

PERIODIC SOLUTIONS OF VOLTERRA TYPE INTEGRAL EQUATIONS WITH FINITE DELAY

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ABSTRACT. In this paper, we study the existence of periodic solutions of Volterra type integral equations with finite delay. These equations often arise from delay differential equations. Banach fixed point theorem, Krasnosel'skii's fixed point theorem, and a combination of Krasnosel'skii's and Schaefer's fixed point theorems are employed in the analysis. The combination theorem of Krasnosel'skii and Schaefer requires an *a priori* bound on all solutions. We employ Liapunov's direct method to obtain such an *a priori* bound. In the process, we compare these theorems in terms of assumptions and outcomes.

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1. Introduction

To study the qualitative behavior of ordinary or functional differential equations, one normally inverts these into integral equations. The resulting integral equation is frequently a Volterra type equation. The integrals of the Volterra equation can take the form

$$\int_{t-h}^t, \quad \text{or} \quad \int_{t_0}^t, \quad \text{or} \quad \int_{-\infty}^t,$$

depending on the duration of "heredity." For example, the delay functional differential equation

$$x'(t) = ax(t) - q(x(t), x(t-h)) + r(t), \quad h > 0, \quad a \neq 0, \quad (1.1)$$

can be inverted into the integral equation

$$x(t) = x(t-h)e^{ah} - \int_{t-h}^t q(x(s), x(s-h))e^{a(t-s)} ds + p(t), \quad (1.2)$$

if the integration is carried out from $t-h$ to t . Examples of delay functional differential equations and their applications can be found in [14] and the references therein.

In the present paper we consider the following generalization of (1.2):

$$x(t) = f(t, x(t), x(t-h)) - \int_{t-h}^t C(t, s)g(s, x(s), x(s-h))ds, \quad (1.3)$$

where $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, and $C : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, are all continuous. We refer to (1.3) as a Volterra type integral equation with finite delay. We study the existence of continuous periodic solutions of (1.3) under suitable assumptions on the functions f , g , and C .

Generally, a fixed point theorem is used to study the existence of periodic solutions to this type of equation. We employ Banach fixed point theorem, also known as the contraction principle, Krasnosel'skii's fixed point theorem, and a fixed point theorem which is a combination of Krasnosel'skii's theorem and Schaefer's fixed point theorem. This combination theorem was obtained by Burton and Kirk [4]. In our current paper, we refer to this as Theorem Krasnosel'skii-Schaefer. We also refer to Krasnosel'skii's theorem as Theorem Krasnosel'skii, and Schaefer's theorem as Theorem Schaefer. Statements of these theorems are provided at the end of this section.

In the process of obtaining periodic solutions of (1.3), we compare these theorems in terms of assumptions and outcomes. As we know, Banach fixed point theorem gives the uniqueness of the solution, but it restricts the sizes of the functions involved in the equation. In particular, we observed that for equation (1.3), Banach's fixed point theorem requires functions C and g to be small for a given f . Likewise, we found that Krasnosel'skii's theorem places some size or growth restrictions on functions C and g . On the other hand, Theorem Krasnosel'skii-Schaefer does not place any size or growth restrictions on these functions. However, due to Schaefer's fixed point theorem, Theorem Krasnosel'skii-Schaefer requires an *a priori* bound on all solutions. Following a technique similar to that of Burton and Kirk [4], we employed Liapunov's direct method to obtain such an *a priori* bound on all periodic solutions of (1.3). We used a Liapunov functional in the analysis and found that functions C and g need to satisfy certain sign conditions. One might be able to obtain the required *a priori* bound without these sign conditions, by employing a different method, or constructing a suitable Liapunov functional different from ours. Our analysis, therefore, indicates that the use of Theorem Krasnosel'skii-Schaefer to study periodic solutions of equations like (1.3) has potential for yielding better results than the use of Theorem Krasnosel'skii alone.

