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EXTREMAL SOLUTIONS AND CONTINUOUS DEPENDENCE FOR SET DIFFERENTIAL EQUATIONS INVOLVING CAUSAL OPERATORS WITH MEMORY

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ABSTRACT. In this paper we proceed to present results pertaining to extremal solutions and continuous dependence of solutions with respect to initial values and a parameter for set differential equations involving causal operators with memory.

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1. Introduction

A causal operator [1,3] is a nonanticipative operator and differential equations involving causal operators unify a variety of dynamic systems including ordinary differential equations [2], delay differential equations [2] and integro differential equations [5], name a few.

Set differential equations are useful in the study of multi-valued differential equations and multivalued differential inclusions. They include the theory of ordinary differential equations and ordinary differential systems as special cases.

Thus combining these two very general and fruitful areas of research will naturally result in a study that would encompass the study of many types of dynamic systems along with their special cases and that too in a semi-linear metric space.

Hence the study of set differential equations involving causal operators with memory was introduced in [6,7], where in the comparison theorems and local and global existence results including uniqueness were considered.

In this paper, we study the existence of extremal solutions and continuous dependence of solutions relative to initial data and a parameter for set differential equations involving causal operators with memory.

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2. **Preliminaries**

In this section, we give all the results that are needed to prove our main results. We begin with the theorems concerning the scalar differential equation

$$u' = g(t,u), u(t_0) = u_0 \tag{2.1}$$

and then proceed to introduce the literature relating to set differential equations.

The following theorems concerning equation (2.1) (i.e) scalar differential equations are from [2]. We begin with an extremal result.

Theorem 2.1. *Let $g \in C[R_0, R]$ where R_0 is an open set in $\mathbb{R}^2$ and $(t_0, u_0) \in R_0$. Then the IVP* (2.1) *has extremal solutions, that is, that can be extended to the boundary of R_0.*

Lemma 2.2. *Let the hypothesis of Theorem* 2.1 *hold and let $[t_0, T]$ be the largest interval of existence of maximal solution r(t) of IVP* (2.1)*. Suppose $[t_0, t_1]$ is a compact subinterval of $[t_0, T]$.Then there is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the maximal solution $r(t, \epsilon)$ of the IVP*

$$u' = g(t,u) + \epsilon,\ \ u(t_0) = u_0 + \epsilon \tag{2.2}$$

exists over $[t_0, t_1]$ and

$$r(t) = \lim_{\epsilon \to 0} r(t, \epsilon)$$

uniformly on $[t_0, t_1]$.

Theorem 2.3. *Let R_0 be an open (t,u) set in $\mathbb{R}^2$ and let $g \in C[R_0, \mathbb{R}]$. Suppose that $[t_0, t_0 + a]$ is the largest interval in which the maximal solution r(t) of* (2.1) *exists. Let $m \in C[[t_0, t_0 + a], \mathbb{R}], (t, m(t)) \in R_0$, for $t \in [t_0, t_0 + a], m(t_0) \le u_0$ and for a fixed Dini derivative*

$$Dm(t) \le g[t, m(t)],\ \ t \in [t_0, t_0 + a]$$

Then $m(t) \le r(t)$, $t \in [t_0, t_0 + a)$, where $r(t) = r(t, t_0, u_0)$ is the maximal solution of the IVP (2.1) *existing on $[t_0, t_0 + a)$.*

Next we proceed to develop the basic notations, definitions and results related to set differential equations. The set $\mathbb{K}_c(\mathbb{R}^n)$ is introduced and concepts like Hukuhara difference, Hukuhara derivative and integral are described. After giving the necessary preliminaries, we proceed to state the results that have been developed in the earlier papers. The Picard's Theorem is stated in this setup.

