Communications in Applied Analysis **15** (2011), no. 1, 69–88

PRACTICAL STABILITY IN TERMS OF TWO MEASURES FOR IMPULSIVE DIFFERENTIAL EQUATIONS WITH "SUPREMUM"

DRUMI BAINOV[†] AND SNEZHANA HRISTOVA[‡]

[†]Medical Academy of Sofia
PO Box 45, Sofia 1504, Bulgaria
E-mail: drumibainov@yahoo.com

[‡]Department of Applied Mathematics and Modeling, Plovdiv University
Plovdiv 4000, Bulgaria
E-mail: snehri@uni-plovdiv.bg

ABSTRACT. In the paper several types of practical stability for impulsive differential equations with "*supremum*" is introduced. The definitions are based on the application of two different measures for the initial condition and for the solution. This allow us to increase the possibility for applications of the stability. Some sufficient conditions for various types of practical stability in terms of two measures of nonlinear impulsive differential equations with "*supremum*" are obtained. The proofs are based on the application of piecewise continuous Lyapunov functions and Razumikhin method. An example illustrates the practical application of the proved results.

Key words: practical stability, two measures, piecewise continuous Lyapunov functions, impulses, differential equations with "*supremum*."

AMS (MOS) Subject Classification. 34D20.

1. INTRODUCTION

From a practical point a view a physical system is stable if its state remains within a certain bound of the equilibrium for all time. Such bounds depend on the particular physical system and often on the initial conditions and the system disturbances. The state of the system may be mathematically unstable and yet it may oscillate sufficiently near an equilibrium that its performance is considered to be acceptable. Many problems fall into this category including the travel of a space vehicle between two points and the problem, in a chemical process, in the keeping of temperature within certain bounds. To deal with this situation the notation of practical stability is very useful ([9]). At the same time it is well-known ([11]) that stability and even asymptotic stability themselves are neither necessary nor sufficient to ensure practical stability. Recently several results for practical stability for various

Received September 4, 2010

types of differential equations are obtained in [4], [9], [13], [14], [17], [20], [21] , [23]. Also, practical stability for impulsive differential equations is studied in [2], [3], [7], [8], [12], [22], [24].

In the last few decades, great attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many problems in control theory correspond to the maximal deviation of the regulated quantity ([16]). Such kind of problems could be adequately modeled by differential equations that contain the maxima operator. A. D. Mishkis also points out the necessity to study differential equations with "*maxima*" in his survey [15]. Note that various conditions for stability for differential equations with "*maxima*" are obtained by D. D. Bainov et al. ([6], [18], [19]).

In this paper, impulsive differential equations with "*supremum*" are studied, i.e., the differential equation depends on one side on the maximum deviation of the unkknown function on a certain interval, and on the other side it involves impulses at initially fixed points. The definition for practical stability involves two different measures for the initial condition and for the solution of the considered equation. The Razumikhin method and piecewise continuous Lypunov functions are used to obtain sufficient conditions for various types of practical stability of solutions of impulsive differential equations with "*supremum*." Comparison results for scalar impulsive differential equations are applied. The main idea is based on the connections between the practical stability of a comparison scalar impulsive ordinary differential equation and the practical stability in two measures for the impulsive differential equations with "*supremum*." An appropriate example illustrates the application of the obtained sufficient conditions and the main advantages of the considered type of stability.

2. MAIN RESULTS

Let $\mathbb{R}^n$ be n-dimensional Euclidean space with a norm $||x||$, Ω be a bounded domain in $\mathbb{R}^n$ containing the origin and $\mathbb{R}_+ = [0, \infty)$.

Let $\{\tau_k\}_1^\infty$ be a sequence of fixed points in $\mathbb{R}_+$ such that $\tau_{k+1} > \tau_k$ and $\lim_{k\to\infty} \tau_k = \infty$. Let $r > 0$ be a fixed constant.

Consider the following system of nonlinear impulsive differential equations with "*supremum*"

$$x' = f(t, x(t), \sup_{s\in[t-r,t]} x(s)) \quad \text{for} \quad t \geq t_0, \quad t \neq \tau_k, \tag{2.1}$$

$$x(\tau_k + 0) = I_k(x(\tau_k - 0)) \quad \text{for} \quad k = 1, 2, \dots, \tag{2.2}$$

with initial condition

$$x(t) = \varphi(t) \qquad \text{for} \ \ t \in [t_0 - r, t_0], \tag{2.3}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, 3, \ldots$, $t_0 \in \mathbb{R}_+$, and $\varphi : [t_0 - r, t_0] \to \mathbb{R}^n$.

Note that for $x : [t - r, t] \to \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$ we denote

$$\sup_{s\in[t-r,t]} x(s) = \Big(\sup_{s\in[t-r,t]} x_1(s), \sup_{s\in[t-r,t]} x_2(s), \ldots, \sup_{s\in[t-r,t]} x_n(s) \Big).$$

Denote by $PC(X, Y)$ $(X \subset \mathbb{R}, Y \subset \mathbb{R}^n)$ the set of all functions $u : X \to Y$ which are piecewise continuous in X with points of discontinuity of the first kind at the points $\tau_k \in X$ and which are continuous from the left at the points $\tau_k \in X$, and $u(\tau_k) = u(\tau_k - 0)$.

We denote by $PC^1(X, Y)$ the set of all functions $u \in PC(X, Y)$ which are continuously differentiable for $t \in X$, $t \neq \tau_k$.

In our further investigations we will assume that for any initial function $\phi \in PC([t_0-r, t_0], \mathbb{R}^n)$ the solution of the initial value problem for the system of impulsive differential equations with “*supremum*” (2.1)–(2.3) exists on $[t_0, \infty)$.

Let $X \subset \mathbb{R}$. we will define the set

$$Z(X) = \{k \in \mathbb{Z} : \ \tau_k \in X, \} \tag{2.4}$$

and the set of measures:

$$\Gamma = \{h \in C([-r, \infty) \times \mathbb{R}^n, \mathbb{R}_+) : \min_{x\in\mathbb{R}^n} h(t, x) = 0 \text{ for each } t \in [-r, \infty)\}. \tag{2.5}$$

Let $h_0 \in \Gamma$, $t_0 \in \mathbb{R}_+$, $\varphi \in PC([t_0 - r, t_0], \mathbb{R}^n)$. We will use the following notation

$$H_0(t_0, \varphi) = \sup_{s\in[t_0-r,t_0]} h_0(s, \varphi(s)). \tag{2.6}$$

Let $\rho > 0$ be a fixed number and $h \in \Gamma$. Define:

$$S(h, \rho) = \{(t, x) \in [-r, \infty) \times \mathbb{R}^n : h(t, x) < \rho\},$$
$$S^C(h, \rho) = \{(t, x) \in [-r, \infty) \times \mathbb{R}^n : h(t, x) \geq \rho\}.$$

We will introduce the definition of a practical stability for impulsive differential equations with “*supremum*,” based on the ideas of stability in terms of two measures ([10]).

