by $-\rho$ in (4.1).

The case where $f_\infty < \lambda_n < f_0$ can be proved in a similar way using Lemmas 4.2 and 4.5. □

Proof of Theorem 3.2. (i) Assume to contrary that BVP (1.1), (1.2) has a solution $y(t)$ with exactly i zeros in (a,b) for any $i \ge n$. Let $\tilde{w}(t) = w(t)f(y(t))/\phi(y(t))$. Then $\tilde{w}(t)$ is continuous on $[a,b]$ by continuous extension since $f_0 < \infty$. Let θ be the generalized Prüfer angle of $y(t)$ with $\theta(a) = \alpha$. Then θ satisfies the equation

$$\theta'(t) = |C(\theta(t))|^{p+1} + [\tilde{w}(t) - q(t)]\frac{|S(\theta(t))|^{p+1}}{p}. \tag{4.24}$$

Since for $j = 0,1,\ldots,i$, $\theta'(j\pi_p) = 1 > 0$, θ is strictly increasing at $j\pi_p$. By BC (2.1), $\theta(b) = i\pi_p + \beta$. On the other hand, since $\tilde{w}(t) < \lambda_i w(t)$ on $[a,b]$, for $i \ge n$ we have

$$\theta'(t) < |C(\theta(t))|^{p+1} + [\lambda_i w(t) - q(t)]\frac{|S(\theta(t))|^{p+1}}{p} \quad \text{a.e. on } [a,b].$$

Also, the generalized Prüfer angle $\varphi_i(t)$ of the eigenfunction $y_i(t)$ associated with the eigenvalue λ_i of SLP (3.1), (1.2) satisfies

$$\varphi_i'(t) = |C(\varphi_i(t))|^{p+1} + [\lambda_i w(t) - q(t)]\frac{|S(\varphi_i(t))|^{p+1}}{p} \tag{4.25}$$

and $\varphi_i(a) = \theta(a)$. By the theory of differential inequalities we find that $\theta(b) < \varphi_i(b) = i\pi_p + \beta$. We have reached a contradiction.

(ii) Assume to the contrary that BVP (1.1), (1.2) has a solution $y(t)$ with exactly i zeros in (a,b) for some $i \le n$. Let $\theta(t)$ be the Prüfer angle of $y(t)$ with $\theta(a) = \alpha$.

Then as in Part (i), θ satisfies Eq. (4.24) and $\theta(b) = i\pi_p + \beta$. Since $f(y(t))/\phi(y(t)) > \lambda_i$ whenever $y(t) \neq 0$ and $i \leq n$. Thus for $i \leq n$

$$\frac{f(y(t))}{\phi(y(t))}\frac{|S(\theta(t))|^{p+1}}{p} > \lambda_i \frac{|S(\theta(t))|^{p+1}}{p} \text{ a.e. on } [a,b].$$

From (4.24)

$$\theta'(t) > |C(\theta(t))|^{p+1} + [\lambda_i w(t) - q(t)]\frac{|S(\theta(t))|^{p+1}}{p} \text{ a.e. on [a,b].}$$

Note that φ_i satisfies (4.25) and $\varphi_i(a) = \theta(a)$. By the theory of differential inequalities we find that $\theta(b) > \varphi(b) = i\pi_p + \beta$ for $i \leq n$. We have reached a contradiction.

(iii) The assumption in Part (iii) implies that either
(a) $\lambda_n < f(y)/\phi(y) < \lambda_{n+1}$ for some $n \in \mathbb{N}_0$ and all $y \in (0,\infty)$, or
(b) $0 < f(y)/\phi(y) < \lambda_0$ for all $y \in (0,\infty)$, if $k = 0$.
Note that every nontrivial solution has only a finite number of zeros in (a, b); the conclusion follows from Parts (i) and (ii). □

Proof of Corollary 3.3. This is an immediate consequence of Theorems 3.1 and 3.2. □

Proof of Theorem 3.4. Without loss of generality we only prove part (i). The proof for part (ii) is similar.

