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SHELL MIXED FRACTIONAL OSTROWSKI INEQUALITIES

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ABSTRACT. Very general shell mixed Caputo fractional Ostrowski inequalities are presented, radial and non-radial cases. One of them is proved to be sharp and attained. Estimates are with respect to $\|\cdot\|_p$, $1 \leq p \leq \infty$.

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1. Introduction

In 1938, A. Ostrowski [12] proved the following important inequality:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x)\right| \leq \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\cdot(b-a)\|f'\|_\infty, \tag{1}$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to Numerical Analysis and Probability.

This paper is greatly motivated and inspired also by the following result.

Theorem 1.2 (see [1])**.** *Let $f \in C^{n+1}([a, b])$, $n \in \mathbb{N}$ and $x \in [a, b]$ be fixed, such that $f^{(k)}(x) = 0$, $k = 1, \ldots, n$. Then it holds*

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy - f(x)\right| \leq \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+2)!}\cdot\left(\frac{(x-a)^{n+2}+(b-x)^{n+2}}{b-a}\right). \tag{2}$$

Inequality (2) is sharp. In particular, when n is odd is attained by $f^*(y) := (y-x)^{n+1}\cdot(b-a)$, while when n is even the optimal function is

$$\overline{f}(y) := |y-x|^{n+\alpha}\cdot(b-a), \quad \alpha > 1.$$

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Clearly inequality (2) *generalizes inequality* (1) *for higher order derivatives of* f.

Also in [4], see Chapters 24-26, we presented a complete theory of left fractional Ostrowski inequalities.

Here we combine both right and left Caputo fractional derivatives and produce Ostrowski inequalities in the multivariate setting of a shell for radial and non-radial functions. A non-radial case ball result is given at the end. For the non-radial case results we use fractional radial derivatives, left and right, the last being introduced here for the first time. For the basic concepts of fractional calculus used here, we refer to [5], [7]-[10], and [13]. We are also motivated by [2] and [3].

2. **Main Results**

We make

Remark 2.1. Let the spherical shell $A := B(0,R_2) - \overline{B(0,R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$, $x \in \overline{A}$. Consider that $f : \overline{A} \to \mathbb{R}$ is radial; that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall\, x \in \overline{A}$. Here x can be written uniquely as $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$, see ([11], p. 149-150 and [4], p. 421), furthermore for $F : \overline{A} \to \mathbb{R}$ a Lebesgue integrable function we have that

$$\int_A F(x)\,dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega)\, r^{N-1} dr \right) d\omega. \tag{3}$$

Let $d\omega$ be the element of surface measure on S^{N-1} and let

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}.$$

Here $Vol(A) = \frac{\omega_N \left(R_2^N - R_1^N\right)}{N}$, and we assume that $g \in AC^m([R_1, R_2])$ (i.e. $g^{(m-1)} \in AC([R_1, R_2])$), $m = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ ceilling of number), $\alpha > 0$, and $g^{(k)}(r_0) = 0$, $k = 1, \ldots, m-1$, $r_0 \in [R_1, R_2]$.

We get by [7], p. 40, that

$$g(r) - g(r_0) = \frac{1}{\Gamma(\alpha)} \int_{r_0}^{r} (r - J)^{\alpha - 1} D_{*r_0}^{\alpha} g(J)\, dJ, \tag{4}$$

$\forall\, r \in [r_0, R_2]$, where $D_{*r_0}^{\alpha} g$ is the left Caputo fractional derivative of order α, see [7], p. 38. And, by [5],

$$g(r) - g(r_0) = \frac{1}{\Gamma(\alpha)} \int_{r}^{r_0} (J - r)^{\alpha - 1} D_{r_0-}^{\alpha} g(J)\, dJ, \tag{5}$$

(where $D_{r_0-}^{\alpha} g$ is the right Caputo fractional derivative of order α, see [8]-[10]), $\forall$ $r \in [R_1, r_0]$. Here assume $D_{*r_0}^{\alpha} g \in L_\infty([r_0, R_2])$, and $D_{r_0-}^{\alpha} g \in L_\infty([R_1, r_0])$. We obtain

$$|g(r) - g(r_0)| \leq \frac{(r - r_0)^{\alpha}}{\Gamma(\alpha + 1)} \left\| D_{*r_0}^{\alpha} g \right\|_{\infty,[r_0,R_2]}, \tag{6}$$

