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BOUNDEDNESS AND CONTINUITY PROPERTIES OF NONLINEAR COMPOSITION OPERATORS: A SURVEY

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Dedicated to Jeff Webb on the occasion of his retirement

ABSTRACT. In this paper we give an overview of mapping, continuity and boundedness properties of nonlinear composition operators of type $Hf(t) := h(f(t))$ or $Hf(t) := h(t, f(t))$ in the function spaces $C([a, b])$, $C^1([a, b])$, $Lip_\alpha([a, b])$, $AC([a, b])$, $BV([a, b])$, $WBV_p([a, b])$, and $RBV_p([a, b])$.

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1. INTRODUCTION AND PRELIMINARIES

A vast variety of ordinary differential equations, subject to some initial, boundary, or periodicity conditions, may be written as operator equations of the form

$$Lf(t) = h(t, f(t), f'(t), \dots, f^{(n)}(t)), \tag{1.1}$$

where L is some linear differential operator and the (usually, nonlinear) function h is defined on $[a, b] \times \mathbb{R}^{n+1}$, say. In order to apply abstract existence and uniqueness theorems of nonlinear analysis to equation (1.1), one has to find appropriate function spaces such that both the linear operator L on the left hand side of (1.1) and the nonlinear operator generated by the function h on the right hand side of (1.1) are "well behaved" in these spaces. The aim of this survey is a discussion of this problem for several important function spaces, where for simplicity we will restrict ourselves to the case when the function h does not contain derivatives of f.

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So given $h : \mathbb{R} \to \mathbb{R}$, we consider the operator

$$Hf(t) := h(f(t)) \qquad (a \leq t \leq b); \tag{1.2}$$

more generally, given $h : [a,b] \times \mathbb{R} \to \mathbb{R}$, we consider the operator

$$Hf(t) := h(t, f(t)) \qquad (a \leq t \leq b). \tag{1.3}$$

The operator H is usually called the *composition operator* (or *superposition operator* or *substitution operator*) generated by h. (In the Russian literature it is particularly common to call this operator *Nemytskij operator* and to denote it by H, since the Russian letter "H" corresponds to the Latin letter "N". This fact allowed Hercule Poirot to reveal the role of the Princess Natalya Dragomirova in [21].) In what follows, we will refer to (1.2) as the *autonomous case* and to (1.3) as the *non-autonomous case*.

A very natural problem related to the operator (1.2) or (1.3) reads as follows:

- *Given a class X of functions $f : [a,b] \to \mathbb{R}$, find conditions on the function h, possibly both necessary and sufficient, under which the operator H generated by h maps the class X into itself.*

This problem is sometimes referred to as the *composition operator problem* in the literature, see e.g. [18,19]. The solution to this problem for given X is sometimes very easy, sometimes highly nontrivial. For example, the Tietze-Uryson extension lemma for continuous functions implies that the operator (1.3) maps the space $C([a,b])$ into itself if and only if the corresponding function h is continuous on $[a,b] \times \mathbb{R}$. The parallel problem for the space $C^1([a,b])$ of continuously differentiable functions, however, is surprisingly complicated, as we will see in Section 3 below. For example, even a *discontinuous* function $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ may generate an operator H which maps $C^1([a,b])$ into itself, see Example 3.4.

Apart from conditions for the mere inclusion $H(X) \subseteq X$, in applications to nonlinear problems it is of course also important to know conditions, possibly both necessary and sufficient, under which the operator H generated by h is continuous or bounded in the norm of X. (Recall that, in contrast to linear operators, a nonlinear operator may be continuous and unbounded, or bounded and discontinuous.) In case of bounded operators H it is useful to give estimates, or even explicit formulas, for the growth function

$$\mu(r) := \sup \{\|Hf\| : \|f\| \leq r\} \qquad (r > 0) \tag{1.4}$$

of the operator H in the norm of the underlying space.

Finally, when applying fixed point theorems, one has to verify either some compactness condition or some Lipschitz condition for H of the form

$$\|Hf - Hg\| \leq K\|f - g\| \qquad (f, g \in X). \tag{1.5}$$

Unfortunately, in many function spaces the global Lipschitz condition (1.5) leads to a strong degeneracy for h: condition (1.5) holds only if

$$h(u) = \alpha + \beta u \qquad (u \in \mathbb{R}) \tag{1.6}$$

for suitable constants $\alpha, \beta \in \mathbb{R}$ in the autonomous case (1.2), and

$$h(t, u) = \alpha(t) + \beta(t)u \qquad (a \leq t \leq b, u \in \mathbb{R}) \tag{1.7}$$

for suitable functions $\alpha, \beta \in X$ in the non-autonomous case (1.3), respectively. This means that we may apply the Banach contraction mapping principle, say, only if the underlying problem is actually *linear* and therefore not very interesting.

Closer scrutiny of problem (1.1) reveals, however, that it often suffices to impose a *local* Lipschitz condition like

$$\|Hf - Hg\| \leq K(r)\|f - g\| \qquad (f, g \in X,\ \|f\|, \|g\| \leq r), \tag{1.8}$$

where the Lipschitz constant $K(r)$ in (1.8) usually depends on the radius r. Fortunately, it turns out that replacing the global condition (1.5) by the local condition (1.8) does not lead to unpleasant degeneracy phenomena, but considerably enlarges the class of admissible nonlinearities. We will describe this result in several important function spaces. In what follows, we will be working in the following spaces.

- The space $C([a, b])$ of continuous functions with norm

$$\|f\|_C := \max_{a \leq x \leq b} |f(x)|; \tag{1.9}$$

- the space $C^1([a, b])$ of continuously differentiable functions with norm

$$\|f\|_{C^1} := |f(a)| + \|f'\|_C. \tag{1.10}$$

- the space $Lip_\alpha([a, b])$ of Hölder continuous (resp. Lipschitz continuous for $\alpha = 1$) functions with norm

$$\|f\|_{Lip_\alpha} := |f(a)| + lip_\alpha(f) \qquad (0 < \alpha \leq 1), \tag{1.11}$$

  where

$$lip_\alpha(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

  denotes the minimal Hölder constant (resp. Lipschitz constant for $\alpha = 1$) of f;

- the space $BV([a,b])$ of functions of bounded variation with norm

$$\|f\|_{BV} := |f(a)| + \mathrm{Var}(f;[a,b]), \tag{1.12}$$

  where

$$\mathrm{Var}(f;[a,b]) = \sup\left\{\sum_{j=1}^{m} |f(t_j) - f(t_{j-1})|\right\},$$

  with the supremum being taken over all partitions $\{t_0,\dots,t_m\}$ of $[a,b]$, denotes the total variation of f;
- the space $AC([a,b])$ of absolutely continuous functions with either the norm (1.12) or the norm

$$\|f\|_{AC} := |f(a)| + \int_a^b |f'(x)|\,dx; \tag{1.13}$$

- the space $WBV_p([a,b])$ of functions of bounded p-variation in Wiener's sense with norm

$$\|f\|_{WBV_p} := |f(a)| + \mathrm{Var}_p^W(f;[a,b])^{1/p} \qquad (1 \le p < \infty), \tag{1.14}$$

  where

$$\mathrm{Var}_p^W(f;[a,b]) = \sup\left\{\sum_{j=1}^{m} |f(t_j) - f(t_{j-1})|^p\right\},$$

  with the supremum being taken over all partitions $\{t_0,\dots,t_m\}$ of $[a,b]$, denotes the total p-variation in Wiener's sense of f;
- the space $RBV_p([a,b])$ of functions of bounded p-variation in Riesz' sense with norm

$$\|f\|_{RBV_p} := |f(a)| + \mathrm{Var}_p^R(f;[a,b])^{1/p} \qquad (1 \le p < \infty), \tag{1.15}$$

  where

$$\mathrm{Var}_p^R(f;[a,b]) = \sup\left\{\sum_{j=1}^{m} \frac{|f(t_j) - f(t_{j-1})|^p}{|t_j - t_{j-1}|^{p-1}}\right\},$$

  with the supremum being taken over all partitions $\{t_0,\dots,t_m\}$ of $[a,b]$, denotes the total p-variation in Riesz' sense of f.

It is well-known that these spaces are related by the (strict for $p > 1$) inclusions

$$C^1([a,b]) \subset Lip([a,b]) \subset RBV_p([a,b]) \subset AC([a,b]) \subset BV([a,b]) \tag{1.16}$$

and

$$BV([a,b]) \subset WBV_p([a,b]), \quad Lip_{1/p}([a,b]) \subset WBV_p([a,b]). \tag{1.17}$$

(For $\alpha = 1$ we drop the subscript 1 in *Lip*.) All inclusions in (1.16) and (1.17) are *continuous imbeddings*. Obviously, for $p = 1$ both $WBV_1([a,b])$ and $RBV_1([a,b])$ coincide with $BV([a,b])$.

In contrast to the Wiener space $WBV_p([a,b])$, the Riesz space $RBV_p([a,b])$ does not contain any Hölder space. In fact, one may construct a function which belongs to *every* Hölder space $Lip_\alpha([0,1])$ for $0 < \alpha < 1$, but not to $BV([0,1])$, see [9, Exercise 14.29]. By (1.16), this function cannot belong to $RBV_p([0,1])$ for any $p \geq 1$.

In what follows we will restrict ourselves to the case $[a,b] = [0,1]$; this is not a loss of generality, since all function spaces mentioned above may be reduced to this case by means of the map $\ell : [0,1] \to [a,b]$ defined by $\ell(t) := (b-a)t + a$ which is a strictly increasing affine diffeomorphism between $[0,1]$ and $[a,b]$ with inverse $\ell^{-1}(s) = (s-a)/(b-a)$.

