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EXISTENCE OF SOLUTIONS FOR A CLASS OF SINGULAR NONLINEAR THIRD ORDER AUTONOMOUS BOUNDARY VALUE PROBLEMS

JOHN V. BAXLEY[1] AND GREY M. BALLARD[2]

[1]Department of Mathematics, Wake Forest University
Winston-Salem, NC 27106 USA
E-mail: baxley@wfu.edu

[2]Department of Computer Science, University of California
Berkeley, CA 94704 USA
E-mail: ballard@cs.berkeley.edu

To Jeff Webb, with our best wishes for retirement

ABSTRACT. Motivated by a problem which arises in the analysis of stagnation point flow toward a stretching sheet, we consider a general class of problems of the form $y''' = f(y, y', y'')$, $y(0) = a$, $y'(0) = m$, $\lim_{t\to\infty} y'(t) = b$. We give conditions on f which imply existence of at least one solution and obtain a partial uniqueness result.

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1. INTRODUCTION

We study a class of third order boundary value problems of the form

$$y''' = f(y, y', y''), \tag{1.1}$$

$$y(0) = a, \;\; y'(0) = m, \;\; \lim_{t\to\infty} y'(t) = b. \tag{1.2}$$

A special case of this problem with

$$f(y, y', y'') = -yy'' + (y')^2 - b^2, \tag{1.3}$$

$a = 0$, and $m = 1$ was considered in [9], in which the authors show that the problem arises in the analysis of stagnation point flow toward a stretching sheet and provide further references. In [9], it is proved that their example problem has a solution if $b > 0$. This solution $\phi(t)$ is unique and $\phi'(t)$ is monotone increasing if $b > m = 1$. If $0 < b < m = 1$, $\phi'(t)$ is monotone decreasing and $\phi(t)$ is the only solution with a

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monotone derivative. Numerical evidence for multiple solutions was given in case the positive b is less than a critical value approximately 0.169.

Examples of boundary value problems on infinite intervals appeared over thirty years ago in the book [6]; see in particular pp. 72-76 (an example studied more carefully in [1]), pp. 119-121, pp. 126-132, problem 2 on p. 134, and pp. 192-197. A survey of theorems on second order problems with appropriate references can be found in Chapter 8 of [8]. Interesting examples of third order problems appeared recently in a number of applied settings; see e.g. [2, 7, 10] and these have led to mathematical treatments by Paullet and his colleagues [3, 4, 5].

Our goal here is to describe a class of problems which contain the example of [9] as a special case. In addition to broadening the sweep of the conclusions, insight into the features of the example which imply the resulting conclusions are clearly seen. Our approach to the problem, while following [9] in the use of shooting methods, is rather different in detail and, at least to us, seems simpler. Solutions to initial value problems, while guaranteed under our general hypotheses including the example above, may exist only locally, so care should be exercised when shooting on infinite intervals. Such caution lies beneath the surface in [9]; here it will be clearly visible.

Our existence theorem in section 2 (and in section 3) and the uniqueness result in section 4, apply only indirectly to the example in [9] given by (1.3) via the modified form

$$f(y, y', y'') = -\text{sgn}(y'')|y||y''| + (y')^2 - b^2 \tag{1.4}$$

But since all solutions (with $a = 0$, $m = 1$ as in [9]) guaranteed by these existence theorems have $y' > 0$ and hence $y > 0$ on $(0, \infty)$, then the solutions satisfy the original example in [9] given by (1.3).

The uniqueness result in section 4 applies only to the case $b < m$ and includes the example (1.4) to give the existence of at most one solution. For $a = 0$, $m = 1$, $0 < b < 1$, as in [9] and the example (1.4), we conclude the the existence of at most one solution ϕ satisfying $\phi'' \leq 0$, or equivalently ϕ' is non-increasing. Thus $\phi' \geq b > 0$ so $\phi \geq a = 0$ and the motivational example of [1], given by (1.3), has at most one solution with y' non-increasing on $[0, \infty)$.