Let $K_c(\mathbb{R}^n)$ denote the collection of all nonempty, compact and convex subsets of $\mathbb{R}^n$ Define the Hausdorff metric

$$D[A,B] = \max[\sup_{x \in B} d(x,A),\ \ \sup_{y \in A} d(y,B)], \tag{2.3}$$

where $d(x,A) = \inf[d(x,y) : y \in A]$, A, B are bounded sets in $\mathbb{R}^n$. We note that $K_c(\mathbb{R}^n)$ with this metric is a complete metric space.

It is known that if the space $K_c(\mathbb{R}^n)$ is equipped with the natural algebraic operations of addition and non-negative scalar multiplication, then $K_c(\mathbb{R}^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space.

The Hausdorff metric (2.3) satisfies the following properties:

$$D[A+C,B+C]=D[A,B] \text{ and } D[A,B]=D[B,A], \tag{2.4}$$

$$D[\lambda A,\lambda B]=\lambda D[A,B], \tag{2.5}$$

$$D[A,B]\le D[A,C]+D[C,B], \tag{2.6}$$

for all $A,B,C\in K_c(\mathbb{R}^n)$ and $\lambda\in\mathbb{R}_+$.

Let $A,B\in K_c(\mathbb{R}^n)$. The set $C\in K_c(\mathbb{R}^n)$ satisfying $A=B+C$ is known as the Hukuhara difference of the sets A and B and is denoted by the symbol $A-B$. We say that the mapping $F:I\to K_c(\mathbb{R}^n)$ has a Hukuhara derivative $D_HF(t_0)$ at a point $t_0\in I$, if

$$\lim_{h\to 0+}\frac{F(t_0+h)-F(t_0)}{h} \quad \text{and} \quad \lim_{h\to 0+}\frac{F(t_0)-F(t_0-h)}{h}$$

exist in the topology of $K_c(\mathbb{R}^n)$ and are equal to $D_HF(t_0)$. Here I is any interval in $\mathbb{R}$.

With these preliminaries, we consider the set differential equation

$$D_HU=F(t,U),\;\; U(t_0)=U_0\in K_c(\mathbb{R}^n),\;\; t_0\ge 0, \tag{2.7}$$

where $F\in C[\mathbb{R}_+\times K_c(\mathbb{R}^n),K_c(\mathbb{R}^n)]$.

The mapping $U\in C^1[J,K_c(\mathbb{R}^n)]$, $J=[t_0,t_0+a]$ is said to be a solution of (2.5) on J if it satisfies (2.5) on J.
Since $U(t)$ is continuously differentiable, we have

$$U(t)=U_0+\int_{t_0}^{t}D_HU(s)ds,\;\; t\in J. \tag{2.8}$$

Hence, we can associate with the IVP (2.5) the Hukuhara integral

$$U(t)=U_0+\int_{t_0}^{t}F(s,U(s))ds,\;\; t\in J. \tag{2.9}$$

where the integral is the Hukuhara integral which is defined as,

$$\int F(s)ds=\{\int f(s)ds : f \textit{ is any continuous selector of } F\}$$

Observe also that U(t) is a solution of (2.5) on J iff it satisfies (2.7) on J.

We now define a partial to define a partial order in the metric space $K_c(\mathbb{R}^n)$. To do so, we need the definition of a cone in $K_c(\mathbb{R}^n)$, which is given below.

Let $K(K^0)$ be the subfamily of $K_c(\mathbb{R}^n)$ consisting of sets $U\in K_c(\mathbb{R}^n)$ such that any $u\in U$ is a nonnegative (positive) vector of n-components satisfying $u_i\ge 0$ ($u_i>0$) for i=1,2,3,..., n. Then $\mathbb{K}$ is a cone in $K_c(\mathbb{R}^n)$ and K^0 is the nonempty interior of K.

For any U and $V \in K_c(\mathbb{R}^n)$, if there exists a $Z \in K_c(\mathbb{R}^n)$ such that $Z \in K(K^0)$ and $U = V + Z$ then we say that $U \geq V$ $(U > V)$. Similarly we can define $U \leq V$ $(U < V)$.