This survey article is expository, intended to provide insight into the theory, with a particular emphasis on examples and counterexamples. It lies in the nature of a survey that many of the results presented below are well-known. This mainly refers to continuous, Hölder continuous, Lipschitz continuous, or differentiable functions; some results in this direction may be found in the book [6] and the references therein. On the other hand, most results on functions of bounded variation presented below are new. In this survey we intentionally do not consider Sobolev spaces and their generalizations which are of utmost importance in the study of partial differential equations and operators; here an excellent standard reference is Chapter 5 of the monograph [50]. Although we will consider only the scalar case of intervals as domains, some of our results (e.g., those referring to the Hölder class Lip_α) carry over without major changes to higher dimensional domains, and therefore apply to partial differential equations as well.

2. FIVE STRUCTURAL THEOREMS

Instead of proving the same result for several function spaces repeatedly, we begin with some general theorems which allow us to provide a unified approach to some of these spaces. Typical requirements on the function h which will occur several times are the global Lipschitz condition

$$|h(u) - h(v)| \leq k|u - v| \qquad (u, v \in \mathbb{R}) \tag{2.1}$$

in the autonomous case (1.2) and

$$|h(t,u) - h(t,v)| \leq k|u - v| \qquad (0 \leq t \leq 1,\ u, v \in \mathbb{R}) \tag{2.2}$$

in the non-autonomous case (1.3). A local Lipschitz condition which is parallel to (1.8) reads

$$|h(u) - h(v)| \leq k(r)|u - v| \qquad (u, v \in \mathbb{R},\ |u|, |v| \leq r). \tag{2.3}$$

and

$$|h(t,u) - h(t,v)| \leq k(r)|u - v| \qquad (0 \leq t \leq 1,\ u, v \in \mathbb{R},\ |u|, |v| \leq r), \tag{2.4}$$

respectively. We will also be interested in the "interconnections" between the value of $K(r)$ in (1.8) and the values of $k(r)$ in (2.3) or (2.4), respectively. If (2.3) holds, we may also consider the characteristic

$$\tilde{k}(r) := \sup_{|u|\leq r} |h(u)| \qquad (r > 0) \tag{2.5}$$

which is related to $k(r)$ by the trivial estimate $\tilde{k}(r) \leq |h(0)| + k(r)r$.

We start with a somewhat surprising result on the autonomous operator (1.2) which covers several function spaces simultaneously and was proved in [4].

Theorem 2.1. *Any of the following 5 equivalent conditions on the autonomous composition operator* (1.2) *implies condition* (2.3) *on the corresponding function h:*

(a) *The operator* (1.2) *maps $Lip([0,1])$ into itself.*

(b) *The operator* (1.2) *maps $RBV_p([0,1])$ into itself.*

(c) *The operator* (1.2) *maps $AC([0,1])$ into itself.*

(d) *The operator* (1.2) *maps $BV([0,1])$ into itself.*

(e) *The operator* (1.2) *maps $Lip([0,1])$ into $BV([0,1])$.*

We point out that some of the assertions contained in Theorem 2.1 are well known. Thus, the fact that (a) implies (2.3) has been proved in [24], that (b) implies (2.3) in [48], that (c) implies (2.3) in [43], and that (d) implies (2.3) in [25]. The novelty is of course (e) which implies all the other conditions, by (1.16). The proof of this consists in assuming that (2.3) is false and constructing a function $f \in Lip([0,1])$ such that $h \circ f \notin BV([0,1])$. For details we refer to [4, Theorem 6].

Theorem 2.1 will be used in Section 4 in connection with spaces of functions of bounded variation. Here we briefly anticipate a simple example to illustrate Theorem 2.1.

Example 2.2. Consider the "seagull function"

$$h(u) := \min\left\{\sqrt{|u|}, 1\right\}. \tag{2.6}$$

It is not hard to see that $h \in RBV_p([0,1])$ for $1 \leq p < 2$ and

$$\mathrm{Var}_p^R(h;[0,1]) = \frac{1}{2^{p-1}(2-p)}.$$

Consequently, by (1.16) the function h also belongs to $AC([0,1])$ and $BV([0,1])$.

Consider now the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(t) := \begin{cases} t^2 \sin^2 \dfrac{1}{t} & \text{for} \quad 0 < t \leq 1, \\ 0 & \text{for} \quad t = 0. \end{cases}$$

Being differentiable with bounded derivative on $[0,1]$, the function f belongs to $Lip([0,1])$, and so also to $RBV_p([0,1])$, $AC([0,1])$, and $BV([0,1])$, by (1.16). On the other hand, the composed function $Hf = h \circ f = \sqrt{|f|}$ does not even belong to $BV([0,1])$, as is shown in every first-year calculus course. A refinement of Example 2.2 will be given in Example 4.2 below. One might wonder why the space $WBV_p([0,1])$ is not covered by Theorem 2.1. In fact, one may show that (2.3) is also equivalent to the inclusion $H(WBV_p) \subseteq WBV_p$ (see Theorem 2.4 below); this was recently proved in [5]. However, it is *not* true that (2.3) follows from the inclusion $H(Lip) \subseteq WBV_p$ which by (1.17) in case $p > 1$ would be still weaker than (e):

Example 2.3. Consider again the operator H generated by the seagull function (2.6). For any partition $\{t_0, \ldots, t_m\}$ of $[0,1]$, the estimate

$$\sum_{j=1}^{m} |h(f(t_j)) - h(f(t_{j-1}))|^2 = \sum_{j=1}^{m} \left| \sqrt{|f(t_j)|} - \sqrt{|f(t_{j-1})|} \right|^2 \leq \sum_{j=1}^{m} |f(t_j) - f(t_{j-1})|$$

shows that the operator H maps $BV([0,1])$ into $WBV_2([0,1])$, and so also $Lip([0,1])$ into $WBV_2([0,1])$. However, the function (2.6) certainly does not satisfy (2.3).

Of course, the function (2.6) belongs to the Hölder space $Lip_{1/2}([0,1])$. So it is not surprising that, in order to include the family of spaces $WBV_p([0,1])$, we have to replace (2.3) by the local Hölder condition

$$|h(u) - h(v)| \leq k(r)|u - v|^\alpha \qquad (u \in \mathbb{R},\ |u|, |v| \leq r) \tag{2.7}$$

for fixed $\alpha \in (0,1]$. In fact, we obtain then the following result which is parallel to Theorem 2.1.

Theorem 2.4. *Any of the following 4 equivalent conditions on the autonomous composition operator* (1.2) *implies condition* (2.7) *on the corresponding function h:*

(a) *The operator* (1.2) *maps $WBV_p([0,1])$ into $WBV_q([0,1])$ for any $q \geq p/\alpha$.*

(b) *The operator* (1.2) *maps $WBV_p([0,1])$ into $WBV_{p/\alpha}([0,1])$.*

(c) *The operator* (1.2) *maps $BV([0,1])$ into $WBV_{1/\alpha}([0,1])$.*

(d) *The operator* (1.2) *maps $Lip([0,1])$ into $WBV_{1/\alpha}([0,1])$.*

Proof. We only sketch the idea. Since (a) implies (b), (b) implies (c), and (c) implies (d), by (1.17), we only have to prove that (d) implies (2.7). So if we assume that (2.7) is false, we may find sequences $(u_k)_k$ and $(v_k)_k$ such that

$$|u_k - v_k| \leq \frac{1}{k^2}, \quad |h(u_k) - h(v_k)| > k^2 |u_k - v_k|^\alpha \qquad (k = 1, 2, \ldots),$$

and then construct a function $f \in Lip([0,1])$ such that $h \circ f \notin WBV_{1/\alpha}([0,1])$ precisely in the same way as in [4, Theorem 6]. $\square$

Of course, in case $\alpha = 1$ condition (2.7) reduces to condition (2.3), and we may recover from Theorem 2.4 some parts of Theorem 2.1. Thus, (a), (b) and (c) in Theorem 2.4 then all reduce to (d) in Theorem 2.1, while (d) in Theorem 2.4 becomes (e) in Theorem 2.1. For $\alpha = 1$ we also get from (b) the equivalence of (2.3) and the inclusion $H(WBV_p) \subseteq WBV_p$ mentioned above.

Let us now prove two theorems which refer to the global Lipschitz condition (1.5). We start with a rather obvious necessary condition.

Theorem 2.5. *Suppose that the composition operator* (1.3) *generated by some function* $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ *maps a normed space* X *into a normed space* Y *and satisfies the global Lipschitz condition* (1.5). *Assume that the space* X *contains the constant functions, and the space* Y *is imbedded into the space of bounded functions with norm*

$$\|f\|_\infty := \sup_{0\leq x\leq 1} |f(x)|. \tag{2.8}$$

Then the function h *satisfies the global Lipschitz condition* (2.2).

Proof. The proof is almost trivial. From (1.5) and our hypothesis on Y it follows that

$$\|Hf - Hg\|_\infty \leq L\|f - g\|_X \qquad (f, g \in X) \tag{2.9}$$

for some $L > 0$, and taking $f(t) \equiv u$ and $g(t) \equiv v$ in (2.9) yields

$$|h(t,u) - h(t,v)| \leq \|Hf - Hg\|_\infty \leq L\|1\|_X|u - v|,$$

where $\|1\|_X$ denotes the norm of the constant function $f(t) \equiv 1$ in X. □

The degeneracy phenomenon mentioned above which states that the global Lipschitz condition (1.5) necessarily leads to affine functions (1.7) has been observed in various function spaces. It was first proved in [33] for the space $Lip([a,b])$, and afterwards in [31] for the space $Lip_\alpha([a,b])$, in [31] (see also [32]) for the space $C^n([a,b])$ of functions whose n-th derivative is continuous, in [30] for the space $Lip^n_\alpha([a,b])$ of functions whose n-th derivative is Hölder continuous, in [37] for the space $AC([a,b])$, in [52] for the space $AC^n([a,b])$ of functions whose n-th derivative is absolutely continuous, in [27] for the space $Lip^n([a,b])$ of functions whose n-th derivative is Lipschitz continuous, in [42] for the space $RBV_p([a,b])$, and in [39] for the Sobolev space $W^n_p([a,b])$ for $n \geq 3$. This list is not exhaustive; further examples of this kind may be found in [35,36,38,41,44–47].