Definition 1. Let the functions $h, h_0 \in \Gamma$. The system of impulsive differential equations with “*supremum*” (2.1), (2.2) is said to be

(S1) *practically stable with respect to (λ, A) in terms of two measures* if there exists $t_0 \geq 0$ such that for any $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ the inequality $H_0(t_0, \phi) < \lambda$ implies $h(t, x(t; t_0, \phi)) < A$ for $t \geq t_0$, where the constants $\lambda, A : \ 0 < \lambda < A$ are given, the function H_0 is defined by (2.6), and $x(t; t_0, \phi)$ is a solution of (2.1), (2.2), (2.3);

(S2) *uniformly practically stable with respect to* (λ, A) *in terms of two measures* if (S1) holds for all $t_0 \in \mathbb{R}_+$;

(S3) *practically quasi stable with respect to* (λ, A, T) *in terms of two measures* if there exists $t_0 \geq 0$ such that for any $\phi \in PC([t_0-r, t_0], \mathbb{R}^n)$ the inequality $H_0(t_0, \phi) < \lambda$ implies $h(t, x(t; t_0, \phi)) < A$ for $t \geq t_0 + T$, where the positive constants $\lambda, A, T :$ $\lambda < A$ are given;

(S4) *uniformly practically quasi stable with respect to* (λ, A, T) *in terms of two measures* if (S3) holds for all $t_0 \in \mathbb{R}_+$;

(S5) *strongly practically stable with respect to* (λ, A, B, T) *in terms of two measures* if there exists $t_0 \geq 0$ such that for any $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ he inequality $H_0(t_0, \phi) < \lambda$ implies $h(t, x(t; t_0, \phi)) < A$ for $t \geq t_0$ and $h(t, x(t; t_0, \phi)) < B$ for $t \geq t_0 + T$, where the positive constants $\lambda, A, B, T :\ B < \lambda < A$ are given;

(S6) *uniformly strongly practically stable with respect to* (λ, A, B, T) *in terms of two measures* if (S5) holds for all $t_0 \in \mathbb{R}_+$;

(S7) *eventually practically stable in terms of two measures* if for any couple $(\lambda, A) : \quad 0 < \lambda < A$ there exists $\tau(\lambda, A) > 0$ such that for some $t_0 \geq \tau(\lambda, A)$ and $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ the inequality $H_0(t_0, \phi) < \lambda$ implies $h(t, x(t; t_0, \phi)) < A$ for $t \geq t_0$;

(S8) *uniformly eventually practically stable in terms of two measures* if (S7) holds for all $t_0 \geq \tau(\lambda, A)$.

Remark 1. In the case $r = 0$ and $h_0(t, x) = h(t, x) = ||x||$, the above given definitions reduce to definitions for the corresponding types of *practical stability of the zero solution* of impulsive differential equations which will be used in our further investigations.

In the case $r = 0$, $I_k(x) \equiv x$, $k = 1, 2, \ldots$, and $h_0(t, x) = h(t, x) = ||x||$ the above given definitions reduce to definitions for the corresponding types of practical stability of the zero solution of ordinary differential equations, given in the books [9], [13].

In our further investigations we will use the initial value problem for the comparison scalar impulsive differential equation

$$\begin{aligned} &u' = g(t, u), \quad t \geq t_0, \quad t \neq \tau_k, \\ &u(\tau_k + 0) = \xi_k(u(\tau_k)), \quad k = 1, 2, \ldots, \\ &u(t_0) = u_0, \end{aligned} \tag{2.7}$$

where $u, u_0 \in \mathbb{R}$, $g :\ \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\xi_k :\ \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots$.

In our further investigations we will assume that for any initial point $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ the solution of scalar impulsive equation (2.7) exists on $[t_0, \infty)$, $t_0 \geq 0$. For some existence results see the book of D. Bainov et al. [1].

We will study the connection between practical stability of the scalar impulsive differential equation (2.7) and the corresponding practical stability in terms of two measures for the system of impulsive differential equations with "*supremum*" (2.1), (2.2).

Introduce the following notations

$$G_k = \{t \in [-r,\infty):\ t \in (\tau_k, \tau_{k+1})\},\quad k = 1,2,\dots,\qquad \mathcal{G} = \bigcup_{k=1}^{\infty} G_k.$$

We will introduce the class Λ of piecewise continuous Lyapunov functions which will be used to investigate the practical stability of impulsive differential equations with "*supremum*."

Definition 2. We will say that the function $V(t,x) : \Delta \times \Omega \to \mathbb{R}_+$, $\Delta \subset [-r,\infty)$, $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$, belongs to class Λ if:

1. $V(t,x)$ is a continuous function in $(\Delta \cap \mathcal{G}) \times \Omega$ and $V(t,0) \equiv 0$ for $t \in \Delta$;
2. For every $k \in Z(\Delta)$ and $x \in \Omega$ there exist the finite limits

$$V(\tau_k, x) = V(\tau_k - 0, x) = \lim_{t \uparrow \tau_k} V(t,x),\qquad V(\tau_k + 0, x) = \lim_{t \downarrow \tau_k} V(t,x),$$

where the set $Z(\Delta)$ is defined by (2.4).

3. $V(t,x)$ is Lipschitz with respect to its second argument in the set $\Delta \times \Omega$.

Let $V(t,x) : \Delta \times \Omega \to \mathbb{R}_+$, $V \in \Lambda$. For any $t \in \Delta \cap \mathcal{G}$ and any function $\psi \in PC([t-r,t],\Omega)$ we will define a derivative of the function V along a trajectory of the solution of (2.1), (2.2) as follows:

$$\begin{aligned} D_{(2.1),(2.2)}V(t,\psi) = \limsup_{\epsilon \to 0} \frac{1}{\epsilon}\Big[V\Big(t+\epsilon, \psi(t) + \epsilon f(t, \psi(t), \max_{s\in[-r,0]} \psi(t+s))\Big) \\ - V(t,\psi(t))\Big]. \end{aligned} \tag{2.8}$$

Definition 3. Let $h_0 \in \Gamma$. The function $V(t,x) : \Delta \times \Omega \to \mathbb{R}_+$, $V \in \Lambda$, is strongly-h_0-decrescent if there exist a fnction $a \in K$ and a constant $\rho > 0$ such that $h_0(t,x) < \rho$ implies $V(t,x) \le a(h_0(t,x))$, where $(t,x) \in \Delta \times \Omega$.

Let $\rho > 0$ be a given number. Consider the following sets:

$$K = \{a \in C(\mathbb{R}_+, \mathbb{R}_+):\ a(r) \text{ is strictly increasing and } a(0) = 0\};$$

$$\mathcal{K} = \{a \in C(\mathbb{R}_+, \mathbb{R}_+):\ a(r) \text{ is strictly increasing and } a(s) \ge s,\ a(0) = 0\}.$$

Definition 4. Let $h, h_0 \in \Gamma$. The function h is *eventually stronger* than h_0 if, for a couple (λ, A) such that $0 < \lambda < A$, the inequality $h_0(t,x) < \lambda$ for some $(t,x) \in [-r,\infty) \times \mathbb{R}^n$, implies $h(t,x) < A$.

In the further investigations, we will use the following comparison result:

Lemma 1 (S. Hristova [5]). *Let the following conditions be fulfilled:*

1. *The functions* $f \in PC([t_0, T] \times \Omega \times \Omega, \mathbb{R}^n)$ *and* $I_k \in C(\Omega, \Omega)$ *for* $k \in Z([t_0, T))$, *where* $\Omega \subset \mathbb{R}^n$, *and* $t_0, T:\ 0 \le t_0 < T < \infty$ *are constants.*
2. *The function* $\varphi \in PC([t_0 - r, t_0], \Omega)$.
3. *The initial value problem* (2.1), (2.2), (2.3) *has a solution* $x(t) = x(t; t_0, \varphi)$, *such that* $x(t) \in \Omega$ *on* $[t_0 - r, T]$.
4. *The functions* $g \in PC([t_0, T] \times \mathbb{R}_+, \mathbb{R}_+)$, $g(t, 0) \equiv 0$ *for* $t \in [t_0, T]$ *and* $\xi_k \in \mathcal{K}$, $k \in Z((t_0, T))$.
5. *For any initial point* $u_0 \in \mathbb{R}_+$ *the initial value problem for the scalar impulsive differential equation* (2.7) *has a maximal solution* $u^*(t) = u^*(t; t_0, u_0)$, *which is defined for* $t \in [t_0, T]$.
6. *The function* $V : [t_0 - r, T] \times \Omega \to \mathbb{R}_+$, $V \in \Lambda$ *is such that*

   (i) for any number $t \in [t_0, T] :\ \ t \neq \tau_k,\ \ k \in Z((t_0, T))$ *and any function* $\psi \in PC([t - r, t], \Omega)$ *such that* $V(t, \psi(t)) \ge V(t + s, \psi(t + s))$ *for* $s \in [-r, 0)$, *the inequality*

   $$D_{(2.1),(2.2)} V(t, \psi(t)) \le g(t, V(t, \psi(t)))$$

   holds.