Related to Schaefer's theorem, the degree-theoretic work of Granas [11], which also requires an *a priori* bound on all solutions, has been used by many researchers to study the existence of bounded and/or periodic solutions of certain equations (cf. [5-7], [9], [10], and [12]). Recently, some researchers have studied these existence results for functional equations using fixed point theorems on time scales. We refer readers interested in time scales to [1], [16], and the references therein.

We remark that in this paper we used Liapunov's method for the integral equation (1.3). Although Liapunov's direct method has been used extensively for ordinary and functional differential equations, its use on integral equations is relatively new and

somewhat limited. Readers interested in Liapunov's method for integral equations will find [3] a very useful resource.

In a parallel article [15], the author has studied periodic solutions of an integral equation with infinite heredity employing the same fixed point theorems used in this paper.

For results on basic existence theory for Volterra type integral equations, we refer readers to [8], [13], and [18].

Theorem Krasnosel'skii. ([19]) Let M be a closed convex subset of a Banach space S . Suppose A and B map M into S such that

- (i) $x, y \in M$, implies $Ax + By \in M$,
- (ii) A is continuous and AM is contained in a compact subset of S ,
- (iii) B is a contraction mapping.

Then there exists $z \in M$ with $z = Az + Bz$.

Theorem Schaefer. ([19]) Let S be a normed space, H a continuous mapping of S into S which is compact on each bounded subset X of S . Then either (i) the equation $x = \lambda Hx$ has a solution for $\lambda = 1$, or (ii) the set of all such solutions x , for $0 < \lambda < 1$, is unbounded.

Theorem Krasnosel'skii-Schaefer. ([4]) Let S be a Banach space. Suppose A and B map S into S , where B is a contraction, and A is continuous with A mapping bounded sets into compact sets. Then either (i) $x = \lambda B(\frac{x}{\lambda}) + \lambda Ax$ has a solution in S for $\lambda = 1$, or (ii) the set of all such solutions x , for $0 < \lambda < 1$, is unbounded.

2. Solution by Banach Fixed Point Theorem

In this section, we employ Banach fixed point theorem on (1.3), and obtain a unique periodic solution. In addition to the basic continuity conditions on functions f, g , and C , we assume:

(A1) there exists a constant $T > 0$ such that

$$\begin{aligned} f(t+T, x, y) &= f(t, x, y), \\ g(t+T, x, y) &= g(t, x, y), \\ C(t+T, s+T) &= C(t, s); \end{aligned}$$

(A2) there exist positive constants a, b, c , and d such that

$$\begin{aligned} |f(t, x_1, y_1) - f(t, x_2, y_2)| &\leq a|x_1 - x_2| + b|y_1 - y_2|, \\ |g(t, x_1, y_1) - g(t, x_2, y_2)| &\leq c|x_1 - x_2| + d|y_1 - y_2|, \end{aligned}$$

with

$$(a + b) + (c + d)h\bar{C} < 1,$$

where

$$\bar{C} = \max\{|C(t, s)| : 0 \leq t \leq T, -h \leq s \leq T\}.$$

Let P_T be the Banach space of all continuous T -periodic real-valued functions with the supremum norm $\|\cdot\|$.

Theorem 2.1. Suppose (A1) and (A2) hold. Then (1.3) has a unique continuous T -periodic solution.

Proof. For $\varphi \in P_T$ let

$$(P\varphi)(t) = (B\varphi)(t) + (A\varphi)(t), \quad (2.1)$$

where

$$(B\varphi)(t) = f(t, \varphi(t), \varphi(t - h)) \quad (2.2)$$

and

$$(A\varphi)(t) = - \int_{t-h}^t C(t, s)g(s, \varphi(s), \varphi(s - h))ds. \quad (2.3)$$

Clearly, $(B\varphi)(t)$ is continuous and T -periodic in t . Using (A1), one can easily verify that $(A\varphi)(t)$ is T -periodic in t . Since C is continuous, it is uniformly continuous for $0 \leq t \leq T$, $-h \leq s \leq T$. Therefore, we can say that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $u, v \in [0, T]$, $s, t \in [-h, T]$, $|u - v| + |s - t| < \delta$ implies $|C(u, s) - C(v, t)| \leq \varepsilon$.