$\forall\ r \in [r_0, R_2]$, and

$$|g(r) - g(r_0)| \leq \frac{(r_0 - r)^{\alpha}}{\Gamma(\alpha+1)} \left\| D_{r_0-}^{\alpha} g \right\|_{\infty,[R_1,r_0]}, \tag{7}$$

$\forall\ r \in [R_1, r_0]$. Next we see that

$$\left| f(r_0\omega) - \frac{\int_A f(y)\,dy}{Vol(A)} \right| = \left| g(r_0) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s)\, s^{N-1} ds \right| =$$

$$\left(\frac{N}{R_2^N - R_1^N} \right) \left| \int_{R_1}^{R_2} (g(r_0) - g(s))\, s^{N-1} ds \right| \leq$$

$$\left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} |g(r_0) - g(s)|\, s^{N-1} ds = \tag{8}$$

$$\left(\frac{N}{R_2^N - R_1^N} \right) \left\{ \int_{R_1}^{r_0} |g(r_0) - g(s)|\, s^{N-1} ds + \int_{r_0}^{R_2} |g(r_0) - g(s)|\, s^{N-1} ds \right\} \leq$$

$$\left(\frac{N}{R_2^N - R_1^N} \right) \frac{1}{\Gamma(\alpha+1)} \left\{ \left\| D_{r_0-}^{\alpha} g \right\|_{\infty,[R_1,r_0]} \int_{R_1}^{r_0} (r_0 - s)^{\alpha}\, s^{N-1} ds \right. \tag{9}$$

$$\left. + \left\| D_{*r_0}^{\alpha} g \right\|_{\infty,[r_0,R_2]} \int_{r_0}^{R_2} (s - r_0)^{\alpha}\, s^{N-1} ds \right\} =: (*).$$

Here we calculate

$$I_1 := \int_{R_1}^{r_0} (r_0 - s)^{\alpha}\, s^{N-1} ds = \int_{R_1}^{r_0} (r_0 - s)^{\alpha} \left((s - R_1) + R_1 \right)^{N-1} ds =$$

$$\sum_{k=0}^{N-1} \binom{N-1}{k} R_1^k \int_{R_1}^{r_0} (r_0 - s)^{(\alpha+1)-1} (s - R_1)^{N-k-1} ds =$$

$$\sum_{k=0}^{N-1} \frac{(N-1)!}{k!\,(N-k-1)!} R_1^k \frac{\Gamma(\alpha+1)\,(N-k-1)!}{\Gamma(\alpha+1+N-k)} (r_0 - R_1)^{\alpha+N-k}.$$

I.e.

$$I_1 = (N-1)!\,\Gamma(\alpha+1) \sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha+N-k}}{k!\,\Gamma(\alpha+1+N-k)}. \tag{10}$$

Also

$$I_2 := \int_{r_0}^{R_2} s^{N-1} (s - r_0)^{\alpha} ds = (-1)^{N-1} \int_{r_0}^{R_2} \left((R_2 - s) - R_2 \right)^{N-1} (s - r_0)^{\alpha} ds =$$

$$(-1)^{N-1} \sum_{k=0}^{N-1} \frac{(N-1)!}{k!\,(N-k-1)!} (-1)^k R_2^k \int_{r_0}^{R_2} (R_2 - s)^{(N-k)-1} (s - r_0)^{(\alpha+1)-1} ds$$

$$= \sum_{k=0}^{N-1} \frac{(N-1)!}{k!\,(N-k-1)!} (-1)^{N+k-1} R_2^k \frac{(N-k-1)!\,\Gamma(\alpha+1)}{\Gamma(N-k+\alpha+1)} (R_2 - r_0)^{N-k+\alpha}.$$

That is

$$I_2 = (N-1)!\,\Gamma(\alpha+1) \sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha}}{k!\,\Gamma(N-k+\alpha+1)}. \tag{11}$$

Consequently we get

$$(*) = \left(\frac{N!}{R_2^N - R_1^N}\right)\left\{\left\|D_{r_0-}^{\alpha} g\right\|_{\infty,[R_1,r_0]} \sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right. \tag{12}$$

$$\left. + \left\|D_{*r_0}^{\alpha} g\right\|_{\infty,[r_0,R_2]} \sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right\}.$$

