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PERIODIC SOLUTIONS OF LAGRANGIAN SYSTEMS OF RELATIVISTIC OSCILLATORS

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Dedicated to J.R.L. Webb at the occasion of his retirement

ABSTRACT. T-periodic solutions of systems of differential equations of the form

$$(\phi(u'))' = \nabla_u F(x,u) + h(x)$$

where $\phi = \nabla\Phi$, with Φ strictly convex, is a homeomorphism of the ball $B_a \subset \mathbb{R}^n$ onto $\mathbb{R}^n$, are considered under various conditions upon F and h. The approach is mostly variational, but requires the use of results on an auxiliary system based upon fixed point theory and Leray-Schauder degree.
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1. INTRODUCTION

In this note, we consider the existence of solutions of the T-periodic boundary value problem

$$(\phi(u'))' = \nabla_u F(x,u) + h(x), \quad u(0) = u(T),\ u'(0) = u'(T), \tag{1.1}$$

where $\phi : B_a \to \mathbb{R}^n$ is a homeomorphism such that $\phi(0) = 0$, $\phi = \nabla\Phi$ for some real strictly convex function $\Phi \in C^1(B_a) \cap C(\overline{B_a})$ (a situation which occurs for the acceleration in special relativity), $F : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ is such that $\nabla_u F : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ exists and satisfies Carathéodory conditions, and $h \in (L^1(0,T))^n$.

In the case of the classical second order problem

$$u'' = \nabla_u F(x,u) + h(x), \quad u(0) = u(T),\ u'(0) = u'(T), \tag{1.2}$$

some existence results were proved, using the direct method of the calculus of variations, by Berger and Schechter [4, 5], when $F(x,u) - \langle h(x), u\rangle$ is coercive in u uniformly in $x \in [0,T]$, and in [14] when $\int_0^T [F(x,u) - \langle h(x), u\rangle]\,dx$ is coercive and, either $\nabla_u F(x,u)$ is bounded or $F(x,\cdot)$ is convex for a.e. $x \in [0,T]$. Numerous extensions of those results have been given for wider classes of potentials F and for u'' replaced

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by the p-Laplacian $(|u'|^{p-2}u)'$, and we refer in particular to [9],[16]-[24],[25]-[28] and their references.

An existence theorem was also proved for (1.2) in [13, 14], when F is periodic in each variable u_i for a.e. $x \in [0,T]$ and h has mean value zero. Such a result is easily extended to the case of the p-Laplacian as shown in [12]. The problem is more delicate in the case where u'' is replaced by a 'relativistic' differential operator $(\phi(u'))'$ like above. The scalar case was recently considered in [6] and the methodology of this paper is used here to obtain existence results in the case of system (1.1), under conditions upon F of the type covered in [5] and [14]. Like in [6], the treatment is essentially based upon the direct method of the calculus of variations and variational inequalities, but an auxiliary result is used, which is a consequence of some existence theorems proved in [2] using Leray-Schauder's method. A pure variational treatment of those questions remains to be done. On the other hand, no proof based upon topological methods of the results given here is known by now.

In Section 2, we introduce the class of homeomorphisms ϕ occuring in (1.1) and recall some of their properties used in the sequel. Section 3 surveys the approach, introduced in [2, 3] for (not necessarily variational) problems (1.1), and based upon a reduction to a fixed point problem treated with Leray-Schauder degree. An auxiliary existence and uniquess result is proved for subsequent use in the paper. In Section 4, we introduce the action functional

$$\mathcal{I}(u) = \int_0^T [\Phi(u') + F(x,u) + \langle h(x), u\rangle]\, dx$$

whose minima on a suitable closed convex set of the Banach space of Lipschitzian functions should provide solutions of (1.1). We give there a sufficient condition for the existence of a minimum, and the corresponding variational inequality. In Section 5, we show that the minima of $\mathcal{I}$ provide solutions of problem (1.1). Section 6 describes a result of Berger-Schechter's type for (1.1). In Section 7, we show that the sublinearity in u of $\nabla_u F$ in the classical case can be replaced by an arbitrary polynomial growth condition, and that the coercivity condition upon F can be weakened. In the last Sections 8 and 9, we consider the classes of convex and periodic potentials respectively.