By Theorem 3.1, BVP (1.1), (1.2) has two solutions $y^{\pm}_m(t)$ with exactly m zeros in (a, b) and have opposite signs in a right-neighborhood of a. Then there exists $\rho_m > 0$ such that $y^{\pm}_m(t) = y(t, \pm\rho_m)$. It follows that $r_m(a, \pm\rho_m) = \rho_m$ and $\theta(b, \pm\rho_m) = m\pi_p + \beta$. Since $\lambda_n < f_\infty$, from Lemma 4.4, there exists $\rho^* > 0$ such that $\theta(b, \pm\rho) > n\pi_p + \beta$ for all $\rho \in (\rho^*, \infty)$. This implies that $\rho_m \leq \rho^*$. By the continuity of $\theta(b, \pm\rho)$ we see that there exists $\rho_n \in (\rho_m, \rho^*]$ such that $\theta(b, \pm\rho_n) = n\pi_p + \beta$. Therefore, $y^{\pm}_n(t) := y(t, \pm\rho_n)$ are two solutions of BVP (1.1), (1.2) with exactly n zeros in (a, b) and satisfying $r_n(a, \pm\rho_n) = \rho_n$. Since $\rho_m < \rho_n$, the conclusion of part (i) is proved. □

Proof of Corollary 3.5. This is an immediate consequence of Theorems 3.1 and 3.4. □

To prove Theorem 3.6 we need the following result about the dependence of the n-th eigenvalue of SLP (3.1), (1.2) on the right endpoint b, which is an extension of Theorem 2.2, 2.3 in [9] for the case when $p = 1$ with a similar proof.

Lemma 4.6. *For a fixed a, consider the n-th eigenvalue of BVP* (1.1), (1.2) *as a function of b for $b \in (a,\infty)$, denoted by $\lambda_n(b)$. Then:*

(i) for $n \in \mathbb{N}_1$, $\lambda_n(b) \to \infty$ as $b \to a+$;

(ii) for $0 \leq \alpha < \beta \leq \pi_p$, $\lambda_0(b) \to \infty$ as $b \to a+$;

(iii) for $0 < \beta < \alpha < \pi_p$, $\lambda_0(b) \to -\infty$ as $b \to a+$.

Proof of Theorem 3.6. (i) Without loss of generality assume $f_0 < \infty$ and $f_\infty = \infty$. Let $\lambda_n(b)$ be defined as in Lemma 4.6. By Lemma 4.6, for any $n \in \mathbb{N}_1$, there exists

$b_n > a$ such that for any $b \in (a, b_n)$ we have $f_0 < \lambda_n(b) < f_\infty$ and hence $f_0 < \lambda_i(b) < f_\infty$ for all $i \geq n$. Then part (a) follows from Theorem 3.1. The same reason holds for part (b) when $\alpha < \beta$. When $\beta < \alpha$, since $\lambda_0(b) \to -\infty$ as $b \to a+$, there exists $b_0 > a$ such that for $b \in (a, b_0)$, $\lambda_0(b) < f_* := \inf\{f(y)/\phi(y) : y \neq 0\}$. The conclusion follows from Theorem 3.2, (ii).

(ii) By Lemma 4.6, there exists $b_* > a$ such that for any $b \in (a, b_*)$ we have that $\lambda_n(s) > f^* := \sup\{f(y)/\phi(y) : y \neq 0\}$ for $n \in \mathbb{N}_1$, $\lambda_0(s) > f^*$ if $\alpha < \beta$ and $\lambda_0(s) < f_* := \inf\{f(y)/\phi(y) : y \neq 0\}$ if $\beta < \alpha$. The conclusion follows from Theorem 3.2, (i) and (ii). □

Proof of Theorem 3.7. For $(\alpha, \beta) \in [0, \pi_p) \times (0, \pi_p]$ and $i \in \mathbb{N}_0$, we denote by $\lambda_i(\alpha, \beta)$ the i-th eigenvalue of the SLP (3.1), (1.2). Then by Theorem 3.7 in [5] we have that

$$\lim_{\alpha \to \pi_p^-} \lambda_0(\alpha, \beta) = \lim_{\beta \to 0^+} \lambda_0(\alpha, \beta) = -\infty$$

and

$$\lim_{\alpha \to \pi_p^-, \beta \to 0^+} \lambda_0(\alpha, \beta) = \lim_{\alpha \to \pi_p^-, \beta \to 0^+} \lambda_1(\alpha, \beta) = -\infty.$$

The conclusion then follows from Theorem 3.2, (ii). □

Proof of Theorem 3.8. For $s \in \mathbb{R}$ and $i \in \mathbb{N}_0$, we denote by $\lambda_i(s)$ the i-th eigenvalue of the SLP consisting of the equation

$$-[\phi(y')]' + [q(t) + sh(t)]\phi(y) = \lambda w(t)\phi(y) \text{ on } [a, b]$$

and BC (1.2). Let $h_* = \min\{h(t)/w(t) : t \in [a, b]\}$ and denote by $\mu_i(s)$ the i-th eigenvalue of the SLP consisting of the equation

$$-[\phi(y')]' + [q(t) + sh_* w(t)]\phi(y) = \mu w(t)\phi(y) \text{ on } [a, b] \tag{4.26}$$

and BC (1.2).