So far we have proved that ($\forall\ \omega \in S^{N-1}$)

$$\left|f(r_0\omega) - \frac{\int_A f(y)\,dy}{Vol(A)}\right| = \left|g(r_0) - \left(\frac{N}{R_2^N - R_1^N}\right)\int_{R_1}^{R_2} g(s)\, s^{N-1} ds\right| \leq$$

$$\left(\frac{N!}{R_2^N - R_1^N}\right)\left\{\left\|D_{r_0-}^{\alpha} g\right\|_{\infty,[R_1,r_0]} \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right)\right.$$

$$\left. + \left\|D_{*r_0}^{\alpha} g\right\|_{\infty,[r_0,R_2]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right)\right\}. \tag{13}$$

Inequality (13) ($\alpha > 0$) is attained by

$$\overline{g}(s) := \begin{cases} (s - r_0)^{\alpha}, & s \in [r_0, R_2], \\ (r_0 - s)^{\alpha}, & s \in [R_1, r_0]. \end{cases} \tag{14}$$

Here $\overline{g}$ fulfills all assumptions here and

$$\left\|D_{r_0-}^{\alpha} g\right\|_{\infty,[R_1,r_0]} = \left\|D_{*r_0}^{\alpha} g\right\|_{\infty,[r_0,R_2]} = \Gamma(\alpha+1). \tag{15}$$

Hence

$$L.H.S.(13) = \frac{N}{R_2^N - R_1^N}\int_{R_1}^{R_2} \overline{g}(s)\, s^{N-1} ds =$$

$$\frac{N}{R_2^N - R_1^N}\left[\int_{R_1}^{r_0} (r_0 - s)^{\alpha} s^{N-1} ds + \int_{r_0}^{R_2} (s - r_0)^{\alpha} s^{N-1} ds\right] =$$

$$\frac{N}{R_2^N - R_1^N}\left[(N-1)!\Gamma(\alpha+1)\left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right) + \right. \tag{16}$$

$$\left. (N-1)!\Gamma(\alpha+1)\left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right)\right] =$$

$$\left(\frac{N!}{R_2^N - R_1^N}\right)\left[(\Gamma(\alpha+1))\left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right) + \right.$$

$$\left. (\Gamma(\alpha+1))\left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right)\right] = R.H.S.(13),$$

proving it attained and sharp (13).

We have established

Theorem 2.2. *Let $f:\overline{A}\to\mathbb{R}$ be radial; that is, there exists g such that $f(x)=g(r)$, $r=|x|$, $\forall\ x\in\overline{A}$; $\omega\in S^{N-1}$. Assume that $g\in AC^m([R_1,R_2])$, $m=\lceil\alpha\rceil$, $\alpha>0$, and $g^{(k)}(r_0)=0$ ($r_0\in[R_1,R_2]$ fixed), $k=1,\dots,m-1$, and $D^{\alpha}_{r_0-}g\in L_\infty([R_1,r_0])$, $D^{\alpha}_{*r_0}g\in L_\infty([r_0,R_2])$. Then* ($\forall\omega\in S^{N-1}$)

$$\left|f(r_0\omega)-\frac{\int_A f(y)\,dy}{Vol(A)}\right|=\left|g(r_0)-\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2}g(s)\,s^{N-1}ds\right|\leq$$

$$\left(\frac{N!}{R_2^N-R_1^N}\right)\left\{\left\|D^{\alpha}_{r_0-}g\right\|_{\infty,[R_1,r_0]}\left(\sum_{k=0}^{N-1}\frac{R_1^k(r_0-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right)\right.$$

$$\left.+\left\|D^{\alpha}_{*r_0}g\right\|_{\infty,[r_0,R_2]}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}R_2^k(R_2-r_0)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right)\right\}. \tag{17}$$

The last inequality (17) *is sharp, that is attained by*

$$\overline{g}(s):=\begin{cases}(s-r_0)^{\alpha}, & s\in[r_0,R_2],\\ (r_0-s)^{\alpha}, & s\in[R_1,r_0].\end{cases} \tag{18}$$

We need to make

Remark 2.3. Let $\alpha\geq1$. We get easily by (4) that

$$|g(r)-g(r_0)|\leq\frac{(r-r_0)^{\alpha-1}}{\Gamma(\alpha)}\left\|D^{\alpha}_{*r_0}g\right\|_{L_1([r_0,R_2])}, \tag{19}$$

$\forall\ r\in[r_0,R_2]$. Also by (5) we find that

$$|g(r)-g(r_0)|\leq\frac{(r_0-r)^{\alpha-1}}{\Gamma(\alpha)}\left\|D^{\alpha}_{r_0-}g\right\|_{L_1([R_1,r_0])}, \tag{20}$$

$\forall\ r\in[R_1,r_0]$.