To define the causal operator we introduce the following notation. Let $E = C[[t_0, T], K_c(\mathbb{R}^n)]$ and $E_0 = C[[t_0 - h_1, T], K_c(\mathbb{R}^n)]$, where $U \in E_0$ implies $U(t) = \Phi_0(t), t_0 - h_1 \leq t \leq t_0$ and U(t) is any arbitrarily continuous function on $[t_0, T]$.

We define a norm on E as follows: for $U, V \in E$

$$D_0[U, V] = Sup_{t_0 \leq t \leq T} D[U(t), V(t)]$$

where D denotes the Hausdorff Metric.

Definition 2.4. By a causal operator or a Volterra operator or a nonanticipative operator we mean a mappling Q: $E \to E$ satisfying the property that if $U(s) = V(s)$, $t_0 \leq s \leq t < T$ then $(QU)(s) = (QV)(s)$, $t_0 \leq s \leq t < T$. By a causal operator with memory we mean a mapping Q:$E_0 \to E$ such that for $U(s) = V(s)$, $t_0 \leq s \leq t < T$,

$$Q(U, \Phi_0)(s) = Q(V, \Phi_0)(s), \ t_0 \leq s \leq t < T \ and \ \Phi_0 \in C_1 = C[[t_0 - h_1, t_0], \ K_c(\mathbb{R}^n)].$$

We now state the results that have been developed in the setup of set differential equations involving causal operators with memory.

We begin with the following results from [6].

Theorem 2.5. *Assume that*
(i) Q is nondecreasing in U for each $t \in I = [t_0, T]$.
(ii) $D_H V(t) \leq (QV)(t)$
$D_H W(t) \geq (QW)(t)$
where $V, W \in C^1[I, K_c(\mathbb{R}^n)]$ and
(iii) $V(t_0) < W(t_0)$.
Then $V(t) < W(t), \ t \in I$, provided one of the above differential inequalities is strict.

Theorem 2.6. *Assume that $Q(U, \Phi_0) \in C[B, E]$ is continuous and compact, where $B \subseteq E_0$ and $B = \{U \in K_c(\mathbb{R}^n) : D_0[U, \Phi_0(t_0)] \leq b$ and $D_0[U_{t_0}, \Phi_0] = 0, \ t \in I\}$ Then there exists a solution of the IVP*

$$D_H U(t) = Q(U, \Phi_0)(t), \tag{2.10}$$

$$U_{t_0} = \Phi_0 \in C_1, \tag{2.11}$$

on some interval $[t_0, t_0 + \delta]$, where $t_0 + \delta < T$, and $C_1 = C[t_0 - h_1, t_0], K_c(\mathbb{R}^n)]$. We next present the following theorems from [7].

Theorem 2.7. *Assume that $m \in C[I, \mathbb{R}_+], \ g \in C[I \times \mathbb{R}_+, \mathbb{R}_+]$ and for $t \in I$,*

$$D_- m(t) \leq g[t, |\ m\ |_0(t)], \tag{2.12}$$

where $|\ m\ |_0(t) = \sup_{t_0 \leq s \leq t} |\ m(s)\ |$. Suppose that $r(t) = r(t, t_0, u_0)$ is the maximal solution of the scalar differential equation (2.1) *existing on I. Then $m(t_0) \leq u_0$ implies $m(t) \leq r(t), \ t \in I$.*

We now state a comparison theorem that connects an estimate on the solution of the IVP (2.10) and (2.11) with maximal solution of the initial value problelm (2.1)

Theorem 2.8. *Let $Q \in C[E_0, E]$ be a causal map such that for $t \in I$,*

$$D[(QU)(t), (QV)(t)] \leq g(t, D_0[U, V](t)],$$

where $g \in C[I \times \mathbb{R}_+, \mathbb{R}_+]$ .Suppose further that the maximal solution $r(t, t_0, u_0)$ of the scalar differential equation (2.1) *exists on I . Then, if $U(t), V(t)$ are two solutions of* (2.10) *and* (2.11) *with initial function $U_{t_0} = V_{t_0} = \Phi_0 \in C_1$, then we have*

$$D[U(t), V(t)] \leq r(t, t_0, u_0), \ t \in I$$

We are now in a position to state the existence and uniqueness result using successive approximations and generalized Lipschitz condition. Once again the proof is very much similar to the corresponding theorem, Theorem 5.7.3 in [4]. Hence we omit it. Observe that the only difference between the two results is that the following theorem has memory included in its set up.