We are now going to prove a general theorem which allows us to cover at least some of these spaces in a unified approach. To this end we need some special construction based on an idea of [32]. By $P_n([0,1])$ we denote the linear space of all polynomials of degree $\leq n$. In particular, $P_1([0,1])$ is the space of all affine functions; we consider $P_1([0,1])$ equipped with the C^1-norm (1.10).

Fix $x_1, x_2 \in [0,1]$ and $u_1, u_2 \in \mathbb{R}$, where $x_1 \neq x_2$, and define $f \in P_1([0,1])$ by

$$f(x) := \frac{u_1 - u_2}{x_1 - x_2}x + \frac{x_1u_2 - x_2u_1}{x_1 - x_2} = \frac{u_1(x - x_2) + u_2(x_1 - x)}{x_1 - x_2}. \tag{2.10}$$

It is not hard to see that this polynomial satisfies then the conditions

$$f(x_1) = u_1, \quad f(x_2) = u_2, \quad \|f\|_{C^1} = \frac{|u_1 - u_2| + |x_1u_2 - x_2u_1|}{|x_1 - x_2|}.$$

Denoting by g the analogous polynomial with u_1 replaced by v_1 and u_2 replaced by v_2 in (2.10), i.e.,

$$g(x) := \frac{v_1 - v_2}{x_1 - x_2}x + \frac{x_1v_2 - x_2v_1}{x_1 - x_2} = \frac{v_1(x - x_2) + v_2(x_1 - x)}{x_1 - x_2}, \tag{2.11}$$

we have

$$g(x_1) = v_1, \quad g(x_2) = v_2, \quad \|g\|_{C^1} = \frac{|v_1 - v_2| + |x_1v_2 - x_2v_1|}{|x_1 - x_2|}.$$

Consequently,

$$\|f - g\|_{C^1} = \frac{|u_1 - u_2 - v_1 + v_2| + |x_1u_2 - x_2u_1 - x_1v_2 + x_2v_1|}{|x_1 - x_2|}.$$

Moreover, fixing $x \in [0,1]$ and letting $x_1 \to x$ and $x_2 \to x$ we obtain

$$\lim_{x_1,x_2\to x} |x_1 - x_2|\|f - g\|_{C^1} = (1 + |x|)|u_1 - u_2 - v_1 + v_2|. \tag{2.12}$$

Theorem 2.6. *Suppose that the composition operator* (1.3) *generated by some function* $h : [0,1]\times\mathbb{R} \to \mathbb{R}$ *maps a normed space* X *into a normed space* Y *and satisfies the global Lipschitz condition* (1.5)*. Assume that the space* $P_1([0,1])$ *of affine functions with norm* (1.10) *is imbedded into* X*, and* Y *is imbedded into the space* $Lip([0,1])$ *with norm* (1.11)*. Then there exist functions* $\alpha, \beta \in Y$ *such that* (1.7) *holds true.*

Proof. From (1.5) and our hypotheses on X and Y it follows that

$$\|Hf - Hg\|_{Lip} \leq L\|f - g\|_{C^1} \qquad (f, g \in P_1([0,1]))$$

for some $L > 0$. In particular, since constant functions belong to $P_1([0,1])$ we see that $h(\cdot, u) \in Lip([0,1])$ for each $u \in \mathbb{R}$, and so $h(\cdot, u)$ is continuous.

Fix $x_1, x_2 \in [0,1]$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$, where $x_1 \neq x_2$, and define $f, g \in P_1([0,1])$ as in (2.10) and (2.11), respectively. By definition (1.11) of the norm in $Lip([0,1])$ we get the estimates

$$\left|\frac{h(x_1,u_1) - h(x_2,u_2) - h(x_1,v_1) + h(x_2,v_2)}{x_1 - x_2}\right|$$

$$= \left|\frac{h(x_1,f(x_1)) - h(x_2,f(x_2)) - h(x_1,g(x_1)) + h(x_2,g(x_2))}{x_1 - x_2}\right|$$

$$\leq lip(Hf - Hg) \leq \|Hf - Hg\|_{Lip} \leq L\|f - g\|_{C^1}.$$

Consequently,

$$|h(x_1, u_1) - h(x_2, u_2) - h(x_1, v_1) + h(x_2, v_2)| \leq L|x_1 - x_2| \|f - g\|_{C^1}.$$

Fixing now $x \in [0, 1]$ and letting $x_1 \to x$ and $x_2 \to x$ we obtain, by (2.12),

$$|h(x, u_1) - h(x, u_2) - h(x, v_1) + h(x, v_2)| \leq L(1 + |x|)|u_1 - u_2 - v_1 + v_2|. \tag{2.13}$$

Substituting $u_1 := y + z$, $u_2 := y$, $v_1 := z$ and $v_2 := 0$, and observing that the right hand side of (2.13) then becomes zero, we arrive at the equality

$$h(x, y + z) - h(x, y) - h(x, z) = -\alpha(x), \tag{2.14}$$

where we have used the shortcut $\alpha(x) := h(x, 0)$. Assume first that $\alpha(x) \equiv 0$. Then (2.14) shows that the function $h(t, \cdot)$ satisfies, for each $t \in [0, 1]$, the Cauchy functional equation. Moreover, putting $v_1 = v_2 = 0$ in (2.13) we see that $h(t, \cdot)$ is (Lipschitz) continuous on $\mathbb{R}$. It follows that $h(t, u) = \beta(t)u$ for some function $\beta : [0, 1] \to \mathbb{R}$. In the general case when $\alpha(t) \not\equiv 0$ we pass from h to the function $(t, u) \mapsto h(t, u) - \alpha(t)$, and the statement follows. The assertion $\alpha, \beta \in Y$ follows from the fact that $\alpha(t) = h(t, 0)$ and $\beta(t) = h(t, 1) - h(t, 0)$. □

It is interesting to note that Theorem 2.6 may be strengthened in the autonomous case (1.2) in two different directions: Firstly, the space *Lip* may be replaced by the space Lip_α; secondly, just uniform continuity of H suffices to imply degeneracy of h.

Theorem 2.7. *Suppose that the autonomous composition operator* (1.2) *generated by some function* $h : \mathbb{R} \to \mathbb{R}$ *maps a normed space* X *into a normed space* Y *and is uniformly continuous. Assume that the space* $P_1([0, 1])$ *of affine functions with norm* (1.10) *is imbedded into* X*, and* Y *is imbedded into the space* $Lip_\gamma([0, 1])$ *for some* $\gamma \in (0, 1]$ *with norm* (1.11)*. Then there exist constants* $\alpha, \beta > 0$ *such that* (1.6) *holds true.*

Proof. From our assumptions it follows that we can find a $\delta > 0$ such that $\|Hf - Hg\|_{Lip_\gamma} \leq 1$ for all $f, g \in P_1([0, 1])$ satisfying $\|f - g\|_{C^1} \leq \delta$.

Fix $\omega > 0$ and $v \in [-\delta, \delta]$, and define $f, g \in P_1([0, 1])$ by $f(t) := \omega t + v$ and $g(t) := \omega t$. Since $\|f - g\|_{C^1} = |v| \leq \delta$, we know that $lip_\gamma(Hf - Hg) \leq 1$, hence

$$|h(\omega s + v) - h(\omega s) - h(\omega t + v) + h(\omega t)| \leq |s - t|^\gamma.$$

Putting, in particular, $s = u/\omega$ and $t = 0$, we conclude that

$$|h(u + v) - h(u) - h(v) + h(0)| \leq \left|\frac{u}{\omega}\right|^\alpha \to 0 \qquad (\omega \to \infty).$$

As in the proof of Theorem 2.6, we may suppose without loss of generality that $h(0) = 0$. Then the last equality shows that

$$h(u + v) = h(u) + h(v) \qquad (u, v \in \mathbb{R},\ |v| \leq \delta)$$

which by standard arguments implies that $h(u) = \beta u$ with $\beta = h(1)$. Replacing h by the function $u \mapsto h(u) - h(0)$ as above, the statement follows. $\square$

For the next theorem we need the so-called *left regularization* of $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$h^{\#}(t,u) := \begin{cases} h(0,u) & \text{for} \quad t = 0, \\ \lim\limits_{s \to t-} h(s,u) & \text{for} \quad 0 < t \leq 1. \end{cases} \tag{2.15}$$

Of course, the value of the regularization $h^{\#}(t,u)$ is different from $h(t,u)$ only at points t where the function $h(\cdot,u)$ is not left continuous.

Theorem 2.8. *Suppose that the composition operator* (1.3) *generated by some function* $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ *maps a normed space* X *into a normed space* Y *and satisfies the global Lipschitz condition* (1.5)*. Assume that the space* $P_n([0,1])$ *of polynomials, equipped with the norm of* X*, is imbedded into* X*, and* Y *is imbedded into the space* $WBV_p([0,1])$ *for some* $p \geq 1$*, with norm* (1.14)*. Then there exist functions* $\alpha, \beta \in Y$ *such that*

$$h^{\#}(t,u) = \alpha(t) + \beta(t)u \qquad (0 \leq t \leq 1, u \in \mathbb{R}) \tag{2.16}$$

i.e., the function $h^{\#}(t,\cdot)$ *is affine on* $\mathbb{R}$*. If* Y *is imbedded into the space* $C([0,1])$ *with norm* (1.9)*, then even* (1.7) *holds instead of* (2.16)*.*