In $\mathbb{R}^n$, we denote the usual inner product by $\langle \cdot, \cdot \rangle$ and the corresponding Euclidian norm by $|\cdot|$. We denote the usual norm in $L_p^n := (L_p(0,T))^n$ $(1 \le p \le \infty)$ by $\|\cdot\|_p$. C denotes the Banach space of continuous functions from $[0,T]$ into $\mathbb{R}^n$, endowed with the uniform norm $\|\cdot\|_\infty$, $AC \subset C$ the subspace of absolutely continuous functions from $[0,T]$ into $\mathbb{R}^n$, $C^1 \subset AC$ the subspace space of continuously differentiable functions, B_r the open ball of center 0 and radius r in any normed space. For any $u \in L_1^n$, we set

$$\overline{u} = \frac{1}{T}\int_0^T u\, dx, \quad \widetilde{u} = u - \overline{u}.$$

2. A CLASS OF HOMEOMORPHISM

The class of homeomorphisms which occurs in (1.1) is characterized by the following condition

(H_Φ) *ϕ is a homeomorphism from $B_a \subset \mathbb{R}^n$ onto $\mathbb{R}^n$ such that $\phi(0) = 0$, $\phi = \nabla\Phi$, with $\Phi : \overline{B_a} \to]-\infty, 0]$ of class C^1 on B_a, continuous and strictly convex on $\overline{B_a}$.*

So, ϕ is strictly monotone on B_a.

If $\Phi^* : \mathbb{R}^n \to \mathbb{R}$ is the Legendre-Fenchel transform of Φ defined by

$$\Phi^*(v) = \langle \phi^{-1}(v), v\rangle - \Phi[\phi^{-1}(v)] = \sup_{u \in \overline{B_a}} \{\langle u, v\rangle - \Phi(u)\},$$

then Φ^* is also strictly convex,

$$\Phi^*(v) \le a|v| - \inf_{\overline{B_a}} \Phi \circ \phi^{-1} := a|v| + d, \tag{2.1}$$

and, using the negativity of Φ,

$$\Phi^*(v) \ge \sup_{u \in \overline{B_a}} \langle v, u\rangle = a|v|, \tag{2.2}$$

so that Φ^* is coercive on $\mathbb{R}^n$ [14]. Adapting the reasoning of Proposition 2.4 in [14], we obtain that Φ^* is of class C^1. Hence $\phi^{-1} = \nabla\Phi^*$, so that

$$v = \nabla\Phi(u) = \phi(u), \quad u \in B_a \quad \Leftrightarrow \quad u = \phi^{-1}(v) = \nabla\Phi^*(v), \quad v \in \mathbb{R}^n.$$

Given $b \in \mathbb{R}^n$ and $g \in C$, let us define $\psi(b; g)$ by

$$\begin{aligned} \psi(b; g) &= \int_0^T \phi^{-1}[g(x) - b]\, dx = \int_0^T \nabla_b \Phi^*[g(x) - b]\, dx \\ &= \nabla_b \int_0^T \Phi^*[g(x) - b]\, dx = \nabla_b \Psi(b; g), \end{aligned}$$

where $\Psi(b; g)$ is defined by

$$\Psi(b; g) = \int_0^T \Phi^*[g(x) - b]\, dx.$$

The following result is proved in [3].

Lemma 2.1. *If $\phi = \nabla\Phi$, with Φ verifying Assumption (H_Φ), then, for each $g \in C$, the system $\psi(b; g) = 0$ has a unique solution $b := Q_\phi(g)$. Moreover, $Q_\phi : C \to \mathbb{R}^n$ is continuous and takes bounded sets into bounded sets.*

Example 2.2. Let us consider the C^∞-mapping $\Phi : \overline{B_1} \subset \mathbb{R}^n \to \mathbb{R}$, given by

$$\Phi(u) = -\sqrt{1 - |u|^2} \quad (u \in \overline{B_1}), \tag{2.3}$$

so that

$$-1 \le \Phi(u) \le 0 \quad (u \in \overline{B_1}), \quad \phi(u) = \nabla\Phi(u) = \frac{u}{\sqrt{1 - |u|^2}} \quad (u \in B_1).$$

As $|\cdot|^2$ is strictly convex on $\mathbb{R}^n$, it follows that Φ is strictly convex on $\overline{B_1}$. Furthermore, $\phi : B_1 \to \mathbb{R}^n$ is a homeomorphism such that, for any $v \in \mathbb{R}^n$.

$$\phi^{-1}(v) = \frac{v}{\sqrt{1+|v|^2}} = \nabla\Phi^*(v),$$

where $\Phi^*(v) = \sqrt{1+|v|^2}$ is strictly convex and of class C^∞ on $\mathbb{R}^n$. Hence, Assumption (H_Φ) with $a = 1$ holds for Φ given by (2.3).