Theorem 2.9. *Suppose that*
(i) $Q \in C[B, E]$ be a causal map, where $B \subseteq E_0$
with $\mathbb{B} = \{U \in E_0 : D_0[U, \Phi_0(t_0)] \leq b, \ D_0[U_{t_0}, \Phi_0] = 0, \ t \in I\}$
and $D_0[Q(U, \Phi_0), \theta] \leq M_1$ on $\mathbb{B}$.
(ii) $g \in C[I \times \mathbb{R}_+, \mathbb{R}_+]$ $g(t, u) \leq M_2$ on $I \times [0, 2b]$, $g(t, 0) = 0$, $g(t, u)$ be nondecreasing in u for each $t \in I$ and $w(t) = 0$ is the only solution of

$$w' = g(t, w), \ w(t_0) = 0 \text{ on } I \tag{2.13}$$

(iii) $D[Q(U, \Phi_0)(t), Q(V, \Phi_0)(t)] \leq g(t, D_0[U, V](t)]$, on $\mathbb{B}$. Then the successive approximations defined by

$$U_{n+1}(t) == \Phi_0(t_0) + \int_{t_0}^{t} Q(U_n.\Phi_0)(s)ds.$$

$$U_{n+1t_0} = \Phi_0 \in C_1 \ n = 0, 1, 2, 3 \ldots,$$

exist on $I_0 = [t_0, t_0 + \eta]$ where $\eta = min[T - t_0, \frac{b}{2M}]$, $M = \max[M_1, M_2]$ and converge uniformly to a unique solution U(t) of (2.10) *and* (2.11)*.*

We now state the Ascoli-Arzela theorem for the family of subsets of $K_c(\mathbb{R}^n)$

Theorem 2.10. *If $\{U_n(t)\}$ is a sequence of equicontinuous and equibounded multimappings defined on an interval J, we can extract a subsequence that converges uniformly to continuous multimapping $U(t)$ on J.*

3. **Extremal Solutions**

Consider the IVP for set differential equations involving causal operators with memory given by

$$D_H U(t) = Q(U, \Phi_0)(t) \tag{3.1}$$

$$U_{t_0} = \Phi_0 \tag{3.2}$$

Defintion 3.1 Let R(t) be the solution of the IVP (3.1), (3.2) existing on $[t_0, T]$. Then R(t) is the maximal solution of (3.1), (3.2), if for every solution U(t) of (3.1), (3.2) existing on $[t_0, T]$, we have

$$U(t) \leq R(t), \ t \in [t_0, T] \tag{3.3}$$

The minimal solution $\rho(t)$ of (3.1), (3.2) is obtained by reversing the inequality (3.3).

Theorem 3.1. *Assume that $Q : E_0 \to E$ is continuous and compact. Further, suppose that Q is non decreasing in U, that is*

$$U(s) \leq V(s), \ t_0 \leq s \leq t_1 < T \ \textit{implies} \ (QU)(t_1) \leq (QV)(t_1).$$

Then the IVP (3.1), (3.2) *possesses an extremal solution on* $[t_0 - h, t_0 + \delta]$ *for* $t_0 + \delta < T$.

