Communications in Applied Analysis **15** (2011), no. 2, 3 and 4, 183–194

ASYMPTOTIC BEHAVIOR OF n-TH ORDER SUBLINEAR DYNAMIC EQUATIONS ON TIME SCALES

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This paper is dedicated to Professor Jeffrey Webb

ABSTRACT. In this paper, we study the asymptotic behavior of the following n-th order sublinear dynamic equation

$$x^{\Delta^n}(t) + p(t)x^{\alpha}(t) = 0, \quad 0 < \alpha < 1,$$

where $p(t) \geq 0$ on an isolated time scale $\mathbb{T}$, and α is a ratio of odd positive integers. As an application, we obtain

(i) when n is even, every solution $x(k)$ of the difference equation

$$\Delta^n x(k) + p(k)x^{\alpha}(k) = 0, \quad 0 < \alpha < 1,$$

where $p(k) \geq 0$, is oscillatory if and only if

$$\sum_{k=1}^{\infty} k^{\alpha(n-1)} p(k) = \infty.$$

(ii) when n is odd, every solution $x(k)$ of the difference equation is either oscillatory or $\lim_{k\to\infty} x(k) = 0$ if and only if the above sum diverges.

AMS (MOS) Subject Classification. 34K11, 39A10, 39A99.

1. INTRODUCTION

Consider the following n-th order sublinear dynamic equation on a time scale

$$x^{\Delta^n}(t) + p(t)x^{\alpha}(t) = 0, \quad 0 < \alpha < 1, \tag{1.1}$$

Received July 21, 2010

where $p(t) \geq 0$, $n \geq 2$, $\mathbb{T}$ is an isolated time scale, and α is a ratio of odd positive integers.

When $n = 2$, equation (1.1) is the second order sublinear dynamic equation

$$x^{\Delta\Delta}(k) + p(k)x^{\alpha}(k) = 0, \quad 0 < \alpha < 1. \tag{1.2}$$

In [4], the present authors proved that if $\int_{t_0}^{\infty} t^{\alpha} p(t)\Delta t < \infty$, then (1.2) has a solution $x(t)$ with the property that

$$\lim_{t\to\infty} \frac{x(t)}{t} = A \neq 0.$$

In this paper, we extend the results of [4] to the n-th order sublinear dynamic equation (1.1) on an isolated time scale. As an application, we prove that

(i) when n is even, every solution $x(k)$ of the difference equation

$$\Delta^n x(k) + p(k)x^{\alpha}(k) = 0, \quad 0 < \alpha < 1, \tag{1.3}$$

where $p(k) \geq 0$, is oscillatory if and only if

$$\sum_{k=1}^{\infty} k^{\alpha(n-1)} p(k) = \infty. \tag{1.4}$$

(ii) when n is odd, every solution $x(k)$ of the difference equation (1.3) is either oscillatory or $\lim_{k\to\infty} x(k) = 0$ if and only if (1.4) holds. In a landmark paper, Licko and Svec (in the continuous case) consider the convergence of the integral corresponding to (1.4) and the asymptotic behavior of solutions of the continuous version of (1.1).

For completeness, (see [8] and [9] for elementary results concerning time scale calculus), we recall some basic results for dynamic equations and the calculus on time scales. Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. The forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$ we say t is right-dense, while if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$ we say t is left-dense. Given a time scale interval $[c, d]_{\mathbb{T}} := \{t \in \mathbb{T} : c \leq t \leq d\}$ in $\mathbb{T}$ the notation $[c, d]^{\kappa}_{\mathbb{T}}$ denotes the interval $[c, d]_{\mathbb{T}}$ in case $\rho(d) = d$ and denotes the interval $[c, d)_{\mathbb{T}}$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$

the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. We say that $x : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}$ provided

$$x^\Delta(t) := \lim_{s\to t} \frac{x(t)-x(s)}{t-s},$$

exists when $\sigma(t) = t$ (here by $s \to t$ it is understood that s approaches t in the time scale) and when x is continuous at t and $\sigma(t) > t$

$$x^\Delta(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$ the delta derivative is just the forward difference operator. We say that $\mathbb{T}$ is an isolated time scale provided there are no points in $\mathbb{T}$ that are either left-dense or right-dense. The results obtained here contain the usual discrete cases as special cases and generalize these results to several other isolated time scales (for example for the time scale $q^{\mathbb{N}_0} := \{1, q, q^2, \cdots\}$, $q > 1$, which is very important in quantum theory [11]).