(i) Since for $s \leq 0$

$$q(t) + sh(t) \leq q(t) + sh_* w(t),$$

by Theorem 3.5 (iv) in [5], $\lambda_i(s) \leq \mu_i(s)$ for all $s \leq 0$ and $i \geq 0$. Note that Eq. (4.26) is the same as the equation

$$-[\phi(y')]' + q(t)\phi(y) = (\mu - sh_*)w(t)\phi(y) \text{ on } [a, b].$$

Thus, for $s \leq 0$ and $i \geq 0$, $\mu_i(s) - sh_* = \mu_i(0)$, which implies

$$\mu_i(s) = \mu(0) + sh_* \to -\infty \text{ as } s \to -\infty,$$

and hence

$$\lambda_i(s) \to -\infty \text{ as } s \to -\infty, i \geq 0.$$

Hence, for any $n \in \mathbb{N}$ there exists $s_n \leq 0$ such that $\lambda_i < 0$ for all $i \leq n$ and $s < s_n$. Therefore, the conclusion follows from Theorem 3.2.

(ii) Without loss of generality assume $f_0 < \infty$ and $f_\infty = \infty$. Similar to the argument in (i) we have

$$\lambda_i(s) \to \infty \text{ as } s \to \infty, i \geq 0. \tag{4.27}$$

Then for any $n \in \mathbb{N}$ there exists $s_n \geq 0$ such that for any $s > s_n$, we have $f_0 < \lambda_n < f_\infty$ and hence $f_0 < \lambda_i < f_\infty$ for all $i \geq n$. Therefore, the conclusion follows from Theorem 3.1.

(iii) From (4.27), there exists $s_* \geq 0$ such that for any $s > s_*$ we have that $\lambda_n(s) > f^* := \sup\{f(y)/\phi(y) : y \neq 0\}$. Therefore, the conclusion follows from Theorem 3.2. □

Proof of Theorem 3.9. For $s \geq 0$ and $i \in \mathbb{N}_0$, we denote by $\lambda_i(s)$ the i-th eigenvalue of the SLP consisting of the equation

$$-[\phi(y')]' + q(t)\phi(y) = \lambda[w(t) + sh(t)]\phi(y) \text{ on } [a, b]$$

and BC (1.2). Let $h_* = \min\{h(t)/w(t) : t \in [a, b]\}$, and denote by $\mu_i(s)$ the i-th eigenvalue of the SLP consisting of the equation

$$-[\phi(y')]' + q(t)\phi(y) = \mu(1 + sh_*)w(t)\phi(y) \text{ on } [a, b]$$

and BC (1.2). Since for $s \geq 0$

$$w(t) + sh(t) \geq (1 + sh_*)w(t),$$

by Theorem 3.5 (v) in [5],

$$\lambda_i(s) \leq \mu_i(s) \quad \text{for all } s \geq 0 \text{ and } i \geq 0, \text{ whenever } \lambda_i(s) \geq 0. \tag{4.28}$$

Note that for $i \geq 0$, $\mu_i(s)(1 + sh_*) = \mu_i(0)$, we have

$$\mu_i(s) = \frac{\mu_i(0)}{1 + sh_*} \to 0 \text{ as } s \to \infty.$$

This together with (4.28) implies that $\lambda_i(s) < f_*$ as $s \to \infty$. Then for any $n \in \mathbb{N}_0$ there exists $s_n \geq 0$ such that $\lambda_i(s) < f_*$ for all $s > s_n$ and $i \leq n$. Therefore, the conclusion follows from Theorem 3.2. □

5. GENERALIZED PROBLEMS

In this section, we extend the results for BVP (1.1), (1.2) to a more general problem with more than one term on the right hand side of the equation. For brevity, we only state the results without proofs. In fact, the proofs can be established in a similar way to those for BVP (1.1), (1.2) given in the previous section though some more technical details are involved.