Proof. Let $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) > 0$ and $|\epsilon| < b/2$. Then consider for each positive integer N, the following IVP corresponding to (3.1), (3.2) given by

$$D_H U(t) = Q(U, \Phi_0)(t) + \frac{\epsilon}{N} \tag{3.4}$$

$$U_{t_0} = \Phi_0 + \frac{\epsilon}{N} \tag{3.5}$$

set

$$Q_N(U, \Phi_0)(t) = Q(U, \Phi_0)(t) + \frac{\epsilon}{N}$$

Then Q_N is continuous, compact on $\mathbb{B} = \mathbb{B}[\Phi_0(t_0), \frac{b}{2}] =$

$$\{U \in E_0 : D_0[U, \Phi_0(t_0) + \frac{\epsilon}{N}] \leq b/2, \ D_0[U_{t_0}, \Phi_0] = 0, \ t \in I]\}$$

and $D_0[Q_N(U, \Phi_0)(t), \theta] \leq K + \frac{\epsilon}{N}$ *on* $\mathbb{B}$, for some constant $K \geq 0$. Hence from Theorem 2.5, we can deduce that there exists a solution $U_N(t, \epsilon) \in \mathbb{B}$ *on* $[t_0 - h_1, t_0 + \delta]$ *with* $t_0 + \delta < T$. Consider $0 < \epsilon_2 < \epsilon_1 < \epsilon$, then

$$U_N(t_0, \epsilon_2) < U_N(t_0, \epsilon_1),$$

$$D_H U_N(t, \epsilon_2) \leq Q(U_N, \Phi_0)(t, \epsilon_2) + \frac{\epsilon_2}{N}$$

and

$$D_H U_N(t, \epsilon_1) > Q(U_N, \Phi_0)(t, \epsilon_1) + \frac{\epsilon_2}{N} \ \textit{on} \ I.$$

On applying Theorem 2.4, to the above inequalities, we arrive at

$$U_N(t, \epsilon_2) < U_N(t, \epsilon_1) \ \textit{on} \ [t_0, T].$$

Since the family of functions $\{U_N(t,\epsilon)\}$ is equicontinuous and uniformly bounded on $[t_0-h_1, t_0+\delta]$, it follows from Arzela-Ascoli theorem in this set up, that is, Theorem 2.9, that there exists a decreasing sequence $\frac{\epsilon}{N_k}$ such that $\frac{\epsilon}{N_k} \to 0\, uniformly\, as\, k \to \infty$

$$R(t) = \lim_{k\to\infty} U_{N_k}(t,\epsilon)$$

exists on $[t_0, T]$. Clearly $R(t_0) = \Phi_0(t_0)\, and\, R_{t_0} = \Phi_0$. Further, the uniform continuity and compactness of $Q(U,\Phi_0)(t)$ yields that $Q(U_{N_k},\Phi_0)(t) \to Q(R,\Phi_0)(t)\ as\ k \to \infty$ uniformly. Thus term by trem integration is applicable and

$$U_{N_k}(t,\epsilon) = \Phi_0(t_0) + \frac{\epsilon}{N_k} + \int_{t_0}^{t} Q(U_{N_k},\Phi_0)(t,\epsilon)\mathrm{d}s$$

as $k \to \infty$ reduces to

$$R(t) = \Phi_0(t_0) + \int_{t_0}^{t} Q(R,\Phi_0)(t)\mathrm{d}s$$

Thus R(t) is a solution of (3.1), (3.2) on $[t_0-h_1, t_0+\delta],\ t_0+\delta < T$. We next claim that R(t) is the required maximal solution of the IVP (3.1), (3.2). To prove our claim, first we note that if U(t) is any solution of the I.V.P. (3.1), (3.2) then

$$U(t_0) = \Phi_0(t_0) < \Phi_0(t_0) + \frac{\epsilon}{N} = U_N(t_0,\epsilon)$$

$$D_H U(t) < Q(U,\Phi_0)(t) + \frac{\epsilon}{N}$$

$$D_H U(t,\epsilon) \geq Q(U,\Phi_0)(t,\epsilon) + \frac{\epsilon}{N}\ on\ [t_0, t_0+\delta]$$

Hence from the differential inequality result,We get

$$U(t) < U_N(t,\epsilon)\ on\ [t_0, t_0+\delta]$$

Thus uniqueness of maximal solution R(t) shows that $U_N(t,\epsilon)$tends uniformly to R(t) on $[t_0, t_0+\delta]$ as $n \to \infty$. This proves that R(t) is the maximal solution of the IVP (3.1), (3.2).