2. LEMMAS

Assume that $\mathbb{T} = \{t_k\}_{k=0}^\infty$ where $1 < t_0 < t_1 < \cdots < t_k \cdots$, with $t_k \to \infty$.

Condition (D): We say that $\mathbb{T}$ satisfies condition (D) if there exists $L > 0$ such that

$$t_{k-1} \geq Lt_k, \quad \text{for all} \quad k \geq 1.$$

Clearly, if $\mathbb{T} = h\mathbb{N}_0$, $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, or $\mathbb{T}$ is the set of harmonic numbers [8, Example 1.45] then $\mathbb{T}$ satisfies condition (D). but it is easy to show that $\mathbb{T} = \{2^{2^k}, k \in \mathbb{N}_0\}$, does not satisfy condition (D).

We will use the following time scale version of Taylor's Theorem.

Lemma 2.1. [8, Theorem 1.113] *Let $n \in \mathbb{N}$. Suppose that f is n times differentiable on $\mathbb{T}^{\kappa^n}$. Let $t_0 \in \mathbb{T}^{\kappa^{n-1}}$, $t \in \mathbb{T}$, and define the functions $h_k(r, s)$ by*

$$h_0(r, s) \equiv 1, \quad \textit{and} \quad h_{k+1}(r, s) = \int_s^r h_k(\tau, s)\Delta\tau, \quad \textit{for} \quad k \in \mathbb{N}_0.$$

Then we have

$$f(t) = \sum_{k=0}^{n-1} h_k(t, t_0)f^{\Delta^k}(t_0) + \int_{t_0}^{\rho^{n-1}(t)} h_{n-1}(t, \sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau.$$

The following lemma gives an estimation for $h_k(t, t_0)$.

Lemma 2.2. *Assume that $\mathbb{T}$ satisfies condition (D). Then for any $m \geq 1$, there exists $\epsilon_m > 0$ such that*

$$h_m(t, t_0) \geq \epsilon_m t^m \tag{2.1}$$

for $t > t_0$.

Proof. We prove this result by induction. When $m = 1$, we have

$$h_1(t, t_0) = t - t_0 = \epsilon_1 t,$$

for $t > t_0$, where $\epsilon_1 = 1 - \frac{t_0}{t_1}$.

Suppose that when $m = k$, (2.1) holds. Then when $m = k + 1$, supposing $\tau_1 = t_l \in \mathbb{T}, l \geq 1$, then we have (note that $\mathbb{T}$ satisfies condition (D))

$$\begin{aligned} h_{k+1}(\tau_1, t_0) &= \int_{t_0}^{\tau_1} h_k(\tau_2, t_0)\Delta\tau_2 \\ &\geq \epsilon_k \int_{t_0}^{\tau_1} \tau_2^k \Delta\tau_2 \\ &= \epsilon_k[t_0^k(t_1 - t_0) + t_1^k(t_2 - t_1) + \cdots + t_{l-1}^k(t_l - t_{l-1})] \\ &\geq \epsilon_k L^k[t_1^k(t_1 - t_0) + t_2^k(t_2 - t_1) + \cdots + t_l^k(t_l - t_{l-1})] \\ &\geq \epsilon_k L^k \int_{t_0}^{t_l} \tau^k d\tau \\ &\geq \epsilon_{k+1} t_l^{k+1}, \end{aligned}$$

for $\tau_1 > t_0$, where $\epsilon_{k+1} = \frac{\epsilon_k L^k}{k+1}\left[1 - \left(\frac{t_0}{t_1}\right)^{k+1}\right]$, which shows that (2.1) holds for $m = k + 1$. □

The following lemmas appear in [4] and [8] (In Lemma 2.3 by $q : \mathbb{T} \to \mathbb{R}$ is rd-continuous we mean q is continuous at right-dense points in $\mathbb{T}$ and at left-dense points in $\mathbb{T}$, left-hand limits of q exist (finite).