Since g is continuous, it is uniformly continuous on a set $[-h, T] \times [-m, m] \times [-m, m]$ for any $m > 0$. Also, g is T -periodic in t . Therefore, g is bounded on that set.

Then for any $\varphi \in P_T$ with $\|\varphi\| \leq m$, there exists a constant $\bar{G} > 0$ such that

$$|g(t, \varphi(t), \varphi(t - h))| \leq \bar{G} \text{ for } t \in R. \quad (2.4)$$

Therefore, one can see that $(A\varphi)(t)$ defined in (2.3) is continuous in t . It follows from (2.1) that $P\varphi \in P_T$ for each $\varphi \in P_T$.

Now, we show that P is a contraction mapping on P_T . Let $\varphi, \psi \in P_T$ with $\|\varphi\| \leq m$, $\|\psi\| \leq m$. Then,

$$\begin{aligned} |(P\varphi)(t) - (P\psi)(t)| &\leq |(B\varphi)(t) - (B\psi)(t)| + |(A\varphi)(t) - (A\psi)(t)| \\ &\leq |f(t, \varphi(t), \varphi(t - h)) - f(t, \psi(t), \psi(t - h))| \\ &\quad + \left| - \int_{t-h}^t C(t, s)g(s, \varphi(s), \varphi(s - h))ds \right. \\ &\quad \left. + \int_{t-h}^t C(t, s)g(s, \psi(s), \psi(s - h))ds \right| \\ &\leq a|\varphi(t) - \psi(t)| + b|\varphi(t - h) - \psi(t - h)| \end{aligned}$$

$$\begin{aligned}
& + \int_{t-h}^t |C(t, s)| |g(s, \varphi(s), \varphi(s-h)) - g(s, \psi(s), \psi(s-h))| ds \\
& \leq (a+b) \|\varphi - \psi\| + \int_{t-h}^t |C(t, s)| (c+d) \|\varphi - \psi\| ds \\
& \leq [(a+b) + (c+d)\bar{C}h] \|\varphi - \psi\|.
\end{aligned}$$

This shows that P is a contraction on P_T by assumption (A2). Therefore, by Banach fixed point theorem, there exists a unique function $\varphi \in P_T$ such that $P\varphi = \varphi$; φ is the unique solution of (1.3). This concludes the proof of Theorem 2.1.

We **remark** that to satisfy the condition $(a+b) + (c+d)\bar{C}h < 1$ of (A2), $(c+d)\bar{C}$ needs to be small if $(a+b)$ is close to one since h is fixed. This means that to apply Banach fixed point theorem on (1.3), one may have to choose functions C and g small for a given f .

3. Solution by Krasnosel'skii's Fixed Point Theorem

In this section, we apply Theorem Krasnosel'skii to (1.3) to obtain a periodic solution. Let P_T be the Banach space defined in Section 2. Let $m > 0$ be any constant. Then

$$M = \{\varphi \in P_T : \|\varphi\| \leq m\} \quad (3.1)$$

is a closed convex subset of P_T . Using (2.3), we define a mapping $A : M \rightarrow P_T$, i.e., for $\varphi \in M$,

$$(A\varphi)(t) = - \int_{t-h}^t C(t, s) g(s, \varphi(s), \varphi(s-h)) ds. \quad (3.2)$$

We have already shown in Section 2 that $(A\varphi)(t)$ is T -periodic and continuous in t , and hence $A\varphi \in P_T$ for $\varphi \in M$. Also, in Section 2 we established that the function g is bounded on $[-h, T] \times [-m, m] \times [-m, m]$. So, for each $\varphi \in M$, there exists $\bar{G} > 0$ satisfying (2.4). Therefore, for each $\varphi \in M$,

$$\begin{aligned}
|(A\varphi)(t)| & \leq \int_{t-h}^t |C(t, s)| |g(s, \varphi(s), \varphi(s-h))| ds \\
& \leq \bar{C}\bar{G}h
\end{aligned} \quad (3.3)$$

where \bar{C} is the constant in (A2). This proves that the set $\{A\varphi : \varphi \in M\}$ is (uniformly) bounded. The arguments used earlier to show the continuity of $(A\varphi)(t)$ in t , will in fact prove that the set $\{A\varphi : \varphi \in M\}$ is equicontinuous. Therefore, by the Arzela-Ascoli theorem, A maps M into a compact set.