Proof. From (1.5) and our hypotheses on X and Y it follows that

$$\|Hf - Hg\|_{WBV_p} \leq L\|f-g\|_X \qquad (f, g \in P_n([0,1])) \tag{2.17}$$

for some $L > 0$. Fix $0 < s < t < 1$, and let $P_m := \{t_0, t_1, \ldots, t_{2m}\}$ be the equidistant partition of the subinterval $[s,t] \subset [0,1]$ defined by

$$t_j - t_{j-1} = \frac{t-s}{2m} \qquad (j = 1, 2, \ldots, 2m).$$

Given $u, v \in \mathbb{R}$ with $u \neq v$, let $f : [0,1] \to \mathbb{R}$ be a polynomial satisfying

$$f(t_{2j}) = v \quad (j = 0, 1, \ldots, m), \qquad f(t_{2j-1}) = \frac{u+v}{2} \quad (j = 1, 2, \ldots, m),$$

and let $g : [0,1] \to \mathbb{R}$ be the polynomial defined by $g(t) := f(t) + \frac{1}{2}(u - v)$. Then we have

$$g(t_{2j-1}) = u, \qquad g(t_{2j}) = \frac{u+v}{2}, \qquad |f(t) - g(t)| \equiv \frac{|u-v|}{2}.$$

Consequently, substituting these functions f and g into (2.17) yields

$$\|Hf - Hg\|_{WBV_p} \leq K|u - v|,$$

where the constant K is given by $K := L\|1\|_X/2$. So for the equidistant partition P_m as above we get

$$\sum_{j=1}^{m} \left|h(t_{2j}, v) - h(t_{2j}, \tfrac{u+v}{2}) - h(t_{2j-1}, \tfrac{u+v}{2}) + h(t_{2j-1}, u)\right|^p$$

$$= \sum_{j=1}^{m} |h(t_{2j}, f(t_{2j})) - h(t_{2j}, g(t_{2j})) - h(t_{2j-1}, f(t_{2j-1})) + h(t_{2j-1}, g(t_{2j-1})|^p$$

$$= \sum_{j=1}^{m} |Hf(t_{2j}) - Hg(t_{2j}) - Hf(t_{2j-1}) + Hg(t_{2j-1})|^p \leq K^p|u - v|^p.$$

Now, since the operator H maps the space of all polynomials into the space $WBV_p([0, 1])$, the function $f(\cdot, z)$ has unilateral limits, for all $z \in \mathbb{R}$, in the interval $[0, 1]$. Therefore we may pass to the left hand limit $s \to t-$ in the last estimate and obtain

$$\sum_{j=1}^{m} \left|h^{\#}(t, v) - h^{\#}(t, \tfrac{u+v}{2}) - h^{\#}(t, \tfrac{u+v}{2}) + h^{\#}(t, u)\right|^p \leq K^p|u - v|^p.$$

Passing now to the limit $m \to \infty$ we further get

$$\sum_{j=1}^{\infty} \left|h^{\#}(t, v) - h^{\#}(t, \tfrac{u+v}{2}) - h^{\#}(t, \tfrac{u+v}{2}) + h^{\#}(t, u)\right|^p \leq K^p|u - v|^p.$$

But this is possible only if

$$h^{\#}(t, v) - h^{\#}(t, \tfrac{u+v}{2}) - h^{\#}(t, \tfrac{u+v}{2}) + h^{\#}(t, u) = 0.$$

This means that the function $h^{\#}(t, \cdot)$ satisfies, for any $t \in [0, 1]$, Cauchy's functional equation

$$2h^{\#}\left(t, \tfrac{u+v}{2}\right) = h^{\#}(t, u) + h^{\#}(t, v) \qquad (0 \leq t \leq 1,\ u, v \in \mathbb{R}).$$

Moreover, the function $h^{\#}(t, \cdot)$ is continuous for any t, by Theorem 2.5. So we conclude that (2.16) holds for some functions $\alpha, \beta : [0, 1] \to \mathbb{R}$ as claimed. The assertion $\alpha, \beta \in Y$ follows from the fact that $\alpha(t) = h^{\#}(t, 0)$ and $\beta(t) = h^{\#}(t, 1) - h^{\#}(t, 0)$.

Finally, the last statement follows from the fact that, if Y is imbedded into $C([0, 1])$, we may carry out all preceding calculations with $h^{\#}$ replaced by h. $\square$

3. THE SPACES C, C^1, AND Lip_α

We start with the space $C([0, 1])$ of continuous functions, where everything is exactly as we expect it.

Theorem 3.1. *The operator* (1.3) *maps the space* $C([0, 1])$ *into itself if and only if the function* h *is continuous on* $[0, 1] \times \mathbb{R}$*. In this case the operator* (1.3) *is automatically bounded and continuous.*

The very simple proof of Theorem 3.1 which builds on the Tietze-Uryson extension lemma shows also that the growth function (1.4) may be explicitly calculated in the space $C([0,1])$ by means of the formula

$$\mu(r) = \max\{|h(t,u)| :\ 0 \leq t \leq 1,\ |u| \leq r\}.$$

The following result was proved in [2], see also [6, Theorem 6.6].

Theorem 3.2. *The operator* (1.3) *satisfies the global Lipschitz condition* (1.5) *in the space* $C([0,1])$ *with norm* (1.9) *if and only if* (2.2) *holds. Similarly, the operator* (1.3) *satisfies the local Lipschitz condition* (1.8) *in the space* $C([0,1])$ *with norm* (1.9) *if and only if* (2.4) *holds.*

In the paper [2] it is also shown that the minimal value of K in (1.5) coincides with the minimal value of k in (2.2), and the minimal value of $K(r)$ in (1.8) coincides with the minimal value of $k(r)$ in (2.4). Moreover, the same numbers k and $k(r)$ are obtained as minimal constants for which the operator (1.3) satisfies a so-called *Darbo condition* which plays a crucial role in the theory and applications of measures of noncompactness and condensing operators (see, e.g., the monographs [1,8,10] or the survey article [3]).

Passing from the space $C([0,1])$ to the space $C^1([0,1])$, the situation changes drastically. Parallel to (2.3) we consider now the local Lipschitz condition

$$|h'(u) - h'(v)| \leq k_1(r)|u-v| \qquad (u, v \in \mathbb{R},\ |u|, |v| \leq r), \tag{3.1}$$

for the derivative of h. We start with the autonomous case (1.2) where we still have the following analogue to Theorem 3.1.

Theorem 3.3. *The autonomous operator* (1.2) *maps the space* $C^1([0,1])$ *into itself if and only if the function h is continuously differentiable on* $\mathbb{R}$*. In this case the operator* (1.2) *is automatically bounded and continuous.*

Theorem 3.3 follows easily from the chain rule. Moreover, the chain rule also allows us to find estimates for the growth function (1.4) in the space $C^1([0,1])$. Indeed, putting in analogy to (2.5)

$$\tilde{k}_1(r) := \sup_{|u|\leq r} |h'(u)| \qquad (r > 0), \tag{3.2}$$

the estimate $\|f\|_{C^1} \leq r$ implies for $g = Hf = h \circ f$ the estimates $|g(0)| = |h(f(0))| \leq \tilde{k}(r)$, with $\tilde{k}(r)$ as in (2.5), and

$$\|g'\|_C = \max_{0\leq x\leq 1} |h'(f(x))f'(x)| \leq \|f'\|_C \max_{0\leq x\leq 1} |h'(f(x))| \leq r\tilde{k}_1(r),$$

where we have used the mean value theorem. Consequently, we get the upper estimate

$$\mu(r) \leq \max\{\tilde{k}(r), r\tilde{k}_1(r)\}$$

for (1.4). Surprisingly enough, Theorem 3.3 is *not* true in the non-autonomous case; the following example is taken from [28], see also [6, Section 8.2].

Example 3.4. Let $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined by

$$h(t,u) := \begin{cases} 0 & \text{if} \quad u \leq 0, \\ 3\dfrac{u^2}{t} - 2\dfrac{u^3}{t\sqrt{t}} & \text{if} \quad 0 < u < \sqrt{t}, \\ 1 & \text{if} \quad u \geq \sqrt{t}. \end{cases} \tag{3.3}$$

A rather cumbersome but straightforward calculation shows then that the operator H generated by this function maps $C^1([0,1])$ into itself, but h is *discontinuous* at $(0,0)$, and so H does not map $C([0,1])$ into itself!

We point out that the non-autonomous operator (1.3) generated by the function (3.3) is even *bounded* in the norm (1.10). In fact, calculating the derivative of $g = Hf$ for $f \in C^1([0,1])$ we obtain

$$g'(t) = \frac{\partial h}{\partial t}(t, f(t)) + \frac{\partial h}{\partial u}(t, f(t))f'(t) = 3\frac{f(t)}{t}\left(1 - \frac{f(t)}{\sqrt{t}}\right)\left(2f'(t) - \frac{f(t)}{t}\right)$$

which remains bounded for $0 < f(t) < \sqrt{t}$. On the other hand, the operator (1.3) with h from Example 3.4 is *not continuous* in the norm (1.10). For example, the constant functions $f_n(t) \equiv 1/n$ which trivially satisfy $\|f_n\|_{C^1} \to 0$ as $n \to \infty$, are mapped by H into the functions

$$Hf_n(t) = h(t, \tfrac{1}{n}) = \begin{cases} 1 & \text{if} \quad 0 \leq t \leq \dfrac{1}{n^2}, \\ \dfrac{1}{n^2 t}\left(3 - \dfrac{2}{n\sqrt{t}}\right) & \text{if} \quad \dfrac{1}{n^2} < t \leq 1 \end{cases}$$

which satisfy $\|Hf_n\|_{C^1} \geq |Hf_n(0)| = 1$.

The discontinuity of H in Example 3.4 is precisely the reason for the pathological behaviour of the function h as the following theorem shows whose proof may be found in [6, Theorem 8.1].

Theorem 3.5. *The operator* (1.3) *maps the space* $C^1([0,1])$ *into itself and is continuous with respect to the norm* (1.10) *if and only if the function* h *is continuously differentiable on* $[0,1] \times \mathbb{R}$.

So if we *add* continuity of H in the norm (1.10), we get the result that we expect in the space $C^1([0,1])$. However, concerning Lipschitz continuity we get a result which is in sharp contrast to Theorem 3.2.