Lemma 2.3. *Assume $q(t) \geq 0$, $y(t)$ are rd-continuous, $y(t_0) \geq 0$ and $0 < \alpha < 1$. If $y(t) \geq 0$, satisfies*

$$y(t) \leq C + \int_{t_0}^{t} q(s)y^\alpha(s)\Delta s$$

for $t \in [t_0, \infty)_\mathbb{T}$, where $C \geq y(t_0)$ is a constant, then

$$y(t) \leq \left[C^{1-\alpha} + (1-\alpha)\int_{t_0}^{t} q(s)\Delta s\right]^{\frac{1}{1-\alpha}},$$

for $t \in [t_0, \infty)_\mathbb{T}$.

Lemma 2.4 (L'Hopital's Rule)**.** *Assume f and g are differentiable on $\mathbb{T}$ with*

$$\lim_{t\to\infty} g(t) = \infty.$$

Suppose that

$$g(t) > 0, \quad g^\Delta(t) > 0, \quad \textit{for large} \quad t.$$

Then $\lim_{t\to\infty} \frac{f^\Delta(t)}{g^\Delta(t)} = r \in \bar{\mathbb{R}}$ implies $\lim_{t\to\infty} \frac{f(t)}{g(t)} = r$.

3. A NONOSCILLATION THEOREM

Assume that $\mathbb{T} = \{t_k\}_{k=0}^{\infty}$ where $1 < t_0 < t_1 < \cdots < t_k \cdots$, with $t_k \to \infty$, and satisfies condition (D).

Theorem 3.1. *Suppose that* $0 < \alpha < 1$ *is a quotient of odd positive integers, and*

$$\int_{t_0}^{\infty} t^{\alpha(n-1)} p(t)\Delta t < \infty.$$

Then equation (1.1) *has a nonoscillatory solution* $x(t)$ *satisfying*

$$\lim_{t\to\infty} \frac{x(t)}{h_{n-1}(t,t_0)} = a \neq 0.$$

In particular, if $x(t) > 0$ *for large* t*, we have* $\liminf_{t\to\infty} \frac{x(t)}{t^{n-1}} = a > 0$*; and if* $x(t) < 0$ *for large* t*, we have* $\limsup_{t\to\infty} \frac{x(t)}{t^{n-1}} = a < 0$.

Proof. Assume $x(t)$ is a solution of (1.1). By the time scale version of Taylor's Formula (Lemma 2.1), we have

$$\begin{aligned} x(t) &= \sum_{k=0}^{n-1} h_k(t,t_0)x^{\Delta^k}(t_0) + \int_{t_0}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\tau))x^{\Delta^n}(\tau)\Delta\tau \\ &= \sum_{k=0}^{n-1} h_k(t,t_0)x^{\Delta^k}(t_0) - \int_{t_0}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\tau))p(\tau)x^{\alpha}(\tau)\Delta\tau. \end{aligned} \tag{3.1}$$

For $k = 1,2,3$, it is easy to show that $0 \le h_k(t,t_0) \le t^k$, for $t > t_0$.
For $k = 3,4,\cdots,n-1$, we have

$$h_k(t,t_0) = \int_{t_0}^{t}\int_{t_0}^{\tau_1}\cdots\int_{t_0}^{\tau_{k-2}} (\tau_{k-1}-t_0)\Delta\tau_{k-1}\cdots\Delta\tau_2\Delta\tau_1.$$

Since $t_0 \le \tau_{k-1} \le \cdots \le \tau_2 \le \tau_1 \le t$, it is easy to see that

$$0 \le h_k(t,t_0) \le (t-t_0)^k < t^k. \tag{3.2}$$

Note that since $\mathbb{T}$ is an isolated time scale, we get that for $\tau \le \rho^{n-1}(t)$,

$$\sigma(\tau) \le \sigma(\rho^{n-1}(t)) = \rho^{n-2}(t) \le t.$$

So we also get that for $\tau \le \rho^{n-1}(t)$

$$0 \le h_{n-1}(t,\sigma(\tau)) \le (t-\sigma(\tau))^{n-1} \le t^{n-1}. \tag{3.3}$$

From (3.1), (3.2) and (3.3), we get that

$$|x(t)| \le Ct^{n-1} + t^{n-1}\int_{t_0}^{t} p(\tau))|x(\tau)|^{\alpha}\Delta\tau, \qquad t \ge t_0,$$

where C is a positive constant.