Now, we show that mapping A of (3.2) is continuous. For that, pick $\varphi, \psi \in M$. Then for $0 \leq u \leq T$,

$$|(A\varphi)(u) - (A\psi)(u)|$$

$$\leq \int_{t-h}^t |C(t, s)| |g(s, \varphi(s), \varphi(s-h)) - g(s, \psi(s), \psi(s-h))| ds \quad (3.4)$$

Once again, g is continuous and periodic in t implies that g is uniformly continuous on $[-h, T] \times [-m, m] \times [-m, m]$. Since C is bounded on $[0, T] \times [-h, T]$, we can make the right side of (3.4) as small as we wish. This proves that mapping A is continuous. Thus, condition (ii) of Theorem Krasnosel'skii is satisfied.

Using (2.2) we define $B : M \rightarrow P_T$, i.e., for $\varphi \in M$,

$$(B\varphi)(t) = f(t, \varphi(t), \varphi(t-h)).$$

We know $(B\varphi)(t)$ is continuous and T -periodic in t and hence $B\varphi \in P_T$ for each $\varphi \in M$.

Assume that

(A3) there exist positive constants a and b such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq a|x_1 - x_2| + b|y_1 - y_2|, \quad a + b < 1,$$

for all $t \in R$.

Clearly, when (A3) holds then B is a contraction, which satisfies condition (iii) of Theorem Krasnosel'skii. Let

$$\bar{m} = \max\{|f(t, 0, 0)| : 0 \leq t \leq T\}. \quad (3.5)$$

Assume g is bounded by a constant \bar{G} . Choose a constant m such that

$$\frac{\bar{m} + \bar{C}\bar{G}h}{1 - (a + b)} \leq m, \quad (3.6)$$

where \bar{C} is defined in (A2). Now, consider the set M defined in (3.1) for the m of (3.6). For $\varphi, \psi \in M$, we have

$$|(A\varphi)(t) + (B\psi)(t)| = |f(t, \psi(t), \psi(t-h))| + \left| \int_{t-h}^t C(t, s)g(s, \varphi(s), \varphi(s-h))ds \right|. \quad (3.7)$$

Notice that

$$\begin{aligned} |f(t, \psi(t), \psi(t-h))| &\leq |f(t, \psi(t), \psi(t-h)) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leq a|\psi(t)| + b|\psi(t-h)| + \bar{m} \\ &\leq (a + b)\|\psi\| + \bar{m}. \end{aligned} \quad (3.8)$$

Then, from (3.3), (3.6), (3.7), and (3.8), we obtain

$$|(A\varphi)(t) + (B\psi)(t)| \leq (a + b)m + \bar{m} + \bar{C}\bar{G}h \leq m.$$

This proves that for $\varphi, \psi \in M$ we have $A\varphi + B\psi \in M$, which establishes condition (i) of Theorem Krasnosel'skii.

Now we prove the existence of a periodic solution of (1.3) in the next theorem.

Theorem 3.1. Suppose assumptions (A1) and (A3) hold. Then there exists a continuous T -periodic solution of (1.3).

Proof. From the preceding work it follows from Theorem Krasnosel'skii that there exists a function $\varphi \in M$ such that $\varphi = A\varphi + B\varphi$. This function φ is a solution of (1.3).

Remark. One can see from condition (3.6) that if $(a + b)$ is close to one then the constants \bar{C} and \bar{G} need to be small, which means the functions C and g need to be small. Also, observe that if $(a + b)$ is close to one then $|f(t, 0, 0)|$ needs to be small.