   (ii) $V(\tau_k + 0, I_k(x)) \le \xi_k(V(\tau_k, x)), \quad k \in Z((t_0, T)), x \in \Omega.$

Then the inequality $\sup_{s \in [-r, 0]} V(t_0 + s, \varphi(t_0 + s)) \le u_0$ *implies the inequality* $V(t, x(t)) \le u^*(t)$ *for* $t \in [t_0, T]$.

Remark 2. Lemma 1 is valid when $T = \infty$,i.e. for $t \in [t_0, \infty)$.

We will obtain sufficient conditions for practical stability in terms of two measures for impulsive differential equations with "*supremum*." We will use Lyapunov functions from class Λ. The proof is based on Razumikhin method and a comparison method employing scalar impulsive differential equations.

In the case where the Lyapunov function satisfies the desired conditions globally, we obtain the following result:

Theorem 1. *Let the following conditions be fulfilled:*

1. *The function* $f \in PC[\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$ *and* $f(t, 0, 0) \equiv 0$.
2. *The functions* $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ *and* $I_k(0) = 0$ *for* $k \in Z(\mathbb{R}_+)$.
3. *The functions* $h_0, h \in \Gamma$.
4. *There exists a function* $V(t, x) : [-r, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ *with* $V \in \Lambda$ *such that*
   (*i*) $b(h(t, x)) \le V(t, x) \le a(h_0(t, x))$ *for* $(t, x) \in [-r, \infty) \times \mathbb{R}^n$ *where* $a, b \in K$ *;*
   (*ii*) *for any number* $t \in \mathbb{R}_+ :\ \ t \neq \tau_k,\ \ k \in Z(\mathbb{R}^+)$ *and any function* $\psi \in PC([t - r, t], \mathbb{R}^n)$ *such that* $V(t, \psi(t)) > V(t + s, \psi(t + s))$ *for* $s \in [-r, 0)$,

the inequality

$$D_{(2.1),(2.2)}V(t,\psi(t)) \leq g(t, V(t,\psi(t)))$$

holds, where $g \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ *and* $g(t,0) \equiv 0$*;*

(*iii*) $V(\tau_k + 0, I_k(x)) \leq \xi_k(V(\tau_k, x)), \quad$ *for* $x \in \mathbb{R}^n, \quad k \in Z(\mathbb{R}^+)$*,* *where* $\xi_k \in \mathcal{K}$*.*

Then

(A) *the practical stability with respect to* $(a(\lambda), b(A))$ *of scalar impulsive differential equation* (2.7) *implies practical stability in terms of two measures with respect to* (λ, A) *of system of impulsive differential equations with "supremum"* (2.1)*,* (2.2) *where the positive constants* $\lambda, A: \ \lambda < A, \quad a(\lambda) < b(A)$ *are given;*

(B) *the uniform practical stability with respect to* $(a(\lambda), b(A))$ *of zero solution of scalar impulsive differential equation* (2.7) *implies uniform practical stability in terms of two measures with respect to* (λ, A) *of system of impulsive differential equations with "supremum"* (2.1)*,* (2.2)*;*

(C) *the practical quasi stability with respect to* $(a(\lambda), b(A), T)$ *of zero solution of scalar impulsive differential equation* (2.7) *implies practical quasi stability in terms of two measures with respect to* (λ, A, T) *of system of impulsive differential equations with "supremum"* (2.1)*,* (2.2)*;*

(D) *the uniform practical quasi stability with respect to* $(a(\lambda), b(A), T)$ *of zero solution of scalar impulsive differential equation* (2.7) *implies uniform practical quasi stability in terms of two measures with respect to* (λ, A, T) *of system of impulsive differential equations with "supremum"* (2.1)*,* (2.2)*;*

(E) *the strong practical stablility with respect to* $(a(\lambda), b(A), b(B), T)$ *of zero solution of scalar impulsive differential equation* (2.7) *implies strong practical stability in terms of two measures with respect to* (λ, A, B, T) *of system of impulsive differential equations with "supremum"* (2.1)*,* (2.2)*;*

(F) *the uniform strong practical stablility with respect to* $(a(\lambda), b(A), b(B), T)$ *of zero solution ofscalar impulsive differential equation* (2.7) *implies uniform strong practical stability in terms of two measures with respect to* (λ, A, B, T) *of system of impulsive differential equations with "supremum"* (2.1)*,* (2.2)*.*

Proof. (A). Let the zero solution of scalar impulsive differential equation (2.7) be practically stable with respect to $(a(\lambda), b(A))$, where $0 < \lambda < A, \quad a(\lambda) < b(A)$. Therefore, there exists a point $t_0 \geq 0$ such that $|u_0| < a(\lambda)$ implies

$$|u(t; t_0, u_0)| < b(A) \qquad \text{for } \ t \geq t_0, \tag{2.9}$$

where $u(t; t_0, u_0)$ is a solution of (2.7).

Choose a function $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ such that

$$H_0(t_0, \phi) < \lambda \tag{2.10}$$

and let $x(t; t_0, \phi)$ be a solution of (2.1), (2.2) with initial condition (2.3).

Let $u_0 = max_{s\in[-r,0]} V(t_0 + s, \phi(t_0 + s))$. From Lemma 1 for $\Delta = [-r, \infty)$ and $\Omega = \mathbb{R}^n$ it follows that

$$V(t, x(t; t_0, \phi)) \leq u^*(t; t_0, u_0) \quad \text{for} \;\; t \geq t_0, \tag{2.11}$$

From condition 4(i) we obtain

$$V(t_0 + s, \phi(t_0 + s)) \leq a((h_0(t_0 + s, \phi(t_0 + s))) \leq a(H_0(t_0, \phi)) < a(\lambda), \quad s \in [-r, 0], \tag{2.12}$$

or $u_0 < a(\lambda)$.

From inequalities (2.11) and (2.12) it follows that

$$V(t, x(t; t_0, \phi)) \leq u^*(t; t_0, u_0) < b(A) \quad \text{for} \;\; t \geq t_0. \tag{2.13}$$

From inequality (2.13) and condition 4(i) we get for $t \geq t_0$

$$b(h(t, x(t; t_0, \phi))) \leq V(t, x(t; t_0, \phi)) \leq u^*(t; t_0, u_0) < b(A), \tag{2.14}$$

or

$$h(t, x(t; t_0, \phi)) < A. \tag{2.15}$$

The proofs of claims (B)-(F) are similar to the one of (A) and we omit them. □

Remark 3. Note that (uniformly) eventually practical stability in terms of two measures implies (uniformly) practical stability in terms of two measures, and (uniformly) practical stability in terms of two measures implies (uniformly) practical quasi stability in terms of two measures.

Corollary 1. *Let the following conditions be fulfilled:*

1. *Conditions* 1, 2, 3 *of Theorem* 1 *are satisfied.*
2. *There exists a function* $V(t, x) : [-r, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ *with* $V \in \Lambda$ *such that*
   (*i*) $b(h(t, x)) \leq V(t, x) \leq a(h_0(t, x))$ *for* $(t, x) \in [-r, \infty) \times \mathbb{R}^n$ *where* $a, b \in K$*;*
   (*ii*) *for any number* $t \in \mathbb{R}_+ : \; t \neq \tau_k, \; k \in Z(\mathbb{R}_+)$ *and any function* $\psi \in PC([t - r, t], \mathbb{R}^n)$ *such that* $V(t, \psi(t)) > V(t + s, \psi(t + s))$ *for* $s \in [-r, 0)$*, the inequality*
   $$D_{(2.1),(2.2)} V(t, \psi(t)) \leq 0$$
   holds;
   (*iii*) $V(\tau_k + 0, I_k(x)) \leq V(\tau_k, x)$ *for* $x \in \mathbb{R}^n, \; k \in Z(\mathbb{R}_+)$.