Here we assume $\left\|D^{\alpha}_{*r_0}g\right\|_{L_1([r_0,R_2])}$, $\left\|D^{\alpha}_{r_0-}g\right\|_{L_1([R_1,r_0])}<\infty$.

Then it holds

$$\left|f(r_0\omega)-\frac{\int_A f(y)\,dy}{Vol(A)}\right|=\left|g(r_0)-\frac{N}{R_2^N-R_1^N}\int_{R_1}^{R_2}g(s)\,s^{N-1}ds\right|\leq$$

$$\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2}|g(r_0)-g(s)|\,s^{N-1}ds= \tag{21}$$

$$\left(\frac{N}{R_2^N-R_1^N}\right)\left\{\int_{R_1}^{r_0}|g(r_0)-g(s)|\,s^{N-1}ds+\int_{r_0}^{R_2}|g(r_0)-g(s)|\,s^{N-1}ds\right\}\leq$$

$$=\left(\frac{N}{R_2^N-R_1^N}\right)\frac{1}{\Gamma(\alpha)}\left\{\left\|D^{\alpha}_{r_0-}g\right\|_{L_1([R_1,r_0])}\left(\int_{R_1}^{r_0}(r_0-s)^{\alpha-1}s^{N-1}ds\right)\right.$$

$$\left.+\left\|D^{\alpha}_{*r_0}g\right\|_{L_1([r_0,R_2])}\int_{r_0}^{R_2}(s-r_0)^{\alpha-1}s^{N-1}ds\right\}=$$

$$\left(\frac{N!}{R_2^N-R_1^N}\right)\left\{\left\|D^{\alpha}_{r_0-}g\right\|_{L_1([R_1,r_0])}\left(\sum_{k=0}^{N-1}\frac{R_1^k(r_0-R_1)^{\alpha+N-k-1}}{k!\Gamma(\alpha+N-k)}\right)\right. \tag{22}$$

$$+\left\|D_{*r_0}^{\alpha}g\right\|_{L_1([r_0,R_2])}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}R_2^k(R_2-r_0)^{N-k+\alpha-1}}{k!\Gamma(N-k+\alpha)}\right)\right\}.$$

We have proved

Theorem 2.4. *Let $f:\overline{A}\to\mathbb{R}$ be radial; that is, there exists g such that $f(x)=g(r)$, $r=|x|$, $\forall\ x\in\overline{A}$; $\omega\in S^{N-1}$. Assume that $g\in AC^m([R_1,R_2])$, $m=\lceil\alpha\rceil$, $\alpha\geq 1$, and $g^{(k)}(r_0)=0$, $k=1,\dots,m-1$; $r_0\in[R_1,R_2]$ be fixed. Assume $D_{r_0-}^{\alpha}g\in L_1([R_1,r_0])$ and $D_{*r_0}^{\alpha}g\in L_1([r_0,R_2])$. Then $(\forall\omega\in S^{N-1})$ we get*

$$\left|f(r_0\omega)-\frac{\int_A f(y)\,dy}{Vol(A)}\right|=\left|g(r_0)-\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2}g(s)\,s^{N-1}ds\right|\leq \tag{23}$$

$$\left(\frac{N!}{R_2^N-R_1^N}\right)\left\{\left\|D_{r_0-}^{\alpha}g\right\|_{L_1([R_1,r_0])}\left(\sum_{k=0}^{N-1}\frac{R_1^k(r_0-R_1)^{\alpha+N-k-1}}{k!\Gamma(\alpha+N-k)}\right)\right.$$

$$\left.+\left\|D_{*r_0}^{\alpha}g\right\|_{L_1([r_0,R_2])}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}R_2^k(R_2-r_0)^{N-k+\alpha-1}}{k!\Gamma(N-k+\alpha)}\right)\right\}.$$

We continue with

Remark 2.5. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha>\frac{1}{q}$. Then by (4) we derive

$$|g(r)-g(r_0)|\leq\frac{1}{\Gamma(\alpha)}\left(\int_{r_0}^{r}(r-J)^{p(\alpha-1)}dJ\right)^{\frac{1}{p}}\left\|D_{*r_0}^{\alpha}g\right\|_{L_q([r_0,R_2])}, \tag{24}$$

$\forall\ r\in[r_0R_2]$. That is

$$|g(r)-g(r_0)|\leq\frac{1}{\Gamma(\alpha)}\frac{(r-r_0)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left\|D_{*r_0}^{\alpha}g\right\|_{L_q([r_0,R_2])}, \tag{25}$$

$\forall\ r\in[r_0,R_2]$. Similarly by (5) we get

$$|g(r)-g(r_0)|\leq\frac{1}{\Gamma(\alpha)}\frac{(r_0-r)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left\|D_{r_0-}^{\alpha}g\right\|_{L_q([R_1,r_0])}, \tag{26}$$

$\forall\ r\in[R_1,r_0]$. Here we assume that $D_{*r_0}^{\alpha}g\in L_q([r_0,R_2])$, and $D_{r_0-}^{\alpha}g\in L_q([R_1,r_0])$.