Set $y(t) = \frac{x(t)}{t^{n-1}}$. Then we have

$$|y(t)| \leq C + \int_{t_0}^{t} \tau^{\alpha(n-1)} p(\tau)|y(\tau)|^{\alpha} \Delta\tau, \qquad t \geq t_0.$$

By Lemma 2.3, we get that

$$\begin{aligned} |y(t)| &\leq \left[C^{1-\alpha} + (1-\alpha)\int_{t_0}^{t} s^{\alpha(n-1)} p(s)\Delta s\right]^{\frac{1}{1-\alpha}} \\ &\leq \left[C^{1-\alpha} + (1-\alpha)\int_{t_0}^{\infty} t^{\alpha(n-1)} p(t)\Delta t\right]^{\frac{1}{1-\alpha}} =: C_1, \end{aligned}$$

where C_1 is a positive constant.

So we have $|y(t)| \leq C_1$, that is $|x(t)| \leq C_1 t^{n-1}$. Since

$$x^{\Delta^{n-1}}(t) = x^{\Delta^{n-1}}(t') - \int_{t'}^{t} p(\tau)x^{\alpha}(\tau)\Delta\tau$$

for $t' \in \mathbb{T}$ and

$$\int_{t'}^{t} p(\tau)|x(\tau)|^{\alpha}\Delta\tau \leq C_1^{\alpha}\int_{t'}^{\infty} p(\tau)\tau^{\alpha(n-1)}\Delta\tau < \infty.$$

We have that $\lim_{t\to\infty} x^{\Delta^{n-1}}(t) = A$ exists. If we now further require that $x(t)$ satisfies

$$x^{\Delta^{n-1}}(t') > C_1^{\alpha}\int_{t'}^{\infty} p(\tau)\tau^{\alpha(n-1)}\Delta\tau,$$

then $\lim_{t\to\infty} x^{\Delta^{n-1}}(t) = A > 0$.

By using the time scales L'Hopital's Rule (Lemma 2.4), we get

$$\lim_{t\to\infty} \frac{x(t)}{h_{n-1}(t,t_0)} = \lim_{t\to\infty} \frac{x^{\Delta}(t)}{h_{n-2}(t,t_0)} = \cdots = \lim_{t\to\infty} x^{\Delta^{n-1}}(t) = A.$$

From Lemma 2.2, we have

$$\frac{x(t)}{\epsilon_{n-1}t^{n-1}} \geq \frac{x(t)}{h_{n-1}(t,t_0)}.$$

So we get that $\liminf_{t\to\infty} \frac{x(t)}{t^{n-1}} \geq \epsilon_{n-1}A > 0$.

□

4. OSCILLATION OF AN N-th ORDER SUBLINEAR DYNAMIC EQUATION

The following lemma for a dynamic equation on an isolated time scale can be regarded as a simple extension of [1, Corollary 1.7.14] (see Ryder and Wend [10] for its continuous version). Its proof is the same as Corollary 1.7.14 in [1], so we omit it.

Lemma 4.1. *Suppose that*

$$\mathbb{T} = \{t_0, t_1, t_2, \cdots, t_k, \cdots\},$$

where $\lim_{k\to\infty} t_k = \infty$. *Let* $x(t)$ *be defined on* $\mathbb{T}$, *with* $x(t) > 0$ *and* $x^{\Delta^n}(t) \leq 0$ *and not identically zero, for large* $t \in \mathbb{T}$. *Then, exactly one of the following is true*
(I) $\lim_{t\to\infty} x^{\Delta^i}(t) = 0$, $1 \leq i \leq n-1$.
(II) there is an odd integer j, $1 \leq j \leq n-1$ *such that* $\lim_{t\to\infty} x^{\Delta^{n-i}}(t) = 0$ *for* $1 \leq i \leq j-1$, $\lim_{t\to\infty} x^{\Delta^{n-j}} x(t) \geq 0$ *(finite),* $\lim_{t\to\infty} x^{\Delta^{n-j-1}}(t) > 0$ *and* $\lim_{t\to\infty} x^{\Delta^i}(t) = \infty$, $0 \leq i \leq n-j-2$.