Then the system of impulsive differential equations with "supremum" (2.1), (2.2) *is uniformly eventually practically stable in terms of two measures.*

Proof. The proof of Corollary 1 follows from the one of Theorem 1 for $g(t,x) \equiv 0$ and $\xi(x) \equiv x$. In this case, the solution of (2.7) is $u(t) = u_0$, and the zero solution of it is uniformly practically stable. □

In the case where the Lyapunov function does not satisfy condition 4 of Theorem 1 globally, we obtain the following sufficient conditions:

Theorem 2. *Let the following conditions be fulfilled:*

1. *The function* $f \in PC[\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$ *and* $f(t,0,0) \equiv 0$.
2. *The functions* $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ *and* $I_k(0) = 0$ *for* $k \in Z(\mathbb{R}_+)$.
3. *The functions* $h_0, h \in \Gamma$ *and there exist positive constants* $\lambda, A: \ \lambda < A$ *such that if* $h(\tau_k, x) < A$ *implies* $h(\tau_k, I_k(x)) \neq A$ *for* $x \in \mathbb{R}^n$, $k \in Z(\mathbb{R}_+)$.
4. *There exists a function* $V(t,x) : [-r,\infty) \times \mathbb{R}^n \to \mathbb{R}_+$ *with* $V \in \Lambda$ *such that*
   (*i*) $b(h(t,x)) \leq V(t,x) \leq a(h_0(t,x))$ *for* $(t,x) \in [-r,\infty) \times \mathbb{R}^n$,
   where $a, b \in K$, $a(\lambda) < b(A)$;
   (*ii*) *for any number* $t \in \mathbb{R}_+ : \ t \neq \tau_k$, $k \in Z(\mathbb{R}_+)$ *and any function* $\psi \in PC([t-r,t], \mathbb{R}^n) : \ h(t+s, \psi(t+s)) < A$, $s \in [-r, 0]$ *such that* $V(t,\psi(t)) > V(t+s, \psi(t+s))$ *for* $s \in [-r, 0)$, *the inequality*

$$D_{(2.1),(2.2)} V(t, \psi(t)) \leq g(t, V(t, \psi(t)))$$

   holds, where $g \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ *and* $g(t,0) \equiv 0$;
   (*iii*) $V(\tau_k + 0, I_k(x)) \leq \xi_k(V(\tau_k, x))$ *for* $(\tau_k, x) \in S(h, A)$, $k \in Z(\mathbb{R}_+)$,
   where $\xi_k \in \mathcal{K}$.

Then

(A) *the practical stability with respect to* $(a(\lambda), b(A))$ *of zero solution of scalar impulsive differential equation* (2.7) *implies practical stability in terms of two measures with respect to* (λ, A) *of system of impulsive differential equations with "supremum"* (2.1), (2.2);

(B) *the uniform practical stability with respect to* $(a(\lambda), b(A))$ *of zero solution of scalar impulsive differential equation* (2.7) *implies uniform practical stability in terms of two measures with respect to* (λ, A) *of system of impulsive differential equations with "supremum"* (2.1), (2.2);

(C) *the practical quasi stability with respect to* $(a(\lambda), b(A), T)$ *of zero solution of scalar impulsive differential equation* (2.7) *implies practical quasi stability in terms of two measures with respect to* (λ, A, T) *of system of impulsive differential equations with "supremum"* (2.1), (2.2);

(D) *the uniform practical quasi stability with respect to* $(a(\lambda), b(A), T)$ *of zero solution of scalar impulsive differential equation* (2.7) *implies uniform practical quasi stability in terms of two measures with respect to* (λ, A, T) *of system of impulsive differential equations with "supremum"* (2.1), (2.2);

(E) *the strong practical stablility with respect to* $(a(\lambda), b(A), b(B), T)$ *of zero solution of scalar impulsive differential equation* (2.7) *implies strong practical stability in terms of two measures with respect to* (λ, A, B, T) *of system of impulsive differential equations with "supremum"* (2.1), (2.2);

(F) *the uniform strong practical stablility with respect to* $(a(\lambda), b(A), b(B), T)$ *of zero solution of scalar impulsive differential equation* (2.7) *implies uniform strong practical stability in terms of two measures with respect to* (λ, A, B, T) *of system of impulsive differential equations with "supremum"* (2.1), (2.2).

Proof. The proofs of claims (A)–(F) are similar and we will give only the proof of (A).

Let zero solution of scalar impulsive differential equation (2.7) be practically stable with respect to the couple $(a(\lambda), b(A))$. Therefore there exists a point $t_0 \geq 0$ such that $|u_0| < a(\lambda)$ implies

$$|u(t; t_0, u_0)| < b(A) \qquad \text{for} \ \ t \geq t_0, \tag{2.16}$$

where $u(t; t_0, u_0)$ is a solution of (2.7).

Choose a function $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ such that

$$H_0(t_0, \phi) < \lambda \tag{2.17}$$

and let $x(t; t_0, \phi)$ be a solution of (2.1), (2.2) with initial condition (2.3).

We will prove that

$$h(t, x(t; t_0, \phi)) < A \tag{2.18}$$

holds for $t \geq t_0$.

From inclusion $(t, \phi(t)) \in S(h_0, \lambda)$ for $t \in [t_0 - r, t_0]$ and conditions 3 and 4(i), it follows that $b\big(h(s, \phi(s))\big) \leq a\big(h_0(s, \phi(s))\big) \leq a(H_0(t_0, \phi)) < a(\lambda) < b(A)$ for $s \in [t_0 - r, t_0]$, i.e., inequality (2.18) holds on $[t_0 - r, t_0]$.

Assume (2.18) does not hold for $t > t_0$. Consider the following three case:

Case 1. Let there exists a point $t^* > t_0$, $t^* \neq \tau_k$, $\ k \in Z((t_0, \infty))$ such that

$$h(t^*, x(t^*; t_0, \phi)) = A \quad \text{and} \quad h(t, x(t; t_0, \phi)) < A \ \ \text{for } t \in [t_0 - r, t^*). \tag{2.19}$$

Let $u_0 = max_{s \in [-r, 0]} V(t_0 + s, \phi(t_0 + s))$. From Lemma 1, for the function $V(t, x)$ defined on the set $\{(t, x) \in [t_0, t^*] \times \mathbb{R}^n : \ h(t, x) \leq A\}$, it follows that

$$V(t, x(t; t_0, \phi)) \leq u^*(t; t_0, u_0) \quad \text{ for } \ t \in [t_0, t^*]. \tag{2.20}$$

From condition 4(i) we obtain

$$V(t_0{+}s, \phi(t_0{+}s)) \leq a(h_0(t_0{+}s, \phi(t_0{+}s))) \leq a(H_0(t_0, \phi)) < a(\lambda), \ \ s \in [-r, 0], \tag{2.21}$$

or $u_0 < a(\lambda)$.

From inequalities (2.16), (2.20), and (2.21) follows that

$$V(t,x(t;t_0,\phi)) \le u^*(t;t_0,u_0) < b(A) \quad \text{for} \;\; t\in[t_0,t^*]. \tag{2.22}$$

From inequality (2.22) and condition 4(i) we get

$$b(A) = b(h(t^*,x(t^*;t_0,\phi))) \le V(t^*,x(t^*;t_0,\phi)) \le u^*(t^*;t_0,u_0) < b(A). \tag{2.23}$$

The obtained contradiction proves the validity of (2.18) for $t > t_0$.

Case 2. Let there exists a number $k \in Z((t_0,\infty))$ such that $h(t,x(t;t_0,\phi)) < A$ for $t \in [t_0 - r, \tau_k)$ and $h(\tau_k, x(\tau_k;t_0,\phi)) = A$. Then as in Case 1 for $t^* = \tau_k$ we obtain a contradiction.