Hence we obtain

$$\left|f(r_0\omega)-\frac{\int_A f(y)\,dy}{Vol(A)}\right|=\left|g(r_0)-\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2}g(s)\,s^{N-1}ds\right|= \tag{27}$$

$$\left(\frac{N}{R_2^N-R_1^N}\right)\left|\int_{R_1}^{R_2}(g(r_0)-g(s))\,s^{N-1}ds\right|\leq$$

$$\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2}|g(r_0)-g(s)|\,s^{N-1}ds=$$

$$\left(\frac{N}{R_2^N-R_1^N}\right)\left\{\int_{R_1}^{r_0}|g(r_0)-g(s)|\,s^{N-1}ds+\int_{r_0}^{R_2}|g(r_0)-g(s)|\,s^{N-1}ds\right\}\leq$$

$$\left(\frac{N}{R_2^N - R_1^N}\right) \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot$$

$$\left\{ \left\| D_{r_0-}^{\alpha} g \right\|_{L_q([R_1,r_0])} \left(\int_{R_1}^{r_0} (r_0 - s)^{\alpha - \frac{1}{q}} s^{N-1} ds \right) \right.$$

$$\left. + \left\| D_{*r_0}^{\alpha} g \right\|_{L_q([r_0,R_2])} \int_{r_0}^{R_2} (s - r_0)^{\alpha - \frac{1}{q}} s^{N-1} ds \right\} = \tag{28}$$

$$\frac{N}{(R_2^N - R_1^N)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot$$

$$\left\{ \left\| D_{r_0-}^{\alpha} g \right\|_{L_q([R_1,r_0])} (N-1)!\Gamma\left(\alpha - \frac{1}{q} + 1\right) \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha - \frac{1}{q} + N - k}}{k!\Gamma\left(\alpha - \frac{1}{q} + 1 + N - k\right)} \right) \right.$$

$$+ \left\| D_{*r_0}^{\alpha} g \right\|_{L_q([r_0,R_2])} (N-1)!\Gamma\left(\alpha - \frac{1}{q} + 1\right) \cdot$$

$$\left. \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha-\frac{1}{q}}}{k!\Gamma\left(N - k + \alpha - \frac{1}{q} + 1\right)} \right) \right\}.$$

We have proved

Theorem 2.6. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let $f : \overline{A} \to \mathbb{R}$ be radial; that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $\forall\, x \in \overline{A}$; $\omega \in S^{N-1}$. Assume that $g \in AC^m([R_1, R_2])$, $m = \lceil \alpha \rceil$, $\alpha > \frac{1}{q}$ and $g^{(k)}(r_0) = 0$, $k = 1, \ldots, m-1$; $r_0 \in [R_1, R_2]$ be fixed. Assume also $D_{r_0-}^{\alpha} g \in L_q([R_1, r_0])$ and $D_{*r_0}^{\alpha} g \in L_q([r_0, R_2])$. Then $(\forall \omega \in S^{N-1})$ we get*

$$\left| f(r_0\omega) - \frac{\int_A f(y)\, dy}{Vol(A)} \right| = \left| g(r_0) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} g(s) s^{N-1} ds \right| \leq \tag{29}$$

$$\frac{\Gamma\left(\alpha + \frac{1}{p}\right) N!}{\Gamma(\alpha)(R_2^N - R_1^N)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot$$

$$\left\{ \left\| D_{r_0-}^{\alpha} g \right\|_{L_q([R_1,r_0])} \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha - \frac{1}{q} + N - k}}{k!\Gamma\left(\alpha + \frac{1}{p} + N - k\right)} \right) \right.$$

$$\left. + \left\| D_{*r_0}^{\alpha} g \right\|_{L_q([r_0,R_2])} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha-\frac{1}{q}}}{k!\Gamma\left(N - k + \alpha + \frac{1}{p}\right)} \right) \right\}.$$