Consider the nonlinear BVP with p-Laplacian consisting of the equation

$$-[\phi(y')]' + q(t)\phi(y) = \sum_{i=1}^{n} w_i(t)f_i(y) \quad \text{on } [a, b], \tag{5.1}$$

where $\phi(y) = |y|^{p-1}y$, $p > 0$, and the general separated boundary condition (1.2). Throughout this section we make the following assumptions: For $i = 1, \dots, n$,
(H1) $q, w_i \in C^1[a,b]$ and $w_i > 0$ on $[a,b]$;
(H2) $f_i \in C(\mathbb{R})$ such that $yf_i(y) > 0$ for $y \neq 0$ and f_i is locally Lipschitz continuous on $\mathbb{R}\backslash\{0\}$;
(H3) There exist limits $f_{i0}, f_{i\infty}$ such that $0 \le f_{i0}, f_{i\infty} \le \infty$, where

$$f_{i0} = \lim_{y\to 0} \frac{f_i(y)}{\phi(y)} \quad \text{and} \quad f_{i\infty} = \lim_{y\to\pm\infty} \frac{f_i(y)}{\phi(y)}.$$

To present our results, we need to compare BVP (5.1), (1.2) with the half-linear SLP consisting of the equation

$$-[\phi(y')]' + q(t)\phi(y) = \lambda \sum_{i=1}^{n} w_i(t)\phi(y) \quad \text{on } [a,b] \tag{5.2}$$

and BC (1.2). Note that Eq. (5.2) is the same equation as Eq. (3.1) except that $w(t)$ is replaced by $\sum_{i=1}^{n} w_i(t)$. Hence, SLP (5.2), (1.2) also has a countably infinite number of real eigenvalues and

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \text{ with } \lambda_n \to \infty.$$

Moreover, any eigenfunctions $y_n = y_n(t, \lambda_n)$ associated with λ_n have exactly n zeros on (a,b).

Let λ_k be the first positive eigenvalue of SLP (5.2), (1.2), and define $\mathbb{N}_k = \{k, k+1, k+2, \dots\}$.

The first theorem is about the existence of certain types of solutions of BVP (5.1), (1.2).

Theorem 5.1. *Assume there exists $n \in \mathbb{N}_k$ such that for any $t \in [a,b]$ either*

$$\sum_{i=1}^{n} w_i(t) f_{i0} < \lambda_n \sum_{i=1}^{n} w_i(t) < \sum_{i=1}^{n} w_i(t) f_{i\infty}$$

or

$$\sum_{i=1}^{n} w_i(t) f_{i\infty} < \lambda_n \sum_{i=1}^{n} w_i(t) < \sum_{i=1}^{n} w_i(t) f_{i0}.$$

Then BVP (5.1), (1.2) has two solutions $y_n^{\pm}(t)$ which have exactly n zeros in (a,b) and have opposite signs in a right-neighborhood of a.

The second theorem is about the nonexistence of nodal solutions of BVP (5.1), (1.2).

Theorem 5.2. *(i) Assume that for some $n \in \mathbb{N}_0$*

$$\sum_{i=1}^{n} w_i(t) \frac{f_i(y)}{\phi(y)} < \lambda_n \sum_{i=1}^{n} w_i(t)$$

for all $t \in [a,b]$ *and* $y \neq 0$. *Then BVP* (5.1), (1.2) *has no solution with exactly* i *zeros in* (a,b) *for any* $i \geq n$.

(ii) Assume that for some $n \in \mathbb{N}_0$

$$\sum_{i=1}^{n} w_i(t)\frac{f_i(y)}{\phi(y)} > \lambda_n \sum_{i=1}^{n} w_i(t)$$

for all $t \in [a,b]$ *and* $y \neq 0$. *Then BVP* (5.1), (1.2) *has no solution with exactly* i *zeros in* (a,b) *for any* $i \leq n$.

(iii) Assume that for any $n \in \mathbb{N}_0$

$$\sum_{i=1}^{n} w_i(t)\frac{f_i(y)}{\phi(y)} \neq \lambda_n \sum_{i=1}^{n} w_i(t)$$

for all $t \in [a,b]$ *and* $y \neq 0$. *Then BVP* (5.1), (1.2) *has no nontrivial solution.*

The combination of Theorem 5.1 and 5.2 leads to the following:

Corollary 5.3. *(i) Assume for all* $t \in [a,b]$ *and* $y \neq 0$

$$\sum_{i=1}^{n} w_i(t) f_{i0} < \sum_{i=1}^{n} w_i(t)\frac{f_i(y)}{\phi(y)} < \sum_{i=1}^{n} w_i(t) f_{i\infty}.$$