In addition, in Case I we know that $(-1)^{i+n-1} x^{\Delta^i}(t) > 0$, *for* $1 \leq i \leq n-1$, $t \in \mathbb{T}$ *and in Case* II, $(-1)^{i+j} x^{\Delta^{n-i}}(t) > 0$, *for* $1 \leq i \leq j, t \in \mathbb{T}$.

The following lemma appears in [9, Theorem 5.37 (i)].

Lemma 4.2 (Leibniz Formula). *If* $f(t,s), f^{\Delta_t}(t,s)$ *are rd-continuous, then*

$$\left[\int_a^t f(t,s)\Delta s\right]^{\Delta_t} = f(\sigma(t),t) + \int_a^t f^{\Delta_t}(t,s)\Delta s.$$

The following lemmas are in [5].

Lemma 4.3. *Suppose that*

$$\mathbb{T} = \{t_0, t_1, t_2, \cdots, t_k, \cdots\},$$

where $1 < t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$. *Then for any* $m \geq 2$, *there exists* $\epsilon_{m-1} > 0$ *such that*

$$\int_{t_{k_0}}^{\sigma(\tau_{m-1})} \int_{t_{k_0}}^{\sigma(\tau_{m-2})} \cdots \int_{t_{k_0}}^{\sigma(\tau_2)} [\sigma(\tau_1) - t_{k_0}]\Delta\tau_1\Delta\tau_2\cdots\Delta\tau_{m-2} \geq \epsilon_{m-1}[\sigma(\tau_{m-1})]^{m-1},$$

for $\tau_{m-1} > t_{k_0}$.

Lemma 4.4. *Suppose that*

$$\mathbb{T} = \{t_0, t_1, t_2, \cdots, t_k, \cdots\}$$

where $\lim_{k\to\infty} t_k = \infty$. *Suppose that* $x(t)$ *is an eventually positive solution of* (1.1).
(i) If $x(t)$ *satisfies Case (I) of Lemma 4.1, then*

$$(-1)^n x^{\Delta}(t) \geq \tag{4.1}$$
$$\int_t^\infty \left\{\int_t^{\sigma(\tau_1)} \left[\int_t^{\sigma(\tau_2)} \cdots \int_t^{\sigma(\tau_{n-3})} (\sigma(\tau) - t)\Delta\tau\Delta\tau_{n-3}\cdots\Delta\tau_3\right]\Delta\tau_2\right\} p(\tau_1)x^\alpha(\tau_1)\Delta\tau_1.$$

(ii) If $x(t)$ *satisfies Case (II) of Lemma 4.1, then*

$$x^{\Delta}(t)$$
$$\geq \int_{T_3}^t \int_{T_3}^{\sigma(\tau_{n-3})} \cdots \int_{T_3}^{\sigma(\tau_2)} [\sigma(\tau_1) - T_3]\Delta\tau_1\cdots\Delta\tau_{n-3} \cdot \int_t^\infty p(s)x^\alpha(s)\Delta s.$$

The following lemmas appear in [1]

Lemma 4.5 (Discrete Kneser's Theorem). *Assume that* $\mathbb{T}=\mathbb{N}_0$. *Let* $x(k)$ *be defined for* $k\geq k_0$, *and* $x(k)>0$ *with* $\Delta^n x(k)$ *of constant sign for* $k\geq a$ *and not identically zero. Then, there exists an integer* j, $0\leq j\leq n$, *with* $(n+j)$ *odd for* $\Delta^n x(k)\leq 0$, *and* $(n+j)$ *even for* $\Delta^n x(k)\geq 0$, *such that*

$$j\leq n-1 \quad \textit{implies} \quad (-1)^{j+i}\Delta^i x(k)>0 \quad \textit{for all} \quad k\geq k_0, \quad j\leq i\leq n-1,$$

and $j\geq 1$ *implies* $\Delta^i x(k)>0$, *for all large* $k\geq k_0, 1\leq i\leq j-1$.