We mention

Corollary 2.7. *($p = q = 2$ case) Let $f : \overline{A} \to \mathbb{R}$ be radial; that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $\forall\, x \in \overline{A}$; $\omega \in S^{N-1}$. Assume that $g \in AC^m([R_1, R_2])$, $m = \lceil \alpha \rceil$, $\alpha > \frac{1}{2}$ and $g^{(k)}(r_0) = 0$, $k = 1, \ldots, m-1$; $r_0 \in [R_1, R_2]$. Assume $D_{r_0-}^{\alpha} g \in L_2([R_1, r_0])$, $D_{*r_0}^{\alpha} g \in L_2([r_0, R_2])$. Then $(\forall \omega \in S^{N-1})$*

$$\left| f(r_0\omega) - \frac{\int_A f(y)\, dy}{Vol(A)} \right| = \left| g(r_0) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} g(s) s^{N-1} ds \right| \leq$$

$$\frac{\Gamma\left(\alpha+\frac{1}{2}\right)N!}{\Gamma(\alpha)\left(R_2^N-R_1^N\right)(2\alpha-1)^{\frac{1}{2}}}\left\{\left\|D_{r_0-}^{\alpha}g\right\|_{L_2([R_1,r_0])}\left(\sum_{k=0}^{N-1}\frac{R_1^k\left(r_0-R_1\right)^{\alpha-\frac{1}{2}+N-k}}{k!\Gamma\left(\alpha+\frac{1}{2}+N-k\right)}\right)\right.$$

$$\left.+\left\|D_{*r_0}^{\alpha}g\right\|_{L_2([r_0,R_2])}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}R_2^k\left(R_2-r_0\right)^{N-k+\alpha-\frac{1}{2}}}{k!\Gamma\left(N-k+\alpha+\frac{1}{2}\right)}\right)\right\}. \tag{30}$$

Finally we deal with non-radial L_∞ Ostrowski inequalities on the shell.

We need (see also [4], p. 421)

Definition 2.8. Let $F:\overline{A}\to\mathbb{R}$, $\alpha>0$, $m=\lceil\alpha\rceil$ such that $F(\cdot\omega)\in AC^m([R_1,R_2])$, $\forall\,\omega\in S^{N-1}$. We call the Caputo left radial fractional derivative the following function

$$\frac{\partial_{*r_0}^{\alpha}F(x)}{\partial r^{\alpha}}:=\frac{1}{\Gamma(m-\alpha)}\int_{r_0}^{r}(r-t)^{m-\alpha-1}\frac{\partial^m F(t\omega)}{\partial r^m}dt, \tag{31}$$

where $x\in\overline{A}$; that is, $x=r\omega$; $r,r_0\in[R_1,R_2]$, r_0 is fixed, $\omega\in S^{N-1}$, $\forall\,r\geq r_0$.

Clearly

$$\frac{\partial_{*r_0}^{0}F(x)}{\partial r^{0}}=F(x), \tag{32}$$

$$\frac{\partial_{*r_0}^{\alpha}F(x)}{\partial r^{\alpha}}=\frac{\partial^{\alpha}F(x)}{\partial r^{\alpha}},\quad \text{if } \alpha\in\mathbb{N}. \tag{33}$$

The above function exists almost everywhere for $x\in\overline{A}$.

We also need

Definition 2.9. Let $F:\overline{A}\to\mathbb{R}$, $\alpha>0$, $m=\lceil\alpha\rceil$ such that $F(\cdot\omega)\in AC^m([R_1,R_2])$, $\forall\,\omega\in S^{N-1}$. We call the Caputo right radial fractional derivative the following function

$$\frac{\partial_{r_0-}^{\alpha}F(x)}{\partial r^{\alpha}}:=\frac{(-1)^m}{\Gamma(m-\alpha)}\int_{r}^{r_0}(J-r)^{m-\alpha-1}\frac{\partial^m F(J\omega)}{\partial J^m}dJ, \tag{34}$$

where $x\in\overline{A}$; that is, $x=r\omega$; $r,r_0\in[R_1,R_2]$, r_0 is fixed, $\omega\in S^{N-1}$, $\forall\,r\leq r_0$.

Clearly

$$\frac{\partial_{r_0-}^{0}F(x)}{\partial r^{0}}=F(x), \tag{35}$$

$$\frac{\partial_{r_0-}^{\alpha}F(x)}{\partial r^{\alpha}}=(-1)^{\alpha}\frac{\partial^{\alpha}F(x)}{\partial r^{\alpha}},\quad \text{if } \alpha\in\mathbb{N}. \tag{36}$$

The above function exists almost everywhere for $x\in\overline{A}$.