Similarly one can prove the existence of a minimal solution $\rho(t)$ of IVP (3.1), (3.2) by considering the IVP

$$D_H U = Q(U,\Phi_0)(t) - \frac{\epsilon}{N}$$

$$U_{t_0} = \Phi_0 - \frac{\epsilon}{N}$$

and proceeding with the proof as in the eariler fashion. □

Theorem 3.2. *Suppose that the conditions of Theorem* 3.1 *hold and* $M \in C^1[[t_0, T], K_c(\mathbb{R}^n_+)]$ *satisfies*

$$D_H M(t) \leq Q(M,\Phi_0)(t)$$

$$M(t_0) \leq \Phi_0(t_0)$$

Then we have $M(t) \leq R(t),\ t \in [t_0-h_1, T]$, *where R(t) is the maximal solution of* (3.1), (3.2) *existing on* $[t_0-h_1, T]$.

Proof. Let $U(t,\epsilon)$ for $\epsilon > 0$ be any solution of

$$D_H U(t,\epsilon) = Q(U,\Phi_0)(t,\epsilon) + \epsilon$$

$$\Phi_0 = \Phi_0 + \epsilon$$

then $\Phi_0(t_0) = \Phi_0(t_0) + \epsilon$ consider

$$D_H M(t) < Q(M,\Phi)(t) + \epsilon$$

and

$$D_H U(t,\epsilon) \geq Q(U,\Phi_0)(t,\epsilon) + \epsilon$$

with

$$M(t_0) = \Phi_0(t_0) < \Phi_0(t_0) + \epsilon = U(t_0,\epsilon)$$

Now an application of the comparision Theorem 2.4 infers that,

$$M(t) < U(t,\epsilon)$$

Now taking limits as $\epsilon \to 0$ we obtain that $M(t) \leq R(t)$, $t \in [t_0 - h_1, t_0 + \delta]$. Thus the proof is complete. □

4. **Continuous Dependence**

To study the continuous dependence of solutions relative to the initial data, we need the following lemma.

Lemma 4.1. *Let $Q(U,\Phi_0) : E_0 \to E$ be continuous and*

$$G(t,r) = Max\{D[(U,\Phi_0),\theta], t \in I : D_0[U,\Phi_0(t_0)] \leq r\}.$$

Assume that $r^*(t,t_0,0)$ be the maximal solution of $u' = G(t,u)$, $u(t_0) = 0$ on I, and let $U = U(t,t_0,\Phi_0)$ be the solution of (3.1) *and* (3.2). *Then*

$$D[U,\Phi_0(t_0)] \leq r^*(t,t_0,0),\ t \in I \tag{4.1}$$

Proof. Define $m(t) = D[U,\Phi_0(t_0)]$ Then

$$\begin{aligned} D^+ m(t) &\leq D[D_H U(t),\theta] = D[Q(U,\Phi_0)(t),\theta] \\ &\leq Max\ D[Q(U,\Phi_0)(t),\theta], t \in I : D_0[U,\Phi_0(t_0)] \leq m(t)] = G[t,m(t)]. \end{aligned}$$

Then from Theorem 1.4.1 in [2] we get

$$D[U(t),\Phi_0(t_0) \leq r^*(t,t_0,0),\ t \in I.$$

Thus the proof is complete. □

Theorem 4.2. *Let $Q(U,\Phi_0) : \mathbb{B} \to E$, $\mathbb{B} \subseteq E_0$ and satisfy*

$$D[Q(U,\Phi_0)(t),\ Q(V,\Phi_0)(t)] \leq g[t, D_0(U,V)(t)],\ U,V \in \mathbb{B}_0 \tag{4.2}$$

where

$$g \in C[I \times \mathbb{R}_+, \mathbb{R}_+]$$

Assume that $u(t) \equiv 0$ be the unique solution of the scalar differentail equation

$$u' = g(t,u),\ u(t_0) = u_0 \tag{4.3}$$

with $u_0 = 0$.