More details and extensions of these remarks appear in section 5. Note that we have no uniqueness result for the case $b > m$. The proofs in [9] for uniqueness in both cases $b > m$ and $b < m$ depend on behavior of the fifth order equation obtained by differentiating the given third order equation twice. Our result here shows that the uniqueness for the case $b < m$ does not depend on such behavior. It is curious that uniqueness in the case $b > m$ seems to lie deeper than that for the case $b < m$. If one attempts to mimic our proof of Theorem 2 below for the case $b > m$, one will see that the needed inequality is reversed.

Until further notice, we shall assume that $b \geq m$ and let

$$Y_{m,b} = \left\{ \begin{array}{l} [a,\infty),\ \text{if } m > 0, \\ (-\infty,\infty),\ \text{if } m \leq 0 < b, \\ (-\infty,a],\ \text{if } b \leq 0. \end{array} \right\}$$

We then assume that for some $\delta > 0$, the function f in (1.1) satisfies

- H1: $f(y,u,v)$ is continuous and satisfies a Lipschitz condition in (y,u,v) for $y \in Y_{m,b}$, $m-\delta \leq u \leq b+\delta$, $v \geq -\delta$;
- H2: $f(y,u,0) > 0$ if $b < u \leq b+\delta$, for all $y \in Y_{m,b}$;
- H3: $f(y,u,v) < 0$ if $m-\delta \leq u < b$, $v \geq 0$, for all $y \in Y_{m,b}$;
- H4: For $\alpha > 0$, let

$$Z_{m,b} = \left\{ \begin{array}{l} [a, a+\frac{2b(b-m)}{\alpha}],\ \text{if } m \geq 0, \\ [a+\frac{2m(b-m)}{\alpha}, a+\frac{2b(b-m)}{\alpha}],\ \text{if } m < 0 < b, \\ [a+\frac{2m(b-m)}{\alpha}, a],\ \text{if } b \leq 0, \end{array} \right\}$$

  and

$$S_\alpha = \{(y,u,v) : y \in Z_{m,b}, m \leq u \leq b, \frac{\alpha}{2} \leq v \leq \alpha\};$$

  then

$$\lim_{\alpha\to\infty} \frac{\min\{f(y,u,v) : (y,u,v) \in S_\alpha\}}{\alpha^2} = 0,$$

- H5: For any $\gamma > 0$, $m < L < b$

$$\sup\{f(y,u,v) : y \in Y_{m,b}, m \leq u \leq L, 0 \leq v \leq \gamma\} < 0.$$

The purpose of H1 is to permit use of the standard theorems on existence, uniqueness, and continuous dependence of solutions of initial value problems. Since f is continuous, it follows from H2 and H3 above that $f(y,b,0) = 0$ for all $y \in Y_{m,b}$ and moreover that f is non-positive on the set S_α so that H4 is basically a growth restriction on f. Since y and u are restricted to compact sets, at first glance H4 appears to require that the growth of f as a function of v is less than quadratic as $v \to \infty$. However, the definition of S_α requires that $y \to a$ as $\alpha \to \infty$ and in some cases this might allow v to grow faster than α^2. The example mentioned above from [9] is a case in point. The presence of a term like $-\text{sgn}(y'')|y||y''|^p$ would be acceptable if $a = 0$ and $p < 3$. We will return to this thought later.

EXISTENCE FOR THE CASE $m \leq b$

Here is our first theorem.

Theorem 2.1. *Under the hypotheses H1-H5, then for $b > m$ the boundary value problem* (1.1), (1.2) *has a solution $y = \phi(t)$ satisfying $m < \phi'(t) < b$, $\phi''(t) > 0$ for all $t > 0$. If $b = m$, then $y = a + bt$ is a solution.*