We need to make

Remark 2.10. We treat here the general, not necessarily radial, case of f. We apply Theorem 2.2 to $f(r\omega)$, ω fixed, $r\in[R_1,R_2]$, under the following assumptions: $f(\cdot\omega)\in AC^m([R_1,R_2])$, $\forall\,\omega\in S^{N-1}$, $\alpha>0$, $m=\lceil\alpha\rceil$, where $f:\overline{A}\to\mathbb{R}$ is Lebesgue integrable; $\frac{\partial^k f}{\partial r^k}$, $k=1,\ldots,m-1$, vanish on $\partial B(0,r_0)$, r_0 is fixed in $[R_1,R_2]$; and $\frac{\partial_{*r_0}^{\alpha}f}{\partial r^{\alpha}}\in B\left(\overline{A_1}\right)$ (bounded functions), where $A_1:=B(0,R_2)-\overline{B(0,r_0)}$, and $\frac{\partial_{r_0-}^{\alpha}f}{\partial r^{\alpha}}\in$

$B\left(\overline{A_2}\right)$, where $A_2 := B(0,r_0) - \overline{B(0,R_1)}$, along with $D^{\alpha}_{r_0-}f(\cdot\omega) \in L_\infty([R_1,r_0])$, $D^{\alpha}_{*r_0}f(\cdot\omega) \in L_\infty([r_0,R_2])$, $\forall\, \omega \in S^{N-1}$.

Then ($\forall\, \omega \in S^{N-1}$)

$$\left| f(r_0\omega) - \left(\frac{N}{R_2^N - R_1^N}\right)\int_{R_1}^{R_2} f(s\omega)\, s^{N-1} ds \right| \leq$$

$$\left(\frac{N!}{R_2^N - R_1^N}\right)\left\{ \left\| \frac{\partial^{\alpha}_{r_0-}f}{\partial r^{\alpha}} \right\|_{\infty,\overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)} \right) \right.$$

$$\left. + \left\| \frac{\partial^{\alpha}_{*r_0}f}{\partial r^{\alpha}} \right\|_{\infty,\overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)} \right) \right\} =: \lambda_1. \tag{37}$$

Therefore

$$\left| \frac{\int_{S^{N-1}} f(r_0\omega)\, d\omega}{\omega_N} - \frac{N}{(R_2^N - R_1^N)\,\omega_N}\int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(s\omega)\, s^{N-1} ds \right) d\omega \right| \leq \lambda_1. \tag{38}$$

That is

$$\left| \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(r_0\omega)\, d\omega - \frac{\int_A f(x)\, dx}{Vol(A)} \right| \leq \lambda_1. \tag{39}$$

Therefore it holds for $x \in \overline{A}$ that

$$\left| f(x) - \frac{\int_A f(x)\, dx}{Vol(A)} \right| \leq$$

$$\left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(r_0\omega)\, d\omega \right| + \lambda_1. \tag{40}$$

We have proved the following

Theorem 2.11. *Let $f : \overline{A} \to \mathbb{R}$ be Lebesgue integrable with $f(\cdot\omega) \in AC^m([R_1,R_2])$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $\forall\, \omega \in S^{N-1}$; $\frac{\partial^k f}{\partial r^k}$, $k = 1,\dots,m-1$, vanish on $\partial B(0,r_0)$, r_0 fixed in $[R_1,R_2]$; and $\frac{\partial^{\alpha}_{*r_0}f}{\partial r^{\alpha}} \in B\left(\overline{A_1}\right)$, $\frac{\partial^{\alpha}_{r_0-}f}{\partial r^{\alpha}} \in B\left(\overline{A_2}\right)$ along with $D^{\alpha}_{r_0-}f(\cdot\omega) \in L_\infty([R_1,r_0])$, $D^{\alpha}_{*r_0}f(\cdot\omega) \in L_\infty([r_0,R_2])$, $\forall\, \omega \in S^{N-1}$. Then for $x \in \overline{A}$ we have*

$$\left| f(x) - \frac{\int_A f(x)\, dx}{Vol(A)} \right| \leq$$

$$\left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(r_0\omega)\, d\omega \right| \leq$$

$$\left(\frac{N!}{R_2^N - R_1^N}\right)\left\{ \left\| \frac{\partial^{\alpha}_{r_0-}f}{\partial r^{\alpha}} \right\|_{\infty,\overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)} \right) \right.$$

$$\left. + \left\| \frac{\partial^{\alpha}_{*r_0}f}{\partial r^{\alpha}} \right\|_{\infty,\overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)} \right) \right\}. \tag{41}$$