Case 3. Let there exists a natural number k such that $h(t,x(t;t_0,\phi)) < A$ for $t \in [t_0 - r, \tau_k]$ and $h\Big(\tau_k, I_k\big(x(\tau_k;t_0,\phi)\big)\Big) \ge A$. From condition 3 it follows that $h\Big(\tau_k, I_k\big(x(\tau_k;t_0,\phi)\big)\Big) > A$. Then as in Case 1 we prove the validity of inequality (2.22) for $t\in[t_0,\tau_k]$. Applying condition 4(iii) we get

$$\begin{aligned} b(A) &< b\bigg(h\Big(\tau_k, I_k\big(x(\tau_k;t_0,\phi)\big)\Big)\bigg) = b\bigg(h\Big(\tau_k+0, \xi_k\big(x(\tau_k;t_0,\phi)\big)\Big)\bigg) \\ &\le V\Big(\tau_k+0, I_k\big(x(\tau_k;t_0,\phi)\big)\Big) \le \xi_k\big(V(\tau_k,x(\tau_k;t_0,\phi))\big) \\ &\le \xi_k\big(u^*(\tau_k;t_0,u_0)\big) = u^*(\tau_k+0;t_0,u_0) \le b(A). \end{aligned} \tag{2.24}$$

The obtained contradictions prove the validity of (2.18) for $t > t_0$. □

Now we will study eventual practical stabiity of impulsive differential equations with "*supremum.*"

Theorem 3. *Let the following conditions be fulfilled:*

1. *Conditions* 1,2, 3 *of Theorem* 2 *are satisfied.*
2. *There exists a function* $V(t,x) : [-r,\infty)\times\mathbb{R}^n \to \mathbb{R}_+$ *with* $V\in\Lambda$ *such that*
   (i) $b(h(t,x)) \le V(t,x) \le a(h_0(t,x))$ *for* $(t,x)\in[-r,\infty)\times\mathbb{R}^n$ *where* $a,b\in K$;
   (ii) *for any number* $t\in\mathbb{R}_+ :\; t\ne\tau_k,\; k\in Z(\mathbb{R}_+)$ *and any function* $\psi\in PC([t-r,t],\mathbb{R}^n)$ *such that* $(t,\psi(t))\in S(h,\rho)$ *and* $V(t,\psi(t)) > V(t+s,\psi(t+s))$ *for* $s\in[-r,0)$, *the inequality*

   $$D_{(2.1),(2.2)}V(t,\psi(t)) \le g(t,V(t,\psi(t)))$$

   holds, where $g\in PC(\mathbb{R}_+\times\mathbb{R},\mathbb{R}_+)$ *and* $g(t,0)\equiv 0$;
   (iii) $V(\tau_k+0, I_k(x)) \le \xi_k(V(\tau_k,x))$ *for* $(\tau_k,x)\in S(h,\rho),\;\; k\in Z(\mathbb{R}_+)$, *where* $\xi_k\in\mathcal{K}$.

Then

(A) *eventual practical stabilty of zero solution of scalar impulsive differential equation* (2.7) *implies eventual practical stability in terms of two measures of system of impulsive differential equations with "supremum"* (2.1), (2.2)*;*

(B) *uniform eventual practical stabilty of zero solution of scalar impulsive differential equation* (2.7) *implies uniform eventual practical stability in terms of two measures of system of impulsive differential equations with "supremum"* (2.1), (2.2).

Proof. (A). Since h_0 is uniformly finer than h, there exists a constant ρ_0 and a function $\psi \in K$ such that inequality $h_0(t,x) < \rho_0$ implies

$$h(t,x) < \psi(h_0(t,x)). \tag{2.25}$$

Without loss of generality we could assume that $\rho_0 < \rho,\ \psi(\rho_0) \le \rho$. Let the positive constants λ, A be fixed such that $\lambda < A < \rho_0,\ a(\lambda) < b(A)$. Let zero solution of scalar impulsive differential equation (2.7) be eventually practical stable. Therefore for the couple $(a(\lambda), b(A))$ there exists $\tau(\lambda, A) > 0$ such that for some $t_0 \ge \tau(\lambda, A)$ the inequality $|u_0| < a(\lambda)$ implies

$$|u(t;t_0,u_0)| < b(A) \qquad \text{for} \ \ t \ge t_0, \tag{2.26}$$

where $u(t;t_0,u_0)$ is a solution of (2.7).

Choose a function $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ such that

$$H_0(t_0,\phi) < \lambda \tag{2.27}$$

and let $x(t;t_0,\phi)$ be a solution of (2.1), (2.2) with initial condition (2.3). We will prove that

$$h(t, x(t;t_0,\phi)) < A \tag{2.28}$$

holds for $t \ge t_0$.

From inequality (2.27) it follows that $h_0(t,\phi(t)) < \lambda \le \rho_0$ for $t \in [t_0 - r, t_0]$ and according to (2.25), $h(t,\phi(t)) < \psi(h_0(t,\phi(t))) < \psi(\rho_0) \le \rho$. Then, from condition 2$(i)$ of Theorem 3, we obtain that $b\big(h(s,\phi(s))\big) \le a\big(h_0(s,\phi(s))\big) \le a(H_0(t_0,\phi)) < a(\lambda) < b(A)$ for $s \in [t_0 - r, t_0]$, i.e., inequality (2.28) holds on $[t_0 - r, t_0]$.

Assume (2.28) does not hold for $t > t_0$. Consider the following three case:

Case 1. Let there exist a point $t^* > t_0$, $t^* \ne \tau_k$, $\ k \in Z((t_0,\infty))$ such that

$$h(t^*, x(t^*;t_0,\phi)) = A \quad \text{and} \quad h(t,x(t;t_0,\phi)) < A \ \ \text{for}\ t \in [t_0 - r, t^*). \tag{2.29}$$

Let $u_0 = max_{s\in[-r,0]} V(t_0 + s, \phi(t_0 + s))$. From Lemma 1 for the function $V(t,x)$ defined on the set $\{(t,x) \in [t_0,t^*] \times \mathbb{R}^n :\ h(t,x) \le A\}$ it follows that

$$V(t, x(t;t_0,\phi)) \le u^*(t;t_0,u_0) \quad \text{for} \ \ t \in [t_0, t^*]. \tag{2.30}$$

From condition 2(i) we obtain

$$V(t_0+s,\phi(t_0+s)) \le a(h_0(t_0+s,\phi(t_0+s))) \le a(H_0(t_0,\phi)) < a(\lambda), \quad s\in[-r,0],$$

or

$$u_0 < a(\lambda). \tag{2.31}$$

From inequalities (2.26), (2.30), and (2.31) it follows that

$$V(t,x(t;t_0,\phi)) \le u^*(t;t_0,u_0) < b(A) \quad \text{ for } \ t\in[t_0,t^*]. \tag{2.32}$$

From inequality (2.32) and condition 2(i) we get

$$b(A) = b(h(t^*,x(t^*;t_0,\phi))) \le V(t^*,x(t^*;t_0,\phi)) \le u^*(t^*;t_0,u_0) < b(A). \tag{2.33}$$

The obtained contradiction proves the validity of (2.28) for $t>t_0$.

Case 2. Let there exists a number $k\in Z((t_0,\infty))$ such that $h(t,x(t;t_0,\phi))<A$ for $t\in[t_0-r,\tau_k)$ and $h(\tau_k,x(\tau_k;t_0,\phi))=A$. Then as in Case 1 for $t^*=\tau_k$ we obtain a contradiction.