4. Solution by Theorem Krasnosel'skii-Schaefer

In this section, we employ Theorem Krasnosel'skii-Schaefer stated in the introduction section. We continue to assume that functions f, g , and C are all continuous, and assumptions (A1) and (A3) hold.

Let P_T be the Banach space defined in Section 2. Consider the mappings B and A from P_T into P_T defined by (2.2) and (2.3) respectively. We have seen in Section 2 that B is a contraction with contraction constant $(a + b)$ of (A3). Also, we have seen that A is a continuous mapping from P_T into P_T and that A maps bounded sets into compact sets.

Next, notice that if mapping B is defined by

$$(Bx)(t) = f(t, x(t), x(t - h))$$

then for any scalar λ ,

$$\left(\lambda B\left(\frac{x}{\lambda}\right)\right)(t) = \lambda f\left(t, \frac{x(t)}{\lambda}, \frac{x(t - h)}{\lambda}\right). \tag{4.1}$$

Lemma 4.1 ([3, Proposition 6.1.1]). The mapping $(\lambda B(\frac{1}{\lambda}))$ defined in (4.1) is a contraction on P_T .

Proof. First notice that for $x \in P_T$, $(\frac{x}{\lambda}) \in P_T$. So, $B(\frac{x}{\lambda}) \in P_T$ because B is a mapping from P_T into P_T . Therefore, $\lambda B(\frac{x}{\lambda}) \in P_T$. Now, for any $x, y \in P_T$, it follows from (A3) that

$$\begin{aligned} \left|\lambda B\left(\frac{x}{\lambda}\right)(t) - \lambda B\left(\frac{y}{\lambda}\right)(t)\right| &= \left|\lambda f\left(t, \frac{x(t)}{\lambda}, \frac{x(t - h)}{\lambda}\right) - \lambda f\left(t, \frac{y(t)}{\lambda}, \frac{y(t - h)}{\lambda}\right)\right| \\ &= \lambda \left[a \left| \frac{x(t)}{\lambda} - \frac{y(t)}{\lambda} \right| + b \left| \frac{x(t - h)}{\lambda} - \frac{y(t - h)}{\lambda} \right| \right] \\ &\leq (a + b) \|x - y\|. \end{aligned}$$

Therefore, $(\lambda B(\frac{1}{\lambda}))$ is a contraction with contraction constant $a + b < 1$. This concludes the proof of Lemma 4.1.

We have already shown in the previous section that A defined in (2.3) (also in (3.2)) is a continuous mapping on P_T and it maps bounded sets into compact sets.

So, for any λ , $0 < \lambda \leq 1$, the same properties hold for mapping λA . Therefore, by Theorem Krasnosel'skii-Schaefer, the equation

$$x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda Ax$$

has a solution $x \in P_T$ provided the set of all solutions x , $0 < \lambda < 1$, is bounded. This means if we can show that all solutions of

$$x(t) = \lambda f\left(t, \frac{x(t)}{\lambda}, \frac{x(t-h)}{\lambda}\right) - \lambda \int_{t-h}^t C(t, s)g(s, x(s), x(s-h))ds. \quad (4.2)$$

for all λ , $0 < \lambda < 1$, are bounded by a fixed constant, independent of λ , then (1.3) has a continuous T -periodic solution. In the next lemma, we show the existence of such a fixed bound on all periodic solutions of (4.2) for all λ , $0 < \lambda < 1$.

Lemma 4.2. Assume (A1) and (A3) hold. Also, assume

(A4) $xg(t, x, y) \geq 0$, and there exists $\beta > 0$ and $L > 0$ such that

$$[-(1-a)xg(t, x, y) + b|y||g(t, x, y)| + \bar{m}|g(t, x, y)|] \leq L - \beta|g(t, x, y)|,$$

where a and b are the constants defined in (A3), and \bar{m} is the constant defined in (3.5);

(A5) $C(t, t-h) = 0$, $C_s(t, t-h) \geq 0$, $C_{st}(t, s) \leq 0$, and C_s, C_{st} are continuous for $t-h \leq s \leq t$.