We prepare for the proof with two preliminary lemmas. Let IVP be the initial value problem consisting of (1.1) and the initial conditions

$$y(0) = a, \;\; y'(0) = m, \;\; y''(0) = \alpha, \tag{2.1}$$

where $\alpha \geq 0$ is a shooting parameter. We say that a function y is a solution of this initial value problem on an interval $[0, s)$ if y satisfies the differential equation (1.1) on this interval and the initial conditions (2.1), and $(y(t), y'(t), y''(t)))$ is contained in the region described in H1 for each $t \in [0, s)$. By hypothesis H1, IVP has a unique local solution y_α; we let $I_\alpha = [0, c_\alpha)$ be the maximal interval of existence of this solution. Let

$$A = \{\alpha \geq 0 : y''_\alpha \text{ has a zero on } I_\alpha\}.$$

If $\alpha \in A$, then there exists $t_\alpha \in I_\alpha$ which is a root of y''_α. We may assume that t_α is the smallest such root. If $\alpha = 0$, then clearly $t_\alpha = 0$ also and in the case $\alpha > 0$, we have $y''_\alpha(t) > 0$ for $0 \leq t < t_\alpha$.

Lemma 2.2. *Suppose H1-H3 are satisfied. If* $0 < \alpha \in A$, *then* $y'_\alpha(t_\alpha) < b$.

Proof: Suppose $y'_\alpha(t_\alpha) = b$. Then by H2, H3 the function $u(t) = b(t - t_\alpha) + y_\alpha(t_\alpha)$ satisfies (1.1) as well as the conditions $u(t_\alpha) = y_\alpha(t_\alpha)$, $u'(t_\alpha) = b = y'_\alpha(t_\alpha)$, $u''(t_\alpha) = 0 = y''_\alpha(t_\alpha)$, and the uniqueness of the solution of this terminal value problem implies $u(t) \equiv y_\alpha(t)$ on $[0, t_\alpha]$. But this is a contradiction since $u''(0) = 0$ and $y''_\alpha(0) = \alpha > 0$. Now suppose that $y'_\alpha(t_\alpha) > b$. Then from H2,

$$y'''_\alpha(t_\alpha) = f(y_\alpha(t_\alpha), y'_\alpha(t_\alpha), y''_\alpha(t_\alpha)) = f(y_\alpha(t_\alpha), y'_\alpha(t_\alpha), 0) > 0.$$

Thus y''_α is increasing in a left neighborhood of t_α, which implies that $y''_\alpha < 0$ in that neighborhood, a contradiction. Thus $y'_\alpha(t_\alpha) < b$. □

Lemma 2.3. *Suppose* $b > m$ *and H1-H4 are satisfied. Then* A *is bounded above.*

Proof: We use H4 to choose $\alpha > 0$ so large that

$$\frac{\min\{f(y, u, v) : (y, u, v) \in S_\alpha\}}{\alpha^2} > -\frac{1}{4(b - m)}$$

and then show that $\alpha \notin A$. Suppose, for contradiction, that $\alpha \in A$. Since $y''_\alpha(0) = \alpha$, the intermediate value theorem guarantees a solution of $y''_\alpha(x) = \alpha/2$ in $(0, t_\alpha)$; let s_α be the smallest such solution. Since $y''_\alpha > 0$ on $[0, t_\alpha)$, then y'_α is increasing on that interval. Thus from Lemma 2.2,

$$m \leq y'_\alpha(t) < b \tag{2.2}$$

on $[0, s_\alpha]$. Hence from H3,

$$y'''_\alpha(t) = f(y_\alpha(t), y'_\alpha(t), y''_\alpha(t)) < 0$$

on $[0, s_\alpha]$, and we conclude that y''_α is decreasing on $[0, s_\alpha]$ and therefore

$$\alpha \geq y''_\alpha(t) \geq \alpha/2$$

on $[0, s_\alpha]$. Integrating the right side of this inequality gives

$$b > y'_\alpha(s_\alpha) \geq m + \frac{\alpha}{2} s_\alpha,$$

from which we obtain $s_\alpha < 2(b-m)/\alpha$. Then integrating (2.2) gives

$$a + mt \leq y_\alpha(t) \leq a + bt$$

on $[0, s_\alpha]$. If $b \leq 0$, then $y_\alpha(t) \leq a$; if $b > 0$, then $y_\alpha(t) \leq a + bs_\alpha < a + 2b(b-m)/\alpha$. If $m \geq 0$, then $y_\alpha(t) \geq a$; if $m < 0$, then $y_\alpha(t) \geq a + ms_\alpha > a + 2m(b-m)/\alpha$. We have now shown that on the interval $[0, s_\alpha]$, all points $(y_\alpha(t), y'_\alpha(t), y''_\alpha(t))$ belong to the set S_α.