We also make

Remark 2.12. Let $f : \overline{B(0,R)} \to \mathbb{R}$ be a Lebesgue integrable function, that is not necessarily a radial function. Assume $f(\cdot\omega) \in AC([0,R])$, $\forall\ \omega \in S^{N-1}$, $0 < \alpha \leq 1$; $r_0 \in [0,R]$ fixed, $D^{\alpha}_{r_0-} f(\cdot\omega) \in L_\infty([0,r_0])$, $D^{\alpha}_{*r_0} f(\cdot\omega) \in L_\infty([r_0,R])$, $\forall\ \omega \in S^{N-1}$. We further assume that

$$\left\|D^{\alpha}_{r_0-} f(t\omega)\right\|_{\infty,(t\in[0,r_0])}, \left\|D^{\alpha}_{*r_0} f(t\omega)\right\|_{\infty,(t\in[r_0,R])} \leq K, \quad \forall\ \omega \in S^{N-1}, \tag{42}$$

where $K > 0$.

By inequality (16) of [6], applied on $f(\cdot\omega)$, $\forall\ \omega \in S^{N-1}$, we get

$$\left|f(r_0\omega) - \frac{N}{R^N}\int_0^R f(s\omega)\, s^{N-1} ds\right| \leq$$

$$\frac{N!K}{R^N}\left[\frac{r_0^{\alpha+N}}{\Gamma(\alpha+N+1)} + \left(\sum_{k=0}^{N-1}\frac{(-1)^{(N+k-1)} R^k (R-r_0)^{(N+\alpha-k)}}{k!\Gamma(N+\alpha+1-k)}\right)\right] =: \lambda_2. \tag{43}$$

Therefore it holds

$$\left|\frac{\int_{S^{N-1}} f(r_0\omega)\, d\omega}{\omega_N} - \frac{N}{R^N\omega_N}\int_{S^{N-1}}\left(\int_0^R f(s\omega)\, s^{N-1} ds\right) d\omega\right| \leq \lambda_2. \tag{44}$$

That is we got

$$\left|\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\int_{S^{N-1}} f(r_0\omega)\, d\omega - \frac{\int_{B(0,R)} f(x)\, dx}{Vol(B(0,R))}\right| \leq \lambda_2. \tag{45}$$

Consequently we derive

$$\left|f(x) - \frac{\int_{B(0,R)} f(x)\, dx}{Vol(B(0,R))}\right| \leq$$

$$\left|f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\int_{S^{N-1}} f(r_0\omega)\, d\omega\right| + \lambda_2. \tag{46}$$

We have proved

Theorem 2.13. *Let $f : \overline{B(0,R)} \to \mathbb{R}$ be a Lebesgue integrable function, that is not necessarily a radial function. Assume $f(\cdot\omega) \in AC([0,R])$, $R > 0$, $\forall\ \omega \in S^{N-1}$, $0 < \alpha \leq 1$, $r_0 \in [0,R]$ fixed, $D^{\alpha}_{r_0-} f(\cdot\omega) \in L_\infty([0,r_0])$, $D^{\alpha}_{*r_0} f(\cdot\omega) \in L_\infty([r_0,R])$, $\forall$ $\omega \in S^{N-1}$.*

Assume also $\left\|D^{\alpha}_{r_0-} f(t\omega)\right\|_{\infty,(t\in[0,r_0])}$, $\left\|D^{\alpha}_{*r_0} f(t\omega)\right\|_{\infty,(t\in[r_0,R])} \leq K$, $\forall\ \omega \in S^{N-1}$, where $K > 0$. Then

$$\left|f(x) - \frac{\int_{B(0,R)} f(x)\, dx}{Vol(B(0,R))}\right| \leq$$

$$\left|f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\int_{S^{N-1}} f(r_0\omega)\, d\omega\right| + \tag{47}$$

$$\frac{N!K}{R^N}\left[\frac{r_0^{\alpha+N}}{\Gamma(\alpha+N+1)} + \left(\sum_{k=0}^{N-1}\frac{(-1)^{(N+k-1)} R^k (R-r_0)^{(N+\alpha-k)}}{k!\Gamma(N+\alpha+1-k)}\right)\right].$$

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