If the solutions $u(t,t_0,u_0)$ of (4.3) through (t_0,u_0) are continuous w.r.t (t_0,u_0), then the solutions $U(t,t_0,\Phi_0)$ of (3.1), (3.2) are unique and continuous w.r.t the initial values (t_0,Φ_0).

Proof. Since uniqueness follows from the Theorem 2.8 on successive approximations, it is enough to prove that the solutions are continuous w.r.t the initial values (t_0,Φ_0). Consider $U(t) = U(t,t_0,\Phi_0)$ and $V(t) = V(t,t_0,\psi_0)$ be the solutions of (3.1), (3.2) such that $U_{t_0} = \Phi_0,\ V_{t_0} = \Psi_0,\ \Phi_0,\Psi_0 \in C_1$. Set $m(t) = D[U(t),V(t)]$, then

$$\begin{aligned} D^+m(t) &\leq D[D_HU(t), D_HV(t)] = D[Q(U,\Phi_0)(t), Q(V,\Phi_0)(t)] \\ &\leq g[t, D_0[U,V](t)] = g[t, |m|_0(t)] \end{aligned}$$

Then from Theorem 2.6 we conclude that

$$m(t) = D[U(t),V(t)] \leq r(t,t_0,D_0[\Phi_0,\Psi_0]),\ t \in I.$$

Since $\Phi_0 \to \Psi_0, \lim r(t,t_0,D_0[\Phi_0,\Psi_0]) = r(t,t_0,0)$ uniformly on I and by hypothesis $r(t,t_0,0) = 0$. It follows that,

$$\lim_{\Phi_0 \to \Psi_0} U(t,t_0,\Phi_0) = V(t,t_0.\Psi_0)$$

Uniformly and hence continuity of $U(t,t_0,\Phi_0)$ relative to Φ_0 is valid.

To prove the continuity relative to t_0, let $U(t) = U(t,t_0,\Phi_0(t_0)), V(t) = V(t,\tau_0,\Phi_0(t_0))$ be two solutions of the IVP (3.1), (3.2) and let $\tau_0 > t_0$. Now set

$$m(t) = D[U(t),V(t)]$$

then

$$m(\tau_0) = D[U(\tau_0),\Phi_0(t_0))$$

where $V(s_0) = \Phi_0(t_0)$ for $t_0 \leq s \leq \tau_0$. On applying Lemma 4.1, we get that

$$m(\tau_0) \leq r^*(\tau_0,t_0,0),$$

and consequently using the comparision theorem for the scalar differentail equation, we get

$$m(t) \leq \bar{r}(t), t > \tau_0$$

where $\bar{r}(t) = r(t, \tau_0, r^*(\tau_0, t_0, 0))$ is the maximal solution of IVP (4.3) through $(\tau_0, r^*(\tau_0, t_0, 0))$. Since $r^*(\tau_0, t_0, 0) = 0$, We have

$$\lim_{\tau_0 \to t_0} \bar{r}(t, \tau_0, r^*(\tau_0, t_0, 0)) = \bar{r}(t, t_0, 0).$$

By the hypothesis, we have that the unique solution of the IVP(4.3) is the zero solution and hence $\bar{r}(t, t_0, 0) \equiv 0$. This implies that $m(t) \leq 0$, that is, the continuity of solutions relative to t_0 is valid and the proof is complete. □

In the following theorem we will study the continuous dependence of solution of the IVP (3.1), (3.2) relative to a parameter.