Lemma 4.6. *Assume that* $\mathbb{T}=\mathbb{N}_0$. *Let* $x(k)$ *be defined for* $k\geq k_0$, *and* $x(k)>0$ *with* $\Delta^m x(k)\leq 0$ *for* $n\geq k_0$ *and not identically zero. Then, there exists a large* $k_1\geq k_0$ *such that*

$$x(k)\geq \frac{(k-k_1)^{n-1}}{(n-1)!}\Delta^{n-1}x(2^{n-j-1}k), \quad k\geq k_1,$$

where j *is defined in Lemma 4.5. Further, if* $x(k)$ *is increasing, then*

$$x(k)\geq \frac{1}{(n-1)!}\left(\frac{k}{2^{n-1}}\right)^{n-1}\Delta^{n-1}x(k), \quad k\geq 2^{n-1}k_1. \tag{4.2}$$

Theorem 4.7. *Assume that* $\mathbb{T}=\{t_k\}_{k=0}^{\infty}$ *where* $1<t_0<t_1<\cdots<t_k\cdots$, *with* $t_k\to\infty$, $x(t)$ *is an eventually positive solution of* (1.1) *and*

$$\int_{t_0}^{\infty} t^{\alpha(n-1)}p(t)\Delta t=\infty. \tag{4.3}$$

If $x(t)$ *satisfies Case (I) of Lemma 4.1 and* n *is an odd integer, then* $\lim_{t\to\infty}x(t)=0$.

Proof. Since n is odd, (4.1) of Lemma 4.4 reduces to

$$-x^{\Delta}(t)\geq \tag{4.4}$$

$$\int_t^{\infty}\left\{\int_t^{\sigma(\tau_1)}\left[\int_t^{\sigma(\tau_2)}\cdots\int_t^{\sigma(\tau_{n-3})}(\sigma(\tau)-t)\Delta\tau\Delta\tau_{n-3}\cdots\Delta\tau_3\right]\Delta\tau_2\right\}p(\tau_1)x^{\alpha}(\tau_1)\Delta\tau_1.$$

and this implies that $x(t)$ is nonincreasing for $t\geq T$. Let $\lim_{t\to\infty}x(t)=L$. We shall prove that $L=0$. Suppose $L>0$. We take T so large that $x(t)\geq\frac{L}{2}$ for $t\geq T$. Integrating (4.4) from T to t, then using integration by parts once, where we use the Leibniz Formula (Lemma 4.2) several times yields

$$\begin{aligned}
& x(T)-x(t)\\
\geq\ & \Big[(s-T)\int_s^{\infty}\int_s^{\sigma(\tau_1)}\int_s^{\sigma(\tau_2)}\cdots\int_s^{\sigma(\tau_{n-3})}(\sigma(\tau)-s)\Delta\tau\Delta\tau_{n-3}\cdots\Delta\tau_2\\
\cdot\ & p(\tau_1)x^{\alpha}(\tau_1)\Delta\tau_1\Big]_{s=T}^{t}\\
+\ & \int_T^t[\sigma(s)-T]\int_s^{\infty}\int_s^{\sigma(\tau_1)}\int_s^{\sigma(\tau_2)}\cdots\int_s^{\sigma(\tau_{n-4})}(\sigma(\tau_{n-3})-s)\Delta\tau_{n-3}\cdots\Delta\tau_2\\
\cdot\ & p(\tau_1)x^{\alpha}(\tau_1)\Delta\tau_1\Delta s
\end{aligned}$$

$$\geq \int_T^t [\sigma(s)-T] \int_s^\infty \int_s^{\sigma(\tau_1)} \int_s^{\sigma(\tau_2)} \cdots \int_s^{\sigma(\tau_{n-4})} (\sigma(\tau_{n-3})-s)\Delta\tau_{n-3}\cdots\Delta\tau_2$$
$$\cdot \quad p(\tau_1)x^\alpha(\tau_1)\Delta\tau_1\Delta s.$$

Repeating the above procedure we get

$$x(T) \geq x(T)-x(t) \geq$$
$$\int_T^t \int_T^{\sigma(s)} [\sigma(v_1)-T]\Delta v_1 \cdot \int_s^\infty \int_s^{\sigma(\tau_1)} \cdots \int_s^{\sigma(\tau_{n-5})} (\sigma(\tau_{n-4})-s)\Delta\tau_{n-4}\cdots\Delta\tau_2$$
$$\cdot \quad p(\tau_1)x^\alpha(\tau_1)\Delta\tau_1\Delta s.$$