Theorem 3.6. *The operator* (1.3) *satisfies the global Lipschitz condition* (1.5) *in the space* $C^1([0,1])$ *with norm* (1.10) *if and only if* (1.7) *holds for suitable functions* $\alpha, \beta \in C^1([0,1])$. *On the other hand, the autonomous operator* (1.2) *satisfies the local*

Lipschitz condition (1.8) *in the space* $C^1([0,1])$ *with norm* (1.10) *if and only if* (3.1) *holds.*

Proof. Clearly, if h has the form (1.7) then H satisfies (1.5) with $K \leq \|\beta\|_C + \|\beta'\|_C$. The converse implication follows from Theorem 2.5. The assertion about local Lipschitz conditions has been proved in [4]. $\square$

Using the characteristic $\tilde{k}_1(r)$ defined in (3.2), in [4] it is also shown that $K(r)$ from (1.8) may be estimated by

$$K(r) \leq \max\{rk_1(r), \tilde{k}_1(r)\}$$

if $k_1(r)$ from (3.1) is known, and $k_1(r)$ may be estimated by

$$k_1(r) \leq \frac{2K(2r)+1}{r} \tag{3.4}$$

if $K(r)$ is known. The last estimate has an interesting consequence. If (1.5) holds, i.e. K can be chosen independent of r, then (3.4) implies that $k_1(r) \to 0$ as $r \to \infty$. Hence (3.1) implies that that h' is actually constant which means that h has the form (1.6), so we have recovered *en passant* the result from [32,34] (or the first part of Theorem 3.6 in the autonomous case). The same argument shows even more precisely that, for h being non-affine, the function $K = K(r)$ in (1.8) must not only depend on r but even satisfy

$$\liminf_{r\to\infty} \frac{K(r)}{r} > 0,$$

i.e., be of superlinear growth for large values of r.

Now we pass to spaces of Hölder continuous functions where also some unexpected features occur. In the autonomous case (1.2), a necessary and sufficient acting condition has been proved in [24]; we give a simple alternative proof building on Theorem 2.4.

Theorem 3.7. *The autonomous operator* (1.2) *maps the space* $Lip_\alpha([0,1])$ $(0 < \alpha \leq 1)$ *into itself if and only if the function h satisfies condition* (2.3).

Proof. If h satisfies the Lipschitz condition (2.3), an easy computation shows that $H(Lip_\alpha([0,1])) \subseteq Lip_\alpha([0,1])$. Conversely, if H maps $Lip_\alpha([0,1])$ into itself, then H also maps $Lip([0,1])$ into $WBV_{1/\alpha}([0,1])$, by (1.17), and the assertion follows from Theorem 2.4. $\square$

We point out that the space $Lip_\alpha([a,b])$ coincides, for $0 < \alpha < 1$, with the Besov space $B^\alpha_{\infty,\infty}([a,b])$ (in the sense of equivalent norms). Therefore for the proof of Theorem 3.7 we could have also referred to Theorem 5.3.1 in the book [50] which in turn builds on results of Bourdaud and Kateb [15–17].

In the non-autonomous case (1.3) only a very "clumsy" necessary and sufficient condition on h is known under which H maps $Lip_\alpha([0,1])$ into itself, see [6, Theorem 7.1], which reads as follows.

Theorem 3.8. *The operator* (1.3) *maps the space* $Lip_\alpha([0,1])$ $(0<\alpha\le 1)$ *into itself if and only if for all* $(s_0,u_0)\in[0,1]\times\mathbb{R}$ *and all* $r>0$ *we find* $k(r)>0$ *and* $\delta>0$ *such that*

$$|h(s,u)-h(t,v)|\le k(r)\left\{|s-t|^\alpha+\frac{|u-v|}{r}\right\} \tag{3.5}$$

for all $(s,u),(t,v)\in[0,1]\times\mathbb{R}$ *satisfying* $|s-s_0|\le\delta$, $|t-s_0|\le\delta$, $|u-u_0|\le r|s-s_0|^\alpha$, *and* $|v-u_0|\le r|t-s_0|^\alpha$.

If we add *boundedness* of the operator (1.3), however, we get the following more transparent necessary and sufficient condition, see [6, Theorem 7.3] for the proof.

Theorem 3.9. *The operator* (1.3) *maps the space* $Lip_\alpha([0,1])$ *into itself and is bounded with respect to the norm* (1.11) *if and only if the function* h *satisfies the condition*

$$|h(s,u)-h(t,v)|\le k(r)\{|s-t|^\alpha+|u-v|\}\qquad (0\le s,t\le 1,\ |u|,|v|\le r) \tag{3.6}$$

for all $s,t\in[0,1]$. *In particular,* h *is then necessarily continuous on* $[0,1]\times\mathbb{R}$.

In Chapter 7 of the monograph [6] it is also shown that, under the hypotheses of Theorem 3.9, the growth function (1.4) of H in the space $Lip_\alpha([0,1])$ satisfies the two-sided estimate

$$\frac{1}{2^{1-\alpha}+1}k(r)\le\mu(r)\le k(r),$$

with $k(r)$ as in (3.6); in particular,

$$\frac{1}{2}k(r)\le\mu(r)\le k(r)$$

in the space $Lip([0,1])$. Such estimates may be useful for determining a priori estimates or invariant balls when applying fixed point theorems.

Roughly speaking, Theorem 3.9 shows that a mixed Hölder-Lipschitz condition on h guarantees that H maps $Lip_\alpha([0,1])$ into itself *and is bounded*. The following example which is quite similar to Example 3.4 is taken from [13,14], see also [6, Section 7.3].

Example 3.10. Let $h:[0,1]\times\mathbb{R}\to\mathbb{R}$ be defined by

$$h(t,u):=\begin{cases}0 & \text{if}\quad u\le t^{\alpha/2},\\[2mm] \dfrac{1}{u^{2/\alpha}}-\dfrac{t}{u^{4/\alpha}} & \text{if}\quad u>t^{\alpha/2}.\end{cases}$$

Again, a cumbersome calculation shows then that the operator (1.3) generated by this function maps $Lip_\alpha([0,1])$ into itself, but the function h is *discontinuous* at $(0,0)$, and so H does not map $C([0,1])$ into itself!

In contrast to Example 3.4, the reason for the pathological behaviour of the function h in Example 3.10 is the lack of boundedness of the corresponding composition operator H in the norm (1.11), as Theorem 3.9 shows. This may also be proved directly. For example, the constant functions $f_c(t) \equiv c$ $(0 < c \leq 1)$ which all belong to the unit ball in $Lip_\alpha([0,1])$ are mapped by H into the functions

$$Hf_c(t) = h(t,c) = \begin{cases} \dfrac{1}{c^{2/\alpha}} - \dfrac{t}{c^{4/\alpha}} & \text{if} \quad t^{\alpha/2} < c, \\ 0 & \text{if} \quad t^{\alpha/2} \geq c \end{cases}$$

which satisfy $\|Hf_c\|_{Lip_\alpha} \geq |Hf_c(0)| = c^{-2/\alpha} \to \infty$ as $c \to 0$.

From Theorem 3.9 it follows that *the boundedness of the operator H implies the continuity of the function h*. The following remarkable theorem shows that in case of the autonomous operator (1.2) we get the boundedness of H, and so also the continuity of h, as an additional "fringe benefit", see [11,12] or [6, Theorem 7.5] for the proof.

Theorem 3.11. *Suppose that the autonomous operator* (1.2) *maps the space* $Lip_\alpha([0,1])$ $(0 < \alpha \leq 1)$ *into itself. Then H is bounded with respect to the norm* (1.11)*, and h is continuous on* $\mathbb{R}$.

Interestingly, even in the autonomous case the condition $H(Lip_\alpha) \subseteq Lip_\alpha$ does not imply the continuity of H in the norm (1.11), see [11] or [6, Section 7.4].

Example 3.12. Consider the function $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$h(u) := \min\{|u|, 1\}. \tag{3.7}$$

By Theorem 2.1 and Theorem 3.11, the operator H generated by h maps the space $Lip([0,1])$ into itself and is bounded in the norm (1.11). Nevertheless, H is not continuous at $f(t) := t$, say. To see this, consider the sequence $(f_n)_n$ defined by $f_n(t) := t + 1/n$. Clearly, $\|f_n - f\|_{Lip} \to 0$ as $n \to \infty$. On the other hand, from $h(f(t)) = t$ for $0 \leq t \leq 1$ and

$$h(f_n(t)) = \begin{cases} t + \dfrac{1}{n} & \text{for} \quad 0 \leq t \leq \tau_n, \\ 1 & \text{for} \quad \tau_n < t \leq 1, \end{cases}$$

where $\tau_n := (n-1)/n$, it follows that

$$lip(Hf_n - Hf) \geq \frac{|h(f_n(\tau_n)) - h(f_n(1)) - h(f(\tau_n)) + h(f(1))|}{1 - \tau_n} = \frac{1 - \tau_n}{1 - \tau_n} = 1,$$

and so $\|Hf_n - Hf\|_{Lip} \not\to 0$ as $n \to \infty$.

It turns out that the discontinuity of the operator (1.2) which is generated by the function (3.7) in the norm of the space $Lip([0,1])$ is explained by the non-differentiability of this function at zero. In fact, the following somewhat surprising result was proved in [23], see also [24].