Then for any λ , $0 < \lambda \leq 1$, if a T -periodic function x satisfies (4.2) then there exists a positive constant K , independent of λ , such that $\|x\| < K$.

Proof. Let x be a T -periodic solution of (4.2). Define a Liapunov functional

$$V(t) := V(t, x(\cdot)) = \lambda^2 \int_{t-h}^t C_s(t, s) \left(\int_s^t g(u, x(u), x(u-h))du \right)^2 ds. \quad (4.3)$$

One can easily verify that $V(t)$ is T -periodic in t . Differentiating (4.3) and using (A5) we get

$$V'(t) \leq 2\lambda^2 g(t, x(t), x(t-h)) \int_{t-h}^t C_s(t, s) \int_s^t g(u, x(u), x(u-h))duds$$

Now, using integration by parts, and (A5) again, we obtain

$$V'(t) \leq 2\lambda^2 g(t, x(t), x(t-h)) \int_{t-h}^t C(t, s)g(s, x(s), x(s-h))ds.$$

Then from (4.2), we have

$$V'(t) \leq 2\lambda g(t, x(t), x(t-h)) \left(\lambda f\left(t, \frac{x(t)}{\lambda}, \frac{x(t-h)}{\lambda}\right) - x(t) \right).$$

From (3.8) one can see that

$$\left| \lambda f\left(t, \frac{x(t)}{\lambda}, \frac{x(t-h)}{\lambda}\right) \right| \leq a|x(t)| + b|x(t-h)| + \bar{m}$$

where \bar{m} is the constant defined in (3.5). So,

$$\begin{aligned} V'(t) &\leq 2\lambda[|g(t, x(t), x(t-h))|\{a|x(t)| + b|x(t-h)| + \bar{m}\} \\ &\quad -x(t)g(t, x(t), x(t-h))] \\ &= 2\lambda[-(1-a)x(t)g(t, x(t), x(t-h)) + b|x(t-h)||g(t, x(t), x(t-h))| \\ &\quad + \bar{m}|g(t, x(t), x(t-h))|] \\ &\leq \lambda[2L - 2\beta|g(t, x(t), x(t-h))|]. \end{aligned}$$

Assumption (A4) is used in the last step of the above inequality.

Now, integrating both sides of the above inequality from 0 to T , and using the T -periodicity of V yields

$$0 = V(T) - V(0) \leq -2\lambda\beta \int_0^T |g(t, x(t), x(t-h))| dt + 2\lambda LT.$$

Therefore,

$$\int_0^T |g(s, x(s), x(s-h))| ds \leq \frac{LT}{\beta}.$$

Since $x \in P_T$ and $g \in P_T$, there exists a constant N such that

$$\int_{t-h}^t |g(s, x(s), x(s-h))| ds \leq N.$$

So, for any T -periodic solution $x(t)$, (4.2) gives

$$\begin{aligned} |x(t)| &\leq \left| \lambda f \left(t, \frac{x(t)}{\lambda}, \frac{x(t-h)}{\lambda} \right) \right| + \lambda \left| \int_{t-h}^t C(t, s) g(s, x(s), x(s-h)) ds \right| \\ &\leq a|x(t)| + b|x(t-h)| + \bar{m} + \lambda \bar{C} \int_{t-h}^t |g(s, x(s), x(s-h))| ds \\ &\leq (a+b)||x|| + \bar{m} + \lambda \bar{C} N. \end{aligned}$$

This implies

$$||x|| \leq \frac{\bar{m} + \bar{C} N}{1 - (a+b)} := K.$$

This concludes the proof of Lemma 4.2.

Therefore, we have the following theorem.

Theorem 4.1. Suppose assumptions (A1), (A3) - (A5) hold. Then there exists a continuous T -periodic solution of (1.3).

Proof. Proof of this theorem follows from Theorem Krasnosel'skii - Schaefer. All required work has already been shown.