Proceeding in this manner we get using finite mathematical induction

$$\begin{aligned} x(T) \quad \geq \quad & \int_T^t \int_T^{\sigma(s)} \int_T^{\sigma(v_{n-4})} \cdots \int_T^{\sigma(v_2)} [\sigma(v_1)-T]\Delta v_1 \cdots \Delta v_{n-4} \\ & \int_s^\infty [\sigma(\tau_1)-s] p(\tau_1)x^\alpha(\tau_1)\Delta\tau_1\Delta s. \end{aligned}$$

Further integration by parts gives us

$$\begin{aligned} & x(T) \\ \geq \quad & \{\int_T^s \int_T^{\sigma(v_{n-3})} \int_T^{\sigma(v_{n-4})} \cdots \int_T^{\sigma(v_2)} [\sigma(v_1)-T]\Delta v_1 \cdots \Delta v_{n-4}\Delta v_{n-3} \\ & \int_s^\infty [\sigma(\tau_1)-s]\cdot p(\tau_1)x^\alpha(\tau_1)\Delta\tau_1\}_{s=T}^t \\ + \quad & \int_T^t \int_T^{\sigma(s)} \int_T^{\sigma(v_{n-3})} \cdots \int_T^{\sigma(v_2)} [\sigma(v_1)-T]\Delta v_1 \cdots \Delta v_{n-3} \\ & \int_s^\infty p(\tau_1)x^\alpha(\tau_1)\Delta\tau_1\Delta s \\ \geq \quad & \int_T^t \int_T^{\sigma(s)} \int_T^{\sigma(v_{n-3})} \cdots \int_T^{\sigma(v_2)} [\sigma(v_1)-T]\Delta v_1 \cdots \Delta v_{n-3} \\ & \int_s^\infty p(\tau_1)x^\alpha(\tau_1)\Delta\tau_1\Delta s \\ = \quad & \{\int_T^s \int_T^{\sigma(v_{n-2})} \int_T^{\sigma(v_{n-3})} \cdots \int_T^{\sigma(v_2)} [\sigma(v_1)-T]\Delta v_1 \cdots \Delta v_{n-3}\Delta v_{n-2} \\ \cdot \quad & \int_s^\infty p(\tau_1)x^\alpha(\tau_1)\Delta\tau_1\}_{s=T}^t \\ + \quad & \int_T^t \int_T^{\sigma(s)} \int_T^{\sigma(v_{n-2})} \cdots \int_T^{\sigma(v_2)} [\sigma(v_1)-T]\Delta v_1 \cdots \Delta v_{n-2} p(s)x^\alpha(s)\Delta s \\ \geq \quad & \int_T^t \int_T^{\sigma(s)} \int_T^{\sigma(v_{n-2})} \cdots \int_T^{\sigma(v_2)} [\sigma(v_1)-T]\Delta v_1 \cdots \Delta v_{n-2} p(s)x^\alpha(s)\Delta s \\ \geq \quad & \int_{\sigma(T)}^t \int_T^{\sigma(s)} \int_T^{\sigma(v_{n-2})} \cdots \int_T^{\sigma(v_2)} [\sigma(v_1)-T]\Delta v_1 \cdots \Delta v_{n-2} p(s)x^\alpha(s)\Delta s. \end{aligned}$$

Applying Lemma 4.3 and later using $0 < \alpha < 1$ we obtain

$$\begin{aligned} x(T) &\geq \epsilon_{n-1} \int_{\sigma(T)}^{t} \sigma^{n-1}(s)p(s)x^{\alpha}(s)\Delta s \\ &\geq \epsilon_{n-1} \left(\frac{L}{2}\right)^{\alpha} \int_{\sigma(T)}^{t} \sigma^{n-1}(s)p(s)\Delta s \\ &\geq \epsilon_{n-1} \left(\frac{L}{2}\right)^{\alpha} \int_{\sigma(T)}^{t} \sigma^{\alpha(n-1)}(s)p(s)\Delta s \\ &\geq \epsilon_{n-1} \left(\frac{L}{2}\right)^{\alpha} \int_{\sigma(T)}^{t} s^{\alpha(n-1)}p(s)\Delta s. \end{aligned}$$

Letting $t \to \infty$, we get a contradiction of (4.3), and hence we must have that $L = 0$.