Theorem 3.13. *The autonomous operator* (1.2) *maps the space* $Lip_\alpha([0,1])$ *into itself and is continuous with respect to the norm* (1.11) *if and only if the function* h *is continuously differentiable on* $\mathbb{R}$*.*

At this point let us take a breath and summarize the strange boundedness and continuity behaviour of the composition operators (1.2) and (1.3) in a series of tables. First we describe all possible equivalences and implications in the space $C([a,b])$, then in the space $C^1([a,b])$, and then in the space $Lip_\alpha([a,b])$. Whenever we only have an implication in one direction, our counterexamples show that the implication in the opposite direction is false. While there is no difference between the autonomous and non-autonomous case in the space $C([a,b])$, this difference becomes essential in the space $C^1([a,b])$ and even dramatic in the space $Lip_\alpha([a,b])$.

$$\boxed{\begin{array}{ccccc} H \text{ bounded in } C & \Leftrightarrow & H(C) \subseteq C & \Leftrightarrow & H \text{ continuous in } C \\ & & \Updownarrow & & \\ & & h \in C([a,b] \times \mathbb{R}) & & \end{array}}$$

Table 1: The space $C([a,b])$ (autonomous and non-autonomous case)

$$\boxed{\begin{array}{ccccc} H \text{ bounded in } C^1 & \Leftrightarrow & H(C^1) \subseteq C^1 & \Leftrightarrow & H \text{ continuous in } C^1 \\ & & \Updownarrow & & \\ & & h \in C^1(\mathbb{R}) & & \end{array}}$$

Table 2: The space $C^1([a,b])$ (autonomous case)

$$\boxed{\begin{array}{ccccc} H \text{ bounded in } C^1 & \Rightarrow & H(C^1) \subseteq C^1 & \Leftarrow & H \text{ continuous in } C^1 \\ \Uparrow & & & & \Updownarrow \\ h \in C^1([a,b] \times \mathbb{R}) & & & & h \in C^1([a,b] \times \mathbb{R}) \end{array}}$$

Table 3: The space $C^1([a,b])$ (non-autonomous case)

$$\begin{array}{ccccc} H \text{ bounded in } Lip_\alpha & \Leftrightarrow & H(Lip_\alpha) \subseteq Lip_\alpha & \Leftarrow & H \text{ continuous in } Lip_\alpha \\ & & \Updownarrow & & \Updownarrow \\ & & h \in Lip_{loc}(\mathbb{R}) & \Leftarrow & h \in C^1(\mathbb{R}) \end{array}$$

Table 4: The space $Lip_\alpha([a,b])$ (autonomous case)

$$\begin{array}{ccccc} H \text{ bounded in } Lip_\alpha & \Rightarrow & H(Lip_\alpha) \subseteq Lip_\alpha & \Leftarrow & H \text{ continuous in } Lip_\alpha \\ \Updownarrow & & \Updownarrow & & \Uparrow \\ \text{see (3.6)} & \Rightarrow & \text{see (3.5)} & \Leftarrow & h \in C^1([a,b] \times \mathbb{R}) \end{array}$$

Table 5: The space $Lip_\alpha([a,b])$ (non-autonomous case)

Concerning global and local Lipschitz continuity, the operator H exhibits in the space $Lip_\alpha([0,1])$ the same behaviour as in the space $C^1([0,1])$.

Theorem 3.14. *The operator* (1.3) *satisfies the global Lipschitz condition* (1.5) *in the space* $Lip_\alpha([0,1])$ *with norm* (1.11) *if and only if* (1.7) *holds for suitable functions* $\alpha, \beta \in Lip_\alpha([0,1])$. *On the other hand, the autonomous operator* (1.2) *satisfies the local Lipschitz condition* (1.8) *in the space* $Lip_\alpha([0,1])$ *with norm* (1.11) *if and only if* (3.1) *holds.*

Proof. Clearly, if (1.7) holds, a straightforward calculation shows that

$$|Hf(0)| \leq |\alpha(0)| + |\beta(0)|\,|f(0)|, \quad |Hf(x) - Hf(y)| \leq \|\beta\|_C |f(x) - f(y)|,$$

and so the operator (1.3) satisfies (1.5) with $k := \max\{|\alpha(0)|, 2\|\beta\|_C\}$. The fact that (1.5) implies (1.7) was proved in case $\alpha = 1$ in [33], and in case $0 < \alpha < 1$ in [31].

Suppose now that the derivative h' of h satisfies the local Lipschitz condition (3.1), and consider again the constant $\tilde{k}_1(r)$ given in (3.2). Fix $f, g \in Lip_\alpha([0,1])$ with $\|f\|_{Lip_\alpha} \leq r$ and $\|g\|_{Lip_\alpha} \leq r$. For $s, t \in [0,1]$ with $s \neq t$ we get then

$$|Hf(s) - Hg(s) - Hf(t) + Hg(t)| = |h(f(s)) - h(g(s)) - h(f(t)) + h(g(t))|$$

$$\leq 2k_1(r)(|f(s) - f(t)| + |g(s) - g(t)|)\|f - g\|_C + \tilde{k}_1(r)|f(s) - g(s) - f(t) + g(t)|.$$

Dividing by $|s - t|^\alpha$ and passing to the supremum over $s \neq t$ we arrive at

$$lip_\alpha(Hf - Hg) \leq 2k_1(r)\,[lip_\alpha(f) - lip_\alpha(g)]\,\|f - g\|_C + \tilde{k}_1(r) lip_\alpha(f - g)$$

$$\leq \max\,\{4rk_1(r), \tilde{k}_1(r)\}\|f - g\|_{Lip_\alpha}.$$

The fact that (1.8) implies the differentiability of h and condition (3.1) for h' was proved in [4]. $\square$

At this point we can make the same remarks as after Theorem 3.6. Namely, it was shown in [4] that, if the autonomous operator H satisfies (1.8) for some $K(r)$, then the derivative h' of the corresponding function h satisfies (3.1) for some $k_1(r)$ such that

$$k_1(r) \leq \frac{K(r+R)}{R} \tag{3.8}$$

for any $R \geq 2r$. In particular, in case of the global Lipschitz condition (1.5), i.e., $K(r) \equiv K$, letting $R \to \infty$ in (3.8) leads to $k_1(r) = 0$ which means that h has the form (1.6). So we regain the degeneracy results from [31] and [33] in this way. More generally, in the same manner one can show that for non-affine h any function $K = K(r)$ satisfying (1.8) must grow at least linearly at ∞.

4. THE SPACES AC, BV, RBV_p AND WBV_p

From Theorem 2.1 we may conclude that condition (2.3) is necessary for the operator (1.2) to map any intermediate space X between $Lip([0,1])$ and $BV([0,1])$ into itself. We have already used this fact for $X = Lip_\alpha([0,1])$ in Theorem 3.7. Likewise, the chain of inclusions (1.16) shows that the following is true.

Theorem 4.1. *The autonomous operator* (1.2) *maps any of the spaces* $RBV_p([0,1])$, $AC([0,1])$, $BV([0,1])$, *or* $WBV_p([0,1])$ *into itself if and only if the function* h *satisfies condition* (2.3). *In this case the operator* (1.2) *is automatically bounded.*

Proof. Suppose first that h satisfies the Lipschitz condition (2.3), and let X be any of the spaces $RBV_p([0,1])$, $AC([0,1])$, $BV([0,1])$, or $WBV_p([0,1])$. Since all functions in X are bounded, for any $f \in X$ there is some $r > 0$ such that $|f(x)| \leq r$ for $0 \leq x \leq 1$, and so

$$\sum_{k=1}^{m} |h(f(b_k)) - h(f(a_k))| \leq k(r) \sum_{k=1}^{m} |f(b_k) - f(a_k)| \tag{4.1}$$

for any collection of non-overlapping intervals $[a_1, b_1], \ldots, [a_m, b_m] \subseteq [0,1]$. This estimate shows that, whenever f belongs to X, then $Hf = h \circ f$ belongs to X as well.

Conversely, let now be $X \in \{RBV_p([0,1]), AC([0,1]), BV([0,1])\}$, and suppose that H maps X into itself. Then H also maps $Lip([0,1])$ into $BV([0,1])$, by (1.17), and the assertion follows from Theorem 2.1. The assertion for the space $WBV_p([0,1])$ for $p > 1$ follows from Theorem 2.4 (b) for $\alpha = 1$. The estimate (4.1) also shows that H is bounded in these spaces. □

If we consider the functions h, f, and f_n as in Example 3.12, a trivial calculation shows that

$$\mathrm{Var}(Hf_n - Hf; [0,1]) = \frac{1}{n} \to 0 \quad (n \to \infty)$$

as well as

$$\mathrm{Var}_p^R(Hf_n - Hf; [0,1]) = \mathrm{Var}_p^W(Hf_n - Hf; [0,1]) = \frac{1}{n^p} \quad (n \to \infty),$$

and so the operator H generated by the function (3.7) is continuous in the norms (1.12), (1.14) and (1.15). Loosely speaking, this shows that only in the Hölder space $Lip_\alpha([0,1])$ the operator (1.2) has a particularly poor continuity behaviour. As far as we know, it is an open problem whether or not the operator (1.2) is automatically in the spaces covered by Theorem 4.1. Moreover, continuity conditions in these spaces, both necessary and sufficient, are not known.

We have illustrated Theorem 2.1 which is the essence of Theorems 3.7 and 4.1, by means of Example 2.2. Of course, the seagull function h from Example 2.2 does not belong to $Lip([0,1])$, but only to $Lip_{1/2}([0,1])$, and so also to $WBV_2([0,1])$, by (1.17). A slight modification leads to a more general example of a function $h \in Lip_\alpha([0,1])$, for fixed $\alpha \in (0,1)$, such that neither $H(AC) \subseteq AC$ nor $H(BV) \subseteq BV$ nor $H(WBV_p) \subseteq WBV_p$.

Example 4.2. For $0 < \alpha < 1$, consider the generalized seagull function $h_\alpha : \mathbb{R} \to \mathbb{R}$ defined by $h_\alpha(u) := \min\{|u|^\alpha, 1\}$, and denote the corresponding autonomous operator (1.2) by H_α. In contrast to Example 2.2, we consider now the zigzag function $f : [0,1] \to \mathbb{R}$ defined for $p \geq 1$ by

$$f(t) := \begin{cases} 0 & \text{if } t = 0 \text{ or } t = \dfrac{1}{2n-1} \ (n \in \mathbb{N}), \\ \dfrac{1}{n^{1/p\alpha}} & \text{if } t = \dfrac{1}{2n} \ (n \in \mathbb{N}), \\ \text{linear} & \text{otherwise.} \end{cases}$$

A straightforward calculation shows that

$$\mathrm{Var}_p^W(f; [0,1]) = 2\sum_{n=1}^{\infty} \frac{1}{n^{1/\alpha}} < \infty,$$

since $\alpha < 1$, and so $f \in WBV_p([0,1])$ (in particular, $f \in BV([0,1])$ for $p = 1$). Moreover, f is certainly continuous and has the Luzin property (i.e., maps nullsets into nullsets). So from the classical Vitali-Banach-Zaretskij theorem [26] it follows that $f \in AC([0,1])$. On the other hand, the fact that $h_\alpha(f(\frac{1}{2n})) = \frac{1}{n^{1/p}}$ and $h_\alpha(f(\frac{1}{2n-1})) = 0$ implies that

$$\mathrm{Var}_p^W(H_\alpha f; [0,1]) \geq 2\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

This shows that $H_\alpha f \notin WBV_p([0,1])$, and so $H_\alpha f \notin BV([0,1])$ and $H_\alpha f \notin AC([0,1])$ either.