Case 3. Let there exists a natural number $k\in Z((t_0,\infty))$ such that

$$h(t,x(t;t_0,\phi))<A,\ t\in[t_0-r,\tau_k] \quad h\Big(\tau_k,I_k\big(x(\tau_k;t_0,\phi)\big)\Big)\ge A.$$

Since $A<\rho$, from condition 3 of Theorem 2 it follows that $h\Big(\tau_k,I_k\big(x(\tau_k;t_0,\phi)\big)\Big)>A$. The rest of the proof is similar to the proof of Case 3 of Theorem 2.

The proof of claim (B) is similar to the one of (A). □

We will study eventual practical stability in the case where the Lyapunov function does not satisfy the condition 2(i) of Theorem 3. Then we will use a perturbing Lyapunov function.

In this case we will use two scalar impulsive differential equations. We will use the (2.7) and

$$\begin{aligned} &v' = g_2(t,v), \quad t\ge t_0, \quad t\ne\tau_k,\\ &v(\tau_k+0)=\eta_k(v(\tau_k)), \quad k=1,2,\dots,\\ &v(t_0)=v_0, \end{aligned} \tag{2.34}$$

where $v,v_0\in\mathbb{R}$, $g_2:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$, $\eta_k\in\mathbb{R}\to\mathbb{R}$. In our further investigations we will assume that for any initial point $(t_0,v_0)\in\mathbb{R}_+\times\mathbb{R}$ the solution of scalar impulsive equation (2.34) exists on $[t_0,\infty)$, $t_0\ge 0$.

Theorem 4. *Let the following conditions be fulfilled:*

1. *Conditions* 1,2, 3 *of Theorem* 2 *are satisfied.*
2. *There exists a function* $V_1(t,x):[-r,\infty)\times\mathbb{R}^n\to\mathbb{R}_+$ *with* $V_1\in\Lambda$ *that is strongly* h_0*-decrescent, and*

(i) for any number $t \in \mathbb{R}_+ : \ t \neq \tau_k, \ k \in Z(\mathbb{R}_+)$ and any function $\psi \in PC([t-r,t],\mathbb{R}^n)$ such that $V_1(t,\psi(t)) \geq V_1(t+s,\psi(t+s))$ for $s \in [-r,0)$ and $(t,\psi(t)) \in S(h,\rho)$ the inequality

$$D_{(2.1),(2.2)}V_1(t,\psi(t)) \leq g(t,V_1(t,\psi(t)))$$

holds, where $g \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ and $g_1(t,0) \equiv 0$.

(ii) $V_1(\tau_k+0, I_k(x)) \leq \xi_k(V_1(\tau_k,x)), \quad (\tau_k,x) \in S(h,\rho), \ k \in Z(\mathbb{R}_+)$ where the functions $\xi_k \in \mathcal{K}$.

3. *For any number $\mu > 0$ there exists a function $V_2^{(\mu)}(t,x) : [-r,\infty) \times \mathbb{R}^n \to \mathbb{R}_+$ with $V_2^{(\mu)} \in \Lambda$ such that:*

(iii) $b(h(t,x)) \leq V_2^{(\mu)}(t,x) \leq a(h_0(t,x))$ for $(t,x) \in [-r,\infty)\times\mathbb{R}^n$, where $a,b \in K$;

(iv) for any number $t \in \mathbb{R}_+ : \ t \neq \tau_k, \ k \in Z(\mathbb{R}_+)$ and any function $\psi \in PC([t-r,t],\mathbb{R}^n)$ such that $(t,\psi(t))) \in S(h,\rho)\bigcap S^C(h_0,\mu)$ and $V(t,\psi(t)) > V(t+s,\psi(t+s))$ for $s \in [-r,0)$ the inequality

$$\begin{aligned} D_{(2.1),(2.2)}V_1(t,\psi(t)) &+ D_{(2.1),(2.2)}V_2^{(\mu)}(t,\psi(t)) \\ &\leq g_2\Big(t, V_1(t,\psi(t)) + V_2^{(\eta)}(t,\psi(t)))\Big) \end{aligned}$$

holds, where $g_2 \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ and $g_2(t,0) \equiv 0$;

(v) $V_1(\tau_k+0, I_k(x)) + V_2^{(\mu)}(\tau_k+0, I_k(x)) \leq \eta_k\Big(V_1(\tau_k,x)+V_2^{(\mu)}(\tau_k,x)\Big)$ for $(\tau_k,x) \in S(h,\rho)\bigcap S^C(h_0,\mu), \ k \in Z(\mathbb{R}_+)$ where the functions $\eta_k \in \mathcal{K}$.

4. *Zero solutions of scalar impulsive differential equations* (2.7) *and* (2.34) *are uniformly eventually practical stable.*

Then the system of impulsive differential equations with "supremum" (2.1), (2.2) *is uniformly eventually practical stable in terms of two measures.*

Proof. Since the function $V_1(t,x)$ is strongly-h_0-decrescent, there exists a constant $\rho_1 > 0$ and a funtion $\psi_1 \in K$ such that $h_0(t,x) < \rho_1$ implies

$$V_1(t,x) \leq \psi_1(h_0(t,x)), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{2.35}$$

Since h_0 is uniformly finer than h, there exists a constant ρ_0 and a function $\psi_2 \in K$ such that inequality $h_0(t,x) < \rho_0$ implies $h(t,x) \leq \psi_2(h_0(t,x))$ where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

We will assume that $\rho_0 < \rho_1 < \rho$ and $\psi_2(\rho_0) < \rho_1$. Choose a function $\psi_3(s) > \max\big(\psi_1(s), \ a(s)\big) : \psi_3 \in K$. Let the couple $(\lambda, A) : 0 < \lambda < A < \rho_0, \ \psi_2(\lambda) \leq A$, $2\psi_3(\lambda) < b(A)$ be fixed.

Since zero solution of scalar impulsive differential equation (2.7) is uniformly eventually practically stable, for the couple $(\psi_1(\lambda), \psi_3(\lambda))$ there exists $\tau_1(\lambda) > 0$ such that for $t_0 \geq \tau_1(\lambda)$ the inequality $|u_0| < \psi_1(\lambda)$ implies

$$u(t;t_0,u_0) < \psi_3(\lambda) \quad \text{ for } t \geq t_0, \tag{2.36}$$

where $u(t;t_0,v_0)$ is the maximal solution of (2.7).

Since zero solution of scalar impulsive differential equation (2.34) is uniformly eventually practically stable, for the couple $(2\psi_3(\lambda), b(A))$ there exists $\tau_2(\lambda, A) > 0$ such that for all $t_0 \geq \tau_2(\lambda, A)$ inequality $|v_0| < 2\psi_3(\lambda)$ implies

$$|v(t; t_0, v_0)| < b(A), \quad t \geq t_0, \tag{2.37}$$

where $v(t; t_0, v_0)$ is the maximal solution of (2.34) with initial condition $v(t_0) = v_0$.

Now let $\tau(\lambda, A) = \max(\tau_1(\lambda), \tau_2(\lambda, A))$. For any $t_0 > \tau(\lambda, A)$ we consider the function $\varphi \in PC([t_0 - r, t_0], \mathbb{R}^n$ such that

$$H_0(t_0, \varphi) < \lambda. \tag{2.38}$$

We will prove that if inequality (2.38) is satisfied then

$$h(t, x(t; t_0, \phi)) < A, \qquad t \geq t_0 - r, \tag{2.39}$$

where $x(t; t_0, \phi)$ is a solution of initial value problem (2.1)-(2.3).

From inequality (2.38) it follows that $h(t_0+s, \phi(t_0+s)) \leq \psi_2(h_0(t_0+s, \phi(t_0+s)) < \psi_2(\lambda) \leq A$ for $s \in [-r, 0]$, i.e., inequality (2.39) holds on $[t_0 - r, t_0]$.

Assume inequality (2.39) is not true for $t > t_0$.