□

Theorem 4.8. *Suppose that* $\mathbb{T} = \mathbb{N}_0$,

$$\sum_{k=1}^{\infty} k^{\alpha(n-1)}p(k) = \infty. \tag{4.5}$$

Then all solutions $x(k)$ *of the* n*-th order sublinear difference equation*

$$\Delta^n x(k) + p(k)x^{\alpha}(k) = 0, \quad 0 < \alpha < 1 \tag{4.6}$$

where $p(k) \geq 0$, *are oscillatory in the case* n *is even, and every solution* $x(k)$ *is either oscillatory or* $\lim_{k\to\infty} x(k) = 0$ *in the case* n *is odd.*

Proof. Let $x(k)$ be a nonoscillatory solution of (4.6). We may assume that $x(k) > 0$ for large k. The case $x(k) < 0$ can be treated similarly.

From (4.6), we get

$$\Delta^n x(k) = -p(k)x^{\alpha}(k) \leq 0 \tag{4.7}$$

for large k. By Lemma 4.5, $\Delta^i x(k)$ is of constant sign for $i = 1, 2, \cdots, n$, and for $n \geq 2$

$$\Delta^{n-1}x(k) > 0, \quad \text{for large} \quad k \geq k_0. \tag{4.8}$$

If $x(k)$ satisfies Case (I) of Lemma 4.1 and n is even or $x(k)$ satisfies Case (II) of Lemma 4.1, from Lemma 4.4, we have that $\Delta x(k) > 0$, so $x(k)$ is increasing. From (4.6) and (4.2), there exists $k_1 > k_0$ such that for $k > k_1 > k_0$, we have

$$\Delta^n x(k) + \frac{p(k)}{[(n-1)!]^{\alpha}} \left(\frac{k}{2^{n-1}}\right)^{\alpha(n-1)} [\Delta^{n-1}x(k)]^{\alpha} \leq 0. \tag{4.9}$$

Let $z(k) = \Delta^{n-1}x(k) > 0$. From (4.9), we have that $\Delta z(k) \leq 0$ and

$$\Delta z(k) + \frac{p(k)}{[(n-1)!]^{\alpha}} \left(\frac{k}{2^{n-1}}\right)^{\alpha(n-1)} z^{\alpha}(k) \leq 0.$$

Note that $0 < \alpha < 1$. Using the continuous mean value theorem, there exists $\xi \in [z(k+1), z(k)]$ such that

$$\begin{aligned} & z^{1-\alpha}(k) - z^{1-\alpha}(k+1) \\ = \; & (1-\alpha)\xi^{-\alpha}[z(k) - z(k+1)] \\ \geq \; & (1-\alpha)z^{-\alpha}(k)[z(k) - z(k+1)] \\ \geq \; & (1-\alpha)\frac{p(k)}{[(n-1)!]^{\alpha}}\left(\frac{k}{2^{n-1}}\right)^{\alpha(n-1)}, \end{aligned}$$

for $k \geq k_1$. It follows that

$$z^{1-\alpha}(k_1) \geq (1-\alpha)\sum_{k=k_1}^{\infty}\frac{p(k)}{[(n-1)!]^{\alpha}}\left(\frac{k}{2^{n-1}}\right)^{\alpha(n-1)}$$

which contradicts (4.5).

If $x(k)$ satisfies Case (I) of Lemma 4.1 and n is a odd, then from Theorem 4.7, we have $\lim_{t\to\infty} x(k) = 0$. □

Using Theorems 3.1, 4.7 and 4.8, we get the following result.

Corollary 4.9. *The following hold:*

(i) When n is even, every solution $x(k)$ of the difference equation

$$\Delta^n x(k) + p(k)x^{\alpha}(k) = 0, \quad 0 < \alpha < 1, \tag{4.10}$$

where $p(k) \geq 0$, is oscillatory if and only if

$$\sum_{k=1}^{\infty} k^{\alpha(n-1)} p(k) = \infty. \tag{4.11}$$

(ii) When n is odd, every solution $x(k)$ of the difference equation (4.10) *is either oscillatory or* $\lim_{k\to\infty} x(k) = 0$ *if and only if* (4.11) *holds.*

ACKNOWLEDGMENTS

Baoguo Jia was supported by the National Natural Science Foundation of China (No. 10971232)

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