By our choice of α using H4, we then calculate

$$-\alpha/2 = y''_\alpha(s_\alpha) - y''_\alpha(0) = \int_0^{s_\alpha} y'''_\alpha(x)dx > -\frac{\alpha^2}{4(b-m)} s_\alpha > -\alpha/2,$$

a contradiction. □

Now for the proof of Theorem 2.1. Clearly $y = a + bt$ is a solution of (1.1) and satisfies (1.2) if $b = m$. So suppose $b > m$. By Lemma 2.3, the set A is bounded above. Since $0 \in A$, then $A \neq \emptyset$. Let $\gamma = \sup A$. We shall show that $\gamma \notin A$, that $I_\gamma = [0, \infty)$ and that $\lim_{t\to\infty} y'_\gamma(t) = b$.

Suppose that $\gamma \in A$. If $\gamma > 0$, then by Lemma 2.2, $y'_\gamma(t_\gamma) < b$; if $\gamma = 0$, this last inequality is trivially true. Therefore $H3$ implies $y'''_\gamma(t_\gamma) < 0$. Thus y''_γ is decreasing in a right neighborhood of t_γ and $y''_\gamma < 0$ in a right neighborhood of t_γ. By continuous dependence, for $\alpha > \gamma$ and sufficiently close to γ, then $y''_\alpha < 0$ at points to the right of t_γ, and hence $\alpha \in A$, a contradiction.

Since $\gamma \notin A$ then $y''_\gamma > 0$ on I_γ and y'_γ is increasing on I_γ. We claim that $y'_\gamma < b$ on I_γ. For otherwise, there exists $t_1 \in I_\gamma$ so that $y'_\gamma(t_1) = b$ and $y'_\gamma(t) < b$ for $0 \leq t < t_1$. Then we can choose $t_1 < t_2 < c_\gamma$ so that $y'_\gamma(t_2) > b$ and $y'_\gamma(t) < b + \delta$ on $[t_1, t_2]$. By continuous dependence, there exists $\alpha \in A$ so that $y''_\alpha(t) > 0$, $y'_\alpha(t) < b + \delta$ on $[0, t_2]$ with $y'_\alpha(t_2) > b$. But this contradicts Lemma 2.2. It then follows from H3 that $y'''_\gamma < 0$ on I_γ so y''_γ is decreasing and positive on I_γ. Thus if $c_\gamma < \infty$, $y_\gamma, y'_\gamma, y''_\gamma$ all have limits from the left as $t \to c_\gamma$ and the solution can be continued beyond c_γ, a contradiction. Thus $I_\gamma = [0, \infty)$.

Now y'_γ is increasing on $[0, \infty)$ and bounded by b. Thus $L = \lim_{t\to\infty} y'_\gamma(t)$ exists and is no larger than b. We complete the proof by showing that $L < b$ is impossible. Since y''_γ is decreasing on $[0, \infty)$, then $\gamma \geq y''_\gamma(t)$ on $[0, \infty)$. From H5, we conclude that y'''_γ is negative and bounded away from zero; thus y''_γ is unbounded below on $[0, \infty)$, a contradiction. □

EXISTENCE FOR THE CASE $m > b$

We consider now the case where $m > b$ in (1.2). If we change variables with $u = -y$, then (1.1) becomes

$$u''' = -f(-u, -u', -u'') \tag{3.1}$$

and (1.2) changes to

$$u(0) = -a, \;\; u'(0) = -m, \;\; \lim_{t\to\infty} u'(0) = -b. \tag{3.2}$$

Now $-m < -b$, so Theorem 2.1 may be applied if $g(y,u,v) = -f(-y,-u,-v)$ satisfies all the required assumptions H1-H5. Note that in this case, the solution ϕ will then satisfy $m > \phi' > b, \phi'' < 0$ for $t > 0$.