Case 1. Let there exist a point $t^* \neq \tau_k, \;\; k \in Z((t_0, \infty))$ such that

$$h(t^*, x(t^*; t_0, \phi)) = A, \qquad h(t, x(t; t_0, \phi)) < A, \;\; t \in [t_0 - r, t^*). \tag{2.40}$$

Define $x(s) = x(s; t_0, \phi), \; s \in [t_0 - r, t^*]$.

If we assume that $h_0(t^*, x(t^*)) \leq \lambda$ then $h(t^*, x(t^*)) \leq \psi_2(h_0(t^*, x(t^*))) \leq \psi_2(\lambda) < A$ which contradicts (2.40). Therefore

$$h_0(t^*, x(t^*)) > \lambda, \quad h_0(t_0, \phi(t_0))) < \lambda. \tag{2.41}$$

There exists a point $t_0^* \in (t_0, t^*)$ such that $h_0(t_0^*, x(t_0^*)) \leq \lambda$ and $h_0(t, x(t)) \geq \lambda$ for $t \in (t_0^*, t^*]$, i.e.

$$(t, x(t)) \in S(h, A) \bigcap S^c(h_0, \lambda), \quad t \in [t_0^*, t^*). \tag{2.42}$$

Let $r_1(t; t_0, u_0)$ be the maximal solution of impulsive differential equation (2.7) where $u_0 = \sup_{s\in[-r,0]} V_1(t_0+s, \varphi(t_0+s))$. Then $u_0 < \psi_1(\lambda)$. From inequalities (2.36) follows that

$$|r_1(t; t_0, u_0)| < \psi_3(\lambda) \;\; \text{for } t \geq t_0. \tag{2.43}$$

From Lemma 1 for the function $V_1(t, x)$ defined on the set $\{(t, x) \in [t_0, t^*] \times \mathbb{R}^n : h(t, x) \leq A\}$ we obtain

$$V_1(s, x(s)) \leq r_1(s; t_0, u_0), \quad s \in [t_0, t^*]. \tag{2.44}$$

If $t_0^* - r < t_0$ then from (2.35) it follows that $V_1(s, x(s)) \leq \psi_1(h_0(s, \phi(s)) \leq \psi_1(\lambda) < \psi_3(\lambda)$ for $s \in [t_0^* - r, t_0]$ and from (2.43) and (2.44) we obtain that $V_1(s, x(s)) < \psi_3(\lambda)$ for $s \in [t_0, t_0^*]$.

If $t_0^* - r \geq t_0$ then from (2.43) and (2.44) we obtain that

$$V_1(t_0^* + s, x(t_0^* + s)) < \psi_3(\lambda) \quad \text{for} \quad s \in [-r, 0]. \tag{2.45}$$

Consider the function $V_2^{(\lambda)}(t,x)$ that is defined in condition 3 of Theorem 4 and define the function

$$m(t,x) = V_1(t,x) + V_2^{(\lambda)}(t,x), \qquad t \geq t_0 - r. \tag{2.46}$$

From inequality (2.38) and condition 3(iii) of Theorem 4 it follows that for $s \in [-r, 0]$

$$V_2^{(\lambda)}(t_0^* + s, x(t_0^* + s)) \leq a(h_0(t_0^* + s, x(t_0^* + s))) \leq a(\lambda) < \psi_3(\lambda). \tag{2.47}$$

From inequalities (2.45) and (2.47) we obtain

$$m(t_0^* + s, x(t_0^* + s)) < 2\psi_3(\lambda) \ \text{ for } s \in [-r, 0]. \tag{2.48}$$

From Lemma 1 for the function $m(t,x)$ defined on the set $\{(t,x) \in [t_0^*, t^*] \times \mathbb{R}^n : h(t,x) \leq A,\ h_0(t,x) \geq \lambda\}$ we get

$$m(t, x(t; t_0, \phi)) \leq r^*(t; t_0^*, v_0^*), \quad t \in [t_0^*, t^*], \tag{2.49}$$

where $r^*(t; t_0^*, v_0^*)$ is the maximal solution of (2.34) for $v_0^* = \sup_{s \in [-r,0]} m(t_0^* + s, x(t_0^* + s; t_0, \phi)$.

From inequality (2.48) follows that $|w_0^*| < 2\psi_3(\lambda)$ and therefore according to inequality (2.37)

$$r^*(t; t_0^*, w_0^*) < b(A), \quad t \geq t_0^*. \tag{2.50}$$

From inequalities (2.49), (2.50), the choice of the point t^*, and condition 3(iii) of Theorem 4 we obtain

$$\begin{aligned} b(A) &> r^*(t^*; t_0^*, w_0^*) \geq m(t^*, x(t^*; t_0, \phi)) \\ &\geq V_2^{(\lambda)}(t^*, x(t^*; t_0, \phi)) \geq b(h(t^*, x(t^*; t_0, \phi))) = b(A). \end{aligned}$$

The obtained contradiction proves the validity of inequality (2.39) for $t \geq t_0$.

Case 2. Let there exists a number $k \in Z((t_0, \infty))$ such that $h(t, x(t; t_0, \phi)) < A$ for $t \in [t_0 - r, \tau_k)$ and $h(\tau_k, x(\tau_k; t_0, \phi)) = A$. Then as in Case 1 for $t^* = \tau_k$ we obtain a contradiction.

Case 3. Let there exists a natural number $k \in Z((t_0, \infty))$ such that

$$h(t, x(t; t_0, \phi)) < A,\ t \in [t_0 - r, \tau_k] \quad h\Big(\tau_k, I_k\big(x(\tau_k; t_0, \phi)\big)\Big) \geq A.$$

Since $A < \rho$ from condition 3 of Theorem 2 follows that $h\Big(\tau_k, I_k\big(x(\tau_k; t_0, \phi)\big)\Big) > A$. The rest of the proof is similar to the proof of Case 3 of Theorem 2.

The obtained contradictions prove the validity of the inequality (2.39). □

3. APPLICATIONS

Now we will illustrate our results.

Consider the following system of nonlinear impulsive differential equations with “*supremum*”

$$\begin{aligned} &x'(t) = y(t)\Big(x^2(t) + y^2(t)\Big)\sin^2 t + e^{-t} \sup_{s\in[t-r,t]} x(s),\\ &y'(t) = -\frac{1}{2}x(t)\Big(x^2(t) + y^2(t)\Big)\sin^2 t + e^{-t} \sup_{s\in[t-r,t]} y(s), \qquad t \geq t_0, t \neq k,\\ &x(k+0) = ax(k), \qquad y(k+0) = by(k), \end{aligned} \tag{3.1}$$

with initial conditions

$$x(t) = \phi_1(t - t_0), \quad y(t) = \phi_2(t - t_0) \quad \text{for } t \in [t_0 - r, t_0], \tag{3.2}$$

where $x, y \in \mathbb{R}$, $r > 0$ is a small constant, $t_0 \geq 0$, and $a, b \in (1, 2)$.

Let $h_0(t, x, y) = |x| + \sqrt{2}|y|$, $h(t, x, y) = x^2 + 2y^2$. Consider $V : \mathbb{R}^2 \to \mathbb{R}_+$, $V(x, y) = \frac{3}{2}(x^2 + 2y^2)$. It is easy to check Condition (i) of Theorem 1 for functions $a(s) = \frac{3}{2}s^2 \in K$ and $b(s) = s \in K$.

Let $t \in \mathbb{R}_+$, $t \neq \tau_k$, $k = 1, 2, \dots$ and $\psi \in PC([t - r, t], \mathbb{R}^2)$, $\psi = (\psi_1, \psi_2)$ be such that

$$\psi_1^2(t) + 2\psi_2^2(t) > \psi_1^2(t + s) + 2\psi_2^2(t + s) \quad \text{for} \quad s \in [-r, 0), \tag{3.3}$$

or $V(\psi_1(t), \psi_2(t)) > V(\psi_1(t + s), \psi_2(t + s))$.