In the non-autonomous case (1.3), only sufficient conditions are known. The following natural condition was formulated and proved in [29].

Theorem 4.3. *Suppose that the function $h(\cdot, u) : [0,1] \to \mathbb{R}$ has bounded variation, uniformly w.r.t. $u \in \mathbb{R}$, and the function $h(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, uniformly w.r.t. $t \in [0,1]$. Then the operator* (1.3) *maps the space $BV([0,1])$ into itself. In this case the operator* (1.3) *is automatically bounded.*

Proof. The proof is almost trivial. By assumption we know that (2.4) is true, where $k(r)$ does not depend on $t \in [0,1]$. Given $f \in BV([0,1])$ and any partition $P = \{t_0, \ldots, t_m\}$ of $[0,1]$, the function $g(t) = Hf(t) = h(t, f(t))$ satisfies

$$\mathrm{Var}(g; [0,1]) \leq k(\|f\|_\infty)\mathrm{Var}(f; [0,1]), \tag{4.2}$$

where $\|f\|_\infty$ denotes the supremum norm (2.8) of f. Now, if $\|f\|_{BV} \leq r$ the estimates (4.2) and $\|f\|_\infty \leq \|f\|_{BV}$ imply the upper estimate

$$\mu(r) \leq k(r)r + \sup_{|u| \leq r} |h(0,u)|$$

for the growth function (1.4) which shows that the operator H is bounded. □

A natural question is whether or not the conditions given in Theorem 4.3 are also necessary for H to map $BV([0,1])$ into itself. The following example shows that the answer is negative.

Example 4.4. Let $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined by

$$h(t,u) := \begin{cases} \operatorname{sgn} u & \text{if } \ t = 0, \\ 0 & \text{if } \ 0 < t \leq 1, \end{cases}$$

where sgn denotes the sign function. Then the corresponding operator (1.3) maps the space $WBV_p([0,1])$, for $1 \leq p < \infty$, into itself, since $\mathrm{Var}_p^W(Hf; [0,1]) = |h(0, f(0))|^p \leq 1$ for any $f \in WBV_p([0,1])$. This equality also shows that $\|Hf\|_{WBV_p} \leq 2$ for all $f \in WBV_p([0,1])$ which shows that the operator H is bounded in the norm (1.14). On the other hand, the function $h(0, \cdot)$ is not even continuous at zero.

Example 4.4 shows that the operator (1.3) is not automatically continuous if it maps the space $WBV_p([0,1])$ into itself. To see this, it suffices to consider the sequence $f_n(t) \equiv 1/n$.

Concerning global Lipschitz continuity, we have a somewhat different degeneracy phenomenon than in the spaces $C^1([0,1])$ and $Lip_\alpha([0,1])$. It is here that we have to use the left regularization $h^\#$ of h introduced in (2.15). Indeed, as an immediate consequence of Theorem 2.8 we get the following

Theorem 4.5. *The operator* (1.3) *satisfies the global Lipschitz condition* (1.5) *in the space $WBV_p([0,1])$ with norm* (1.14) (*in particular, in the space $BV([0,1])$ with norm* (1.12)) *if and only if* (2.16) *holds. On the other hand, the operator* (1.3) *satisfies the global Lipschitz condition* (1.5) *in the space $RBV_p([0,1])$ for $p > 1$ with norm* (1.15) *if and only if* (1.7) *holds for suitable functions $\alpha, \beta \in RBV_p([0,1])$.*

We remark that the first statement in Theorem 4.5 was proved for $BV([a,b])$ in [37]. The following example shows that, in contrast to the situation in C^1, Lip_α, or RBV_p, the Lipschitz condition (1.5) in the norm of BV or WBV_p need not lead to the form (1.7) of the function h itself.

Example 4.6. Let $\{r_0, r_1, r_2, \ldots\}$ be an enumeration of all rational numbers in $[0,1]$ ($r_0 := 0$), and let $\psi : \mathbb{R} \to \mathbb{R}$ be any function satisfying $\psi(0) = 0$ and $|\psi(u) - \psi(v)| \leq L|u - v|$. We define $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ by

$$h(t,u) := \begin{cases} \dfrac{\psi(u)}{2^k} & \text{if } t = r_k, \\ 0 & \text{otherwise.} \end{cases}$$

For any partition $P = \{t_0, \ldots, t_m\}$ of $[0,1]$ and $f \in WBV_p([0,1])$ we have then

$$\sum_{j=1}^{m} |Hf(t_j) - Hf(t_{j-1})|^p \leq 2\sum_{k=0}^{\infty} |h(r_k, f(r_k))|^p \leq 2L^p \sum_{k=0}^{\infty} \frac{|f(r_k)|^p}{2^{pk}} < \infty,$$

which shows that H maps the space $WBV_p([0,1])$ into itself. Furthermore, for $f, g \in WBV_p([0,1])$ and the same partition P as above we obtain the estimate

$$\sum_{j=1}^{m} |Hf(t_j) - Hg(t_j) - Hf(t_{j-1}) + Hg(t_{j-1})|^p \leq 2\sum_{k=0}^{\infty} |h(r_k, f(r_k)) - h(r_k, g(r_k))|^p$$

$$\leq 2\sum_{k=0}^{\infty} \frac{|\psi(f(r_k)) - \psi(g(r_k))|^p}{2^{pk}} \leq 2L^p \sum_{k=0}^{\infty} \frac{|f(r_k) - g(r_k))|^p}{2^{pk}} \leq 2L^p \|f - g\|_{WBV_p}^p.$$

This together with the trivial estimate $|Hf(0) - Hg(0)| \leq L|f(0) - g(0)|$ shows that H satisfies the global Lipschitz condition (1.5) with $K = 2L^p$, although h is not of the form (1.7).

It is not hard to see that $h^\#(t,u) \equiv 0$ for the function h in Example 4.6, in accordance with Theorem 4.5. Concerning the local Lipschitz condition (1.8), we have exactly the same situation as in the spaces $C^1([0,1])$ and $Lip_\alpha([0,1])$. The following result was proved for $BV([0,1])$ and $AC([0,1])$ in [4], for $RBV_p([0,1])$ in [5].

Theorem 4.7. *Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Then the autonomous operator* (1.2) *satisfies the local Lipschitz condition* (1.8) *in the space $BV([0,1])$ with norm* (1.12)*, $AC([0,1])$ with norm* (1.13)*, or $RBV_p([0,1])$ with norm* (1.15) *if and only if* (3.1) *holds.*

Again, at this point we can make the same remarks on the "interdependence" of the Lipschitz constants $K(r)$ and $k_1(r)$ as after Theorem 3.6. In fact, it is proved in [4] that $K(r)$ may be estimated by $K(r) \leq \max\{4rk_1(r), \tilde{k}_1(r)\}$ if $k_1(r)$ is known, where $\tilde{k}_1(r)$ is given by (3.2), and $k_1(r)$ may be estimated by $k_1(r) \leq K(3r+1)$ if $K(r)$ is known.

5. A COMPARISON OF SPACES

For the reader's ease we summarize part of the results scattered over the preceding 3 sections now in some synoptic tables. In the following Table 6 we compare different conditions on $h : [0,1] \to \mathbb{R}$ under which the corresponding operator H maps every function from some space $X = X([0,1])$ whose range is in the domain $[0,1]$ of h, into the same space $X = X([0,1])$. The spaces under consideration are $X \in \{AC, BV, RBV_p, WBV_p\}$. This table exhibits a certain *asymmetry* in such conditions inasmuch as the requirement $h \in X$ is, for these spaces, *never* sufficient for guaranteeing that $H(X) \subseteq X$. The "correct" condition on h is throughout local Lipschitz continuity, even local Hölder continuity does not suffice.

$f \in BV([0,1]),\ \ h \in BV([0,1])$	$\not\Rightarrow$	$h \circ f \in BV([0,1])$	(Example 2.2)
$f \in BV([0,1]),\ \ h \in Lip_\alpha([0,1])$	$\not\Rightarrow$	$h \circ f \in BV([0,1])$	(Example 2.2)
$f \in BV([0,1]),\ \ h \in Lip([0,1])$	$\Leftrightarrow$	$h \circ f \in BV([0,1])$	(Theorem 2.1)
$f \in RBV_p([0,1]),\ \ h \in RBV_p([0,1])$	$\not\Rightarrow$	$h \circ f \in RBV_p([0,1])$	(Example 2.2)
$f \in RBV_p([0,1]),\ \ h \in Lip_\alpha([0,1])$	$\not\Rightarrow$	$h \circ f \in RBV_p([0,1])$	(Example 2.2)
$f \in RBV_p([0,1]),\ \ h \in Lip([0,1])$	$\Leftrightarrow$	$h \circ f \in RBV_p([0,1])$	(Theorem 2.1)
$f \in WBV_p([0,1]),\ \ h \in WBV_p([0,1])$	$\not\Rightarrow$	$h \circ f \in WBV_p([0,1])$	(Example 4.2)
$f \in WBV_p([0,1]),\ \ h \in Lip_\alpha([0,1])$	$\not\Rightarrow$	$h \circ f \in WBV_p([0,1])$	(Example 4.2)
$f \in WBV_p([0,1]),\ \ h \in Lip([0,1])$	$\Leftrightarrow$	$h \circ f \in WBV_p([0,1])$	(Theorem 2.4)
$f \in AC([0,1]),\ \ h \in AC([0,1])$	$\not\Rightarrow$	$h \circ f \in AC([0,1])$	(Example 4.2)
$f \in AC([0,1]),\ \ h \in Lip_\alpha([0,1])$	$\not\Rightarrow$	$h \circ f \in AC([0,1])$	(Example 4.2)
$f \in AC([0,1]),\ \ h \in Lip([0,1])$	$\Leftrightarrow$	$h \circ f \in AC([0,1])$	(Theorem 2.1)

Table 6: Asymmetry in the composition $f \mapsto h \circ f$

The important point in this table is of course that the crucial condition $h \in Lip([0,1])$ in every row where a theorem is cited is also *necessary* for the operator H to map the underlying space into itself. More precisely, the equivalence arrow $\Leftrightarrow$ in the third row, say, means that $h \circ f \in BV([0,1])$ *for all* functions $f \in BV([0,1])$ satisfying $f([0,1]) \subseteq [0,1]$ *if and only if* $h \in Lip([0,1])$, and similarly for the other three equivalence arrows.