Remark. We observed that in this section no growth or size restrictions are placed on the functions C and g , although we had to use some sign conditions as shown in (A4) and (A5). However, we used these sign conditions only to obtain an *a priori* bound employing Liapunov's method, which requires construction of a suitable Liapunov functional. We used one such functional for equation (1.3). It is possible that

one can employ an alternative method to Liapunov's, or perhaps construct an entirely different Liapunov functional that may not need these sign conditions. Therefore, our analysis indicates that the use of Theorem Krasnosel'skii-Schaefer to study periodic solutions of equations like (1.3) has potential for yielding better results than the use of Theorem Krasnosel'skii alone.

Another Remark. In [2], Burton introduced a type of contraction called "large contraction" and proved that Krasnosel'skii's theorem holds if the contraction property of the mapping B is replaced by a large contraction. Later in [17], Liu and Li introduced a more general concept of contraction called "separate contraction." In that article, the authors have shown that every large contraction is a separate contraction, and that Krasnosel'skii's theorem as well as Theorem Krasnosel'skii-Schaefer hold if the mapping B is a separate contraction. Therefore, the results of Sections 3 and 4 of our present paper hold if the function f of equation (1.3) defines a separate contraction or a large contraction. For definitions of separate contraction and large contraction, and for examples of functions f that define these types of contractions, we refer the reader to [17].

REFERENCES

- [1] M. Adivar and Y. Raffoul, Existence results for periodic solutions of integro-dynamic equations on time scales, *Ann. Mat. Pura. Appl.*, 188(2009), 543-559.
- [2] T. A. Burton, Integral equations, implicit functions and fixed points, *Proc. Amer. Math. Soc.*, 124(1996), 2383-2390
- [3] T. A. Burton, *Liapunov Functionals for Integral Equations*, Trafford Publishing, Victoria, Canda, 2008.
- [4] T. A. Burton and C. Kirk, A fixed point theorem of Krasnoselskii Schaefer type, *Math. Nachr.*, 189 (1998), 23-31.
- [5] T. A. Burton, P. W. Eloe, and M. N. Islam, Periodic solutions of linear integrodifferential equations, *Math. Nachr.*, 147 (1990), 175-184.
- [6] T. A. Burton, P. W. Eloe, and M. N. Islam, Nonlinear integrodifferential equations and a priori bounds on periodic solutions, *Ann. Mat. Pura. Appl.*, CLXI (1992), 271-283.
- [7] T. A. Burton, and Bo Zhang, Periodic solutions of abstract differential equations with infinite delay, *J. Differential Equations*, 90 (1991), 357-396.
- [8] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, Cambridge, 1991.
- [9] P. W. Eloe, and J. Henderson, Nonlinear boundary value problems and a priori bounds on solutions, *SIAM J. Math. Anal.*, 15 (1984), 642-647.
- [10] P. W. Eloe, and M. N. Islam, Periodic solutions of nonlinear integral equations with infinite memory, *Applicable Analysis*, 28 (1988), 79-93.
- [11] A. Granas, Sur la méthode de continuité de Poincaré, *C. R. Acad. Sci.*, Paris, 282 (1976), 983-985.
- [12] A. Granas, R. B. Guenther, and J. W. Lee, Nonlinear boundary value problems for some classes of ordinary differential equations, *Rocky Mountain J. Math.*, 10 (1980), 35-58.

- [13] G. Gripenberg, S. O. Londen, and O. Staffans, *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge, 1990.
- [14] J. K. Hale, and S. M. V. Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.
- [15] M. N. Islam, Three fixed point theorems; periodic solutions of a Volterra type integral equation with infinite heredity, *Canadian Mathematical Bulletin*, to appear.
- [16] E. Kaufmann, N. Kosmatov, and Y. Raffoul, The connection between boundedness and periodicity in nonlinear functional neutral dynamic equations on a time scale, *Nonlinear Dyn. Syst. Theory*, 9(2009), 89-98.
- [17] Y. Liu and Z. Li, Krasnoselskii type fixed point theorems and applications, *Proc. Amer. Math. Soc.*, 136(2008), 1213-1220.
- [18] R. K. Miller, *Nonlinear Volterra Integral Equations*, Benjamin, New York, 1971.
- [19] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge, 1980.