UNIQUENESS

Here is our uniqueness result.

Theorem 4.1. *Suppose that* $m \geq b$ *and that* $f(y_1, u_1, v) > f(y_2, u_2, v)$ *whenever* $y_1 > y_2, \;\; u_1 > u_2, \;\; v \leq 0$. *Then there is at most one solution of* (1.1), (1.2) *satisfying* $y''(t) \leq 0$ *for all* $t \geq 0$.

Proof: Suppose that there are two distinct solutions y_1 and y_2 satisfying $y_i''(t) \leq 0$ for all $t \geq 0$ and $i = 1, 2$. Then

$$y_1(0) = y_2(0) = a, \;\; y_1'(0) = y_2'(0) = m, \;\; \lim_{t\to\infty} y_1'(t) = \lim_{t\to\infty} y_2'(t) = b.$$

Certainly, $y_1''(0) \neq y_2''(0)$ since otherwise by uniqueness in initial value problems, $y_1 \equiv y_2$. So we may assume that $y_1''(0) > y_2''(0)$. Now let $u = y_1' - y_2'$. It follows that u has a positive maximum on $(0, \infty)$ and we may let t_1 be the first zero of u' on this interval. Since $u'(0) > 0$, then $u' = y_1'' - y_2''$ is positive on $[0, t_1)$. Hence u is increasing on this interval and so $u = y_1' - y_2' > 0$ on $(0, t_1]$. Then $y_1 - y_2 > 0$ also and

$$u''(t_1) = y_1'''(t_1) - y_2'''(t_1) = f(y_1(t_1), y_1'(t_1), y_1''(t_1)) - f(y_2(t_1), y_2'(t_1), y_2''(t_1)).$$

Then our hypothesis implies that u must have a minimum at t_1, a contradiction. □

EXAMPLE AND DISCUSSION

We return to a slightly more general example than that of [9]:

$$y''' = -\text{sgn}(y'')|y|^q|y''|^p + (y')^2 - b^2, \tag{5.1}$$

with boundary conditions

$$y(0) = 0, \;\; y'(0) = 1, \;\; y''(0) = b. \tag{5.2}$$

Assume first that $b > 1$. Then it is straightforward to check that all the hypotheses H1-H5 are satisfied, where $a = 0$, $m = 1$. Note that for H4, we have

$$\min\{f(y,u,v) > -(2b(b-1))^q \alpha^{p-q} + 1 - b^2.$$

So what is required for H4 is that $p < q+2$. If the boundary condition were $y(0) = 1$, then we would have instead

$$\min\{f(y,u,v) > -\left(1 + \frac{2b(b-1)}{\alpha}\right)^q \alpha^p + 1 - b^2,$$

now we need $p < 2$. The extra freedom allowed is totally a consequence of the zero boundary condition. Thus there is a solution satisfying all the conclusions of Theorem 2.1. This solution $\phi(t)$ has a monotone increasing derivative, which is thus always positive, and $\phi(t)$ is positive and concave up. In particular the example of [9] with $p = q = 1$ is allowed for any value of $y(0) \geq 0$. Note that the solution guaranteed by Theorem 2.1 satisfies $\phi(t) > 0$, $\phi''(t) > 0$ for all $t > 0$ so it is a solution of that original example.

Now consider $0 < b < 1$. One now must check that

$$g(y,u,v) = -f(-y,-u,-v) = \operatorname{sgn}(-v)|y|^p|v|^p - u^2 + b^2$$

satisfies H1-H5, where now $a = 0$, m is replaced by -1, b by $-b$. Again, these hypotheses are easily checked.

Note that the solution guaranteed for (5.1), (5.2) with $p = q = 1$, whether or not $b > 1$ or $0 < b < 1$, also satisfies the original example in [9]. Our uniqueness result also applies to (5.1), (5.2) when $0 < b < 1$, and thus for the original Paullet-Weidman example if $0 < b < 1$.

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