We remark that the crucial local Lipschitz condition (2.3) for h is equivalent to the mapping condition $H(X) \subseteq X$ in many more spaces. For example, this equivalence was proved in [20] for the space $HBV([a,b])$ of functions of bounded harmonic variation, and in [49] (see also [22]) for both the space $\Lambda BV([a,b])$ of functions of bounded Λ-variation in Waterman's sense [53] and the space $\Phi BV([a,b])$ of functions of bounded Φ-variation in Wiener's sense [49,51] whose definition uses the concept of Young functions [54]. Other spaces with this property are described in [25,45, 49], see also the monograph [47].

In view of this one could be inclined to suspect that (2.3) is equivalent to $H(X) \subseteq X$ for virtually all important spaces arising in applications. However, this is far from being true. For instance, Theorem 3.1 shows that (2.3) is *too strong* in the space $X = C([0,1])$, because the mere continuity of h ensures that $H(C) \subseteq C$. On the other hand, there are function spaces X for which we have the opposite effect, i.e., (2.3) is *too weak* to ensure the inclusion $H(X) \subseteq X$; a simple example is $X = C^1([0,1])$. Here is a more interesting and "exotic" example of such a space.

Denote by $X = D^{-1}([a,b])$ the (linear) space of all functions f having a primitive on $[a,b]$, i.e., $f = F'$ for some differentiable function F. Here the class of "admissible" transformations is extremely poor, as the following surprising result from [7] shows.

Theorem 5.1. *The autonomous operator* (1.2) *maps the space* $D^{-1}([a,b])$ *into itself if and only if the function* h *has the form* (1.6)*, i.e., is affine.*

So even a very harmless looking nonlinearity like $h(u) = u^2$ generates a composition operators H which does not always map a function with primitive to a function with primitive. In particular, this means that $D^{-1}([a,b])$ is not an algebra with respect to pointwise multiplication of functions. We illustrate this by means of a typical example which must involve, of course, a *discontinuous* function with primitive. By the Darboux intermediate value theorem, the discontinuity cannot be of first kind (a jump), but must be of second kind (oscillatory).

Example 5.2. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(u) := u^2$, and consider the function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) := \begin{cases} \sin \dfrac{1}{x} & \text{for} \quad 0 < x \leq 1, \\ 0 & \text{for} \quad x = 0. \end{cases}$$

Clearly, f has a discontinuity of second kind at zero. Now, a straightforward calculation shows that $f \in D^{-1}([0,1])$ with primitive

$$F(x) = \begin{cases} x^2 \cos \dfrac{1}{x} - 2 \displaystyle\int_0^x t \cos \dfrac{1}{t}\, dt & \text{for} \quad 0 < x \leq 1, \\ 0 & \text{for} \quad x = 0 \end{cases}$$

on the whole interval $[0,1]$. The function $g = Hf = f^2$ has the form

$$g(x) = \begin{cases} \sin^2 \dfrac{1}{x} & \text{for} \quad 0 < x \leq 1, \\ 0 & \text{for} \quad x = 0. \end{cases}$$

It is not hard to see that g has the primitive

$$G(x) = \frac{x}{2} + \frac{x^2}{4} \sin \frac{2}{x} - \frac{1}{2} \int_0^x t \sin \frac{2}{t} \, dt,$$

i.e., $G' = g$, but *only on the interval* $(0,1]$. Indeed, L'Hospital's rule shows that

$$\lim_{x \to 0+} \frac{G(x)}{x} = \frac{1}{2},$$

and so this primitive has the "wrong derivative" at zero. We conclude that $g \notin D^{-1}([0,1])$ and so $H(D^{-1}) \not\subseteq D^{-1}$ as claimed.

It is interesting (and nontrivial) to ask for which values of $\alpha, \beta \in \mathbb{R}$ the oscillatory function $f_{\alpha,\beta} : [0,1] \to \mathbb{R}$ given by

$$f_{\alpha,\beta}(x) := \begin{cases} x^\alpha \sin x^\beta & \text{for} \quad 0 < x \leq 1, \\ 0 & \text{for} \quad x = 0. \end{cases}$$

is continuous, admits a primitive, or has the intermediate value property. Although this is somewhat beyond the scope of this survey, we summarize the answer to this question in the following

Theorem 5.3. *The function $f_{\alpha,\beta}$ is continuous on $[0,1]$ if and only if $\alpha > 0$, or $\alpha \leq 0$ and $\beta > -\alpha$. The function $f_{\alpha,\beta}$ admits a primitive on $[0,1]$ if and only if $\alpha \geq 0$ and $\beta \neq 0$, or $\alpha < 0$ and $|\beta| > -\alpha$. The function $f_{\alpha,\beta}$ has the intermediate value property on $[0,1]$ if and only if $\alpha \geq 0$ and $\beta \neq 0$, or $\alpha < 0$ and $\beta > -\alpha$, or $\alpha < 0$ and $\beta < 0$.*

To conclude, we give in the following Table 7 (autonomous case) and Table 8 (non-autonomous case) conditions on h under which the operators (1.2) resp. (1.3) map a certain function space into itself, are automatically bounded or continuous, and satisfy a (global or local) Lipschitz condition. We mention only conditions which are *both necessary and sufficient*; conditions which are just sufficient are easily found and therefore not so interesting.

function space X	*$H(X) \subseteq X$ criterion*	*automatic boundedness*	*automatic continuity*	*global Lip-continuity*	*local Lip-continuity*
$C([a,b])$	$h \in C$ (Thm. 3.1)	yes (Thm. 3.1)	yes (Thm. 3.1)	$h \in Lip$ (Thm. 3.2)	$h \in Lip_{loc}$ (Thm. 3.2)
$C^1([a,b])$	$h \in C^1$ (Thm. 3.3)	yes (Thm. 3.3)	yes (Thm. 3.3)	h affine (Thm. 3.6)	$h' \in Lip_{loc}$ (Thm. 3.6)
$Lip_\alpha([a,b])$ $(0 < \alpha \leq 1)$	$h \in Lip_{loc}$ (Thm. 3.7)	yes (Thm. 3.11)	no (Ex. 3.12)	h affine (Thm. 3.14)	$h' \in Lip_{loc}$ (Thm. 3.14)
$AC([a,b])$	$h \in Lip_{loc}$ (Thm. 4.1)	yes (Thm. 4.1)		h affine (Thm. 2.8)	$h' \in Lip_{loc}$ (Thm. 4.7)
$BV([a,b])$	$h \in Lip_{loc}$ (Thm. 4.1)	yes (Thm. 4.1)		$h^\#$ affine (Thm. 4.5)	$h' \in Lip_{loc}$ (Thm. 4.7)
$WBV_p([a,b])$ $(1 < p < \infty)$	$h \in Lip_{loc}$ (Thm. 4.1)	yes (Thm. 4.1)		$h^\#$ affine (Thm. 4.5)	$h' \in Lip_{loc}$ (Thm. 4.7)
$RBV_p([a,b])$ $(1 < p < \infty)$	$h \in Lip_{loc}$ (Thm. 4.1)	yes (Thm. 4.1)		h affine (Thm. 2.8)	$h' \in Lip_{loc}$ (Thm. 4.7)

Table 7: The autonomous operator $Hf(t) = h(f(t))$

function space X	*$H(X) \subseteq X$ criterion*	*automatic boundedness*	*automatic continuity*	*global Lip-continuity*	*local Lip-continuity*
$C([a,b])$	$h \in C$ (Thm. 3.1)	yes (Thm. 3.1)	yes (Thm. 3.1)	$h \in Lip$ (Thm. 3.2)	$h \in Lip_{loc}$ (Thm. 3.2)
$C^1([a,b])$			no (Ex. 3.4)	h affine (Thm. 3.6)	
$Lip_\alpha([a,b])$ $(0 < \alpha \leq 1)$	see (3.4) (Thm. 3.8)	no (Ex. 3.10)	no (Ex. 3.12)	h affine (Thm. 3.13)	
$AC([a,b])$				h affine (Thm. 3.13)	
$BV([a,b])$			no (Ex. 4.4)	$h^\#$ affine (Thm. 4.5)	
$WBV_p([a,b])$ $(1 < p < \infty)$			no (Ex. 4.4)	$h^\#$ affine (Thm. 4.5)	
$RBV_p([a,b])$ $(1 < p < \infty)$				h affine (Thm. 4.5)	

Table 8: The non-autonomous operator $Hf(t) = h(t, f(t))$

We point out again that the two tables only contain conditions which are both necessary and sufficient. Since such conditions are much more difficult to find in the non-autonomous case than in the autonomous case, it is not surprising that the regions of *terra incognita* in Table 8 are more numerous than in Table 7. In some cases we have found counterexamples for the operator (1.3), like the striking Examples 3.4 and 3.10; on the other hand, almost nothing is known for this operator in spaces of absolutely continuous functions or functions of bounded variation.

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