Theorem 4.3. *Suppose that $Q : E^* \to E$ where E^* is an open set in $\bar{E} = E_0 \times \mathbb{R}_+$ that contains a parameter μ and for $\mu = \mu_0$, let $\bar{U}(t) = U(t, t_0, \Phi_0, \mu_0)$ be the solution of*

$$D_H U(t) = Q(U, \Phi_0, \mu)(t), \; U_{t_0} = \Phi_0 \tag{4.4}$$

existing on I. Assume that

$$\lim_{\mu \to \mu_0} Q(U, \Phi_0, \mu)(t) = Q(U, \Phi_0, \mu_0)(t) \tag{4.5}$$

uniformly in $(t, U(t))$ and

$$D[Q(U, \Phi_0, \mu)(t), Q(V, \Phi_0, \mu)(t)] \leq g(t, D_0(U, V)(t)) \tag{4.6}$$

where $U, V \in \mathbb{B}$ [see Theorem 2.8] and $g \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$. Suppose that $u(t) \equiv 0$ is the unique solution of the IVP (4.3) with $u(t_0) = 0$. Then, given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that whenever $|\mu - \mu_0| < \delta(\epsilon)$, the IVP

$$D_H U(t) = Q(U, \Phi_0, \mu)(t), \; U_{t_0} = \Phi_0 \tag{4.7}$$

admits a unique solution $U(t) = U(t, t_0, \Phi_0, \mu)$ satisfying

$$D[U(t), \bar{U}(t)] < \epsilon, \; t \in I.$$

Proof. The uniqueness of the solutions follows from Theorem 2.8, as the hypothesis of that theorem is satisfied. Since $u(t) \equiv 0$ is the only solution of (4.3), by using Lemma 2.2, we deduce that, given any compact interval $[t_0, t_0 + a] \subseteq I$ and any $\epsilon > 0$ there exists a positive number $\eta = \eta(\epsilon) > 0$ such that the extremal solution $r(t, t_0, 0, \eta)$ of $u' = g(t, u) + \eta$ exist on $[t_0, T]$ and satisfies $r(t, t_0, 0, \eta) < \epsilon, t \in I$. Also since

$$\lim_{\mu \to \mu_0} Q(U, \Phi_0, \mu)(t) = Q(U, \Phi_0, \mu_0)(t)$$

uniformly for $(t, U(t))$, $t \in I$, we have given $\eta > 0$ there exist a $\hat{\delta} = \hat{\delta}(\eta) > 0$ such that whenever $|\mu - \mu_0| < \hat{\delta}$, We have

$$D[Q(U, \Phi_0, \mu)(t), Q(U, \Phi_0, \mu_0)(t)] < \eta$$

Now let $\epsilon > 0$ be given. Define,

$$m(t) = D[U(t), \bar{U}(t)]$$

where $U(t)$, $\bar{U}(t)$ are the solutions of the IVPs (4.7) and (4.4) respectively. Set $m(t) = D[U(t), \bar{U}(t)]$, then

$$\begin{aligned} D_{-}m(t) &\leq D[D_H U(t), D_H \bar{U}(t)] \\ &= D[Q(U, \Phi_0, \mu)(t), Q(\bar{U}, \Phi_0, \mu_0)(t)] \\ &\leq D[Q(U, \Phi_0, \mu)(t), Q(\bar{U}, \Phi_0, \mu)(t)] \\ &\qquad + D[Q(\bar{U}, \Phi_0, \mu)(t), Q(\bar{U}, \Phi_0, \bar{\mu})(t)] \\ &\leq g(t, D_0(U, \bar{U})(t)) + \eta = g(t, |m|_0(t)) + \eta \end{aligned}$$

Now using the Theorem 2.3 adjusted to the present situation, we obtain that

$$m(t) \leq r(t, t_0, 0, \eta), \; t \geq t_0$$

This gives, $D[U(t), \bar{U}(t)] < \epsilon$, when ever $|\mu - \mu_0| < \delta$. Then δ depends only on η and η depends only on ϵ hence δ depends on ϵ. Thus the proof is complete. □

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