Let $i = 1, 2$. If there exists a point $\eta \in [t-r, t]$ such that $\sup_{s\in[t-r,t]} \psi_i(s) = \psi_i(\eta)$, then $\big(\sup_{s\in[t-r,t]} \psi_i(s)\big)^2 = (\psi_i(\eta))^2 \leq \sup_{s\in[t-r,t]} \big(\psi_1^2(s) + 2\psi_2^2(s)\big) = \psi_1^2(t) + 2\psi_2^2(t)$.

The above inequality could analogously be proved if $\sup_{s\in[t-r,t]} \psi_i(s) > \psi_i(\eta)$ for all $\eta \in [-r, t]$, i.e., there exists a point $\tau_k \in (t - r, t)$ such that $\sup_{s\in[t-r,t]} \psi_i(s) = \psi_i(\tau_k + 0)$.

Then for $i = 1, 2$ we obtain

$$\begin{aligned} &\psi_i(t) \sup_{s\in[t-r,t]} \psi_i(s) \leq |\psi_i(t)|\ |\sup_{s\in[t-r,t]} \psi_i(s)| = \sqrt{(\psi_i(t))^2}\ \sqrt{(\sup_{s\in[t-r,t]} \psi_i(s))^2}\\ &\leq \frac{3}{2}\Big(\psi_1^2(t) + 2\psi_2^2(t)\Big) = V(\psi_1(t), \psi_2(t)). \end{aligned}$$

Therefore, if inequality (3.3) is fulfilled, we have

$$\begin{aligned} &D_{(3.1)}V(\psi_1(t), \psi_2(t))\\ = \;& 3e^{-t}\Big(\psi_1(t) \max_{s\in[t-r,t]} \psi_1(s) + 2\psi_2(t) \max_{s\in[t-r,t]} \psi_2(s)\Big)\\ &\leq 6e^{-t}V(\psi_1(t), \psi_2(t)). \end{aligned}$$

For any k we obtain

$$V(ax, by) = \frac{3}{2}(a^2x^2 + 2b^2y^2) \leq c^2\frac{3}{2}(x^2 + 2y^2) = c^2V(x, y),$$

where $c = \max(a, b) > 1$.

Now, consider the initial value problem for the scalar comparison impulsive differential equation

$$u' = 6e^{-t}u \ \text{ for } t \neq k, \quad u(k+0) = c^2 u(k), \quad u(t_0) = u_0$$

whose solution is $u(t) = \Big(\prod_{i:\ t_0 \leq t < t_{i+1}} (c^2 - 1)\Big) u_0 e^{6\left(e^{-t_0} - e^{-t}\right)}$ and $|u(t)| \leq |u_0| e^{6e^{-t_0}}$ for $t \geq t_0$. For any numbers $0 < \lambda < A$, we choose a number $\tau > \max\{0, ln6 - ln(ln(\frac{A}{\lambda}))\} > 0$. Note $\tau = \tau(\lambda, A) > 0$. It is easy to check that for $t_0 > \tau$ and $|u_0| < \lambda$ the inequality $|u(t)| < A$ holds, i.e., the zero solution of the scalar comparison equation is uniformly eventually practically stable. Therefore, according to Theorem 2 the system of impulsive differential equations with "*supremum*" (3.1) is uniformly eventually practically stable in terms of two measures, i.e., for any numbers $0 < \lambda < A$, there exists a number $\tau = \tau(\lambda, A) > 0$ such that, if $t_0 > \tau$ then the inequality $\sup_{s \in [-r,0]}(|\phi_1(s)| + 2|\phi_2(s)|) < \lambda$ implies $x^2(t; t_0, \phi) + 2y^2(t; t_0, \phi) < A$, for $t \geq t_0$.

ACKNOWLEDGMENTS

Research was partially supported by RS09FMI018.

REFERENCES

[1] D. D. Bainov and P. S. Simeonov, *Systems with Impulse Effect: Stability, Theory and Applications*, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, UK, 1989.

[2] G. Ballinger and X. Liu, Practical stability of impulsive delay differential equations and applications to control problems, in *Optimization Methods and Applications*, vol. 52 of Applied Optimization, pp. 3-21, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[3] G. Chen, J. Shen, and W. Wang, Practical stability of impulsive functional differential equations, *Nonlinear Studies*, **16**, no. 1 (2009), 57-72.

[4] W. Feng, On practical stability of linear multivariable feedback systems with time-delays, *Automatica* **27**, no. 2, (1991), 389-394.

[5] S. Hristova, Lipschitz stability for impulsive differential equations with "*supremum*" *Intern. Electr. J. Pure Appl. Math.* **1**, no.4, (2010), 345-358.

[6] S. Hristova, Practical stability and cone valued Lyapunov functions for differential equations with "maxima″, *Intern. J. Pure Appl. Math.* **57**, no.3, (2009), 313-324.

[7] V. I. Slyn′ko, Sufficient practical-stability conditions for nonlinear impulsive systems, *Intern. Appl. Mech.*, **40**, no.10, (2004), 1171-1174.

[8] S. Li, X. Song, and An Li, Strict practical stability of nonlinear impulsive systems by employing two Lyapunov-like functions, *Nonlin. Anal., Real World Applications* **9**, no. 5, (2008), 2262-2269.

[9] V. Lakshmikantham, S. Leela and A. A. Martynyuk, *Practical Stability of Nonlinear Systems*, World Scientific, Singapore, 1990.

[10] V. Lakshmikantham and X. Liu, *Stability Analysis in Terms of Two Measures*, World Scientific, Singapore, 1993.

[11] J. La Salle and S. Lefschetz, *Stability by Lyapunov's Direct Method and Applications*, Academic Press, New York, 1961.

[12] Y. Liu, S. Zhao, A new approach to practical stability of impulsive functional differential equations in terms of two measures, *J.Comput. Appl. Math.* **223**, no. 1, (2009), 449-458.

[13] A. A. Martynyuk, Practical stability of motion, Naukova Dumka, Kiev, 1983. (in Russian)

[14] A. A. Martynyuk, On practical stability of hybrid systems, *Appl. Mech.* **25** (1989) 101-107, (in Russian).

[15] A. D. Myshkis, On some problems of the theory of differential equations with deviating argument, *Russian Math. Surveys* **32**, no. 2, (1977), 181-210 (in Russian).

[16] E. P. Popov, *Automatic Regulation and Control*, Moscow, 1966 (in Russian).

[17] X. Song, S. Li and An Li, Practical stability of nonlinear differential equation with initial time difference, *Appl. Math. Comput.* **203**, no.1, (2008), 157-162.

[18] H. D. Voulov and D. D. Bainov, Nonuniform stability for a nonautonomous differential equation with “maxima″, *Rend. Sem. Fac. Sci. Univ. Cagliari* **62**, no. 2, (1992) 101-113.

[19] H. D. Voulov and D. D. Bainov, On the asymptotic stability of differential equations with “maxima″, *Rend. Circ. Mat. Palermo*, (2) **40**, no. 3, (1991) 385-420.

[20] P. Wang and X. Liu, Practical stability of impulsive hybrid differential systems in terms of two measures on time scales, *Nonlinear Anal.* **65**, no.11, (2006), 2035-2042.

[21] X. Xu and G. Zhai, On practical stability and stabilization of hybrid and switched systems, in *Hybrid Systems: Computation and Control*, Springer Berlin/ Heidelberg, 2004, 63-84.

[22] T. Yang and L. O. Chua, Practical stability of impulsive synchronization between two nonautonomous chaotic systems, *Int. J. Bifurcation and Chaos* **10**, no.4, (2000), 859-867.

[23] Y. Zhang and J. Sun, Eventual practical stability of impulsive differential equations with time delay in terms of two measurements, *J. Comput. Appl. Math.* **176**, no.1, (2005), 223-229.

[24] Y. Zhang and J. Sun, Practical stability of impulsive functional differential equations in terms of two measurements, *Comput. Math. Appl.* **48**, no. 10-11, (2004) 1549-1556