Then for $n \in \mathbb{N}_k$, *BVP* (5.1), (1.2) *has two solutions* $y_n^{\pm}(t)$ *which have exactly* n *zeros in* (a,b) *and have opposite signs in a right-neighborhood of* a *if and only if*

$$\sum_{i=1}^{n} w_i(t) f_{i0} < \lambda_n \sum_{i=1}^{n} w_i(t) < \sum_{i=1}^{n} w_i(t) f_{i\infty}.$$

(ii) Assume for all $t \in [a,b]$ *and* $y \neq 0$

$$\sum_{i=1}^{n} w_i(t) f_{i\infty} < \sum_{i=1}^{n} w_i(t)\frac{f_i(y)}{\phi(y)} < \sum_{i=1}^{n} w_i(t) f_{i0}.$$

Then for $n \in \mathbb{N}_k$, *BVP* (5.1), (1.2) *has two solutions* $y_n^{\pm}(t)$ *which have exactly* n *zeros in* (a,b) *and have opposite signs in a right-neighborhood of* a *if and only if*

$$\sum_{i=1}^{n} w_i(t) f_{i\infty} < \lambda_n \sum_{i=1}^{n} w_i(t) < \sum_{i=1}^{n} w_i(t) f_{i0}.$$

The next theorem and corollary are for the existence of multiple and even an infinite number of solutions.

Theorem 5.4. *(i) Assume that for* $t \in [a,b]$ *we have*

$$\sum_{i=1}^{n} w_i(t) f_{i0} < \lambda_m \sum_{i=1}^{n} w_i(t) < \lambda_n \sum_{i=1}^{n} w_i(t) < \sum_{i=1}^{n} w_i(t) f_{i\infty}.$$

Then BVP (5.1), (1.2) *has solutions* $y_m^{\pm}(t)$ *and* $y_n^{\pm}(t)$ *which have exactly* m *and* n *zeros in* (a,b) *and have opposite signs in a right-neighborhood of* a, *respectively, and* *satisfy* $r_m(a) < r_n(a)$, *where* $r_i = (|y_i^{\pm}(t)|^{p+1} + |y_i^{\pm\prime}(t)|^{p+1})^{1/(p+1)}$ *for* $i = m, n$.

(ii) Assume that for $t \in [a, b]$ *we have*

$$\sum_{i=1}^{n} w_i(t) f_{i\infty} < \lambda_m \sum_{i=1}^{n} w_i(t) < \lambda_n \sum_{i=1}^{n} w_i(t) < \sum_{i=1}^{n} w_i(t) f_{i0}.$$

Then BVP (5.1), (1.2) *has solutions* $y_m^{\pm}(t)$ *and* $y_n^{\pm}(t)$ *which have exactly* m *and* n *zeros in* (a, b) *and have opposite signs in a right-neighborhood of* a, *respectively, and satisfy* $r_m(a) > r_n(a)$, *where* $r_i = (|y_i^{\pm}(t)|^{p+1} + |y_i^{\pm\prime}(t)|^{p+1})^{1/(p+1)}$ *for* $i = m, n$.

Corollary 5.5. *Assume either (i) All of* $\{f_{i0}, i = 1, \ldots, n\}$ *are* 0 *and at least one of* $\{f_{i\infty}, i = 1, \ldots, n\}$ *is* ∞ *or (ii) all of* $\{f_{i\infty}, i = 1, \ldots, n\}$ *are* 0 *and at least one of* $\{f_{i0}, i = 1, \ldots, n\}$ *is* ∞. *Then BVP* (5.1), (1.2) *has an infinite number of solutions* $\{y_n^{\pm} : n \in \mathbb{N}_k\}$ *such that* $y_n^{\pm}$ *have exactly* n *zeros in* (a, b) *and have opposite signs in a right-neighborhood of* a *respectively for each* $n \in \mathbb{N}_k$, *and*

$$r_k(a) < r_{k+1}(a) < r_{k+2}(a) < \cdots \quad \textit{if (i) holds}$$

and

$$r_k(a) > r_{k+1}(a) > r_{k+2}(a) > \cdots \quad \textit{if (ii) holds,}$$

where $r_n = (|y_n^{\pm}(t)|^{p+1} + |y_n^{\pm\prime}(t)|^{p+1})^{1/(p+1)}$ *for* $n \in \mathbb{N}_k$.
Furthermore, BVP (5.1), (1.2) *has no solutions with exactly* m *zeros in* (a, b) *for* $m < k$ *if* $\lambda_{k-1} < 0$.

Remark 5.6. All the results on the structural changes for the BVP (1.1), (1.2) can be extended to the general BVP (5.1), (1.2). We omit the details.

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