3. FIXED POINT PROBLEM

We introduce the (possibly) nonlinear operators $N : C \to L^1$ and $f : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ satisfying the following assumptions :

(H_N) $N : C \to L^1$ *is continuous and takes bounded sets into bounded sets,*
(H_f) $f : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ *satisfies the* L^1*-Carathéodory conditions.*

We associate to f its *Nemytskii operator* $N_f : C \to L^1$ defined for a.e. $x \in [0,T]$ by

$$N_f(u)(x) = f[x, u(x)].$$

It is standard to show that N_f is continuous and takes bounded sets into bounded sets.

A solution of problem

$$(\phi(u'))' = N(u), \quad u(0) = u(T),\ u'(0) = u'(T) \tag{3.1}$$

is a function $u \in C^1$ such that $\phi(u') \in AC$ and the equations in (3.1) are satisfied (a.e. for the differential system).

The proof of the following proposition, essentially analogous to that given for the scalar case and for $f = f(x,u,u')$ in [2], is not repeated here. We define $P : C \to \mathbb{R}^n$ by $Pu := u(0)$, $Q : L^1 \to \mathbb{R}^n$ by $Qu := \overline{u}$, $H : L^1 \to AC^1$ by

$$Hu(x) = \int_0^x u(s)\,ds \quad (x \in [0,T]),$$

and we let $C_\# = \{u \in C : u(0) = u(T)\}$. When appropriate, we identify $\mathbb{R}^n$ with the subspace of constant functions in C.

Proposition 3.1. *If Assumptions (H_Φ) and (H_N) hold, u is a solution of problem (3.1) if and only if $u \in C_\#$ is a fixed point of the operator M defined on $C_\#$ by*

$$M(u) = Pu + QF(u) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I-Q)N](u). \tag{3.2}$$

Furthermore, $\|(M(u))'\|_\infty < a$ for all $u \in C_\#$ and M is completely continuous.

Consider now the periodic boundary value problem

$$(\phi(u'))' = f(x,u), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{3.3}$$

when Assumptions (H_Φ) and (H_f) hold. In order to apply Leray-Schauder degree [8] to the equivalent fixed point operator M given in (3.2) with $F = N_f$, we introduce, for $\lambda \in [0,1]$, by analogy to the semilinear situation considered in [10], the family of periodic boundary value problems

$$(\phi(u'))' = \lambda N_f(u) + (1-\lambda)QN_f(u), \quad u(0) = u(T),\ u'(0) - u'(T). \tag{3.4}$$

Notice that (3.4) coincide with (3.3) for $\lambda = 1$. For each $\lambda \in [0,1]$, the fixed point operator (3.2) on $C_\#$ associated to (3.4) is $\mathcal{M}(\lambda,\cdot)$, where $\mathcal{M}$ is defined on $[0,1]\times C_\#$ by

$$\mathcal{M}(\lambda,u) = Pu + QN_f(u) + H\circ\phi^{-1}\circ(I-Q_\phi)\circ[\lambda H(I-Q)N_f](u). \tag{3.5}$$

Using Lemma 2.1 and Arzelá-Ascoli's theorem it is not difficult to see that $\mathcal{M} : [0,1]\times C_\# \to C_\#$ is completely continuous.

A special case of the following existence and uniqueness result will be used in Section 5.

Proposition 3.2. *If assumption (H_Φ) holds, then, for any $e \in L_1^n$, $p > 1$ and any $b \neq 0$, the periodic problem*

$$(\phi(u'))' = b|u|^{p-2}u + e(x), \quad u(0) = u(T),\ u'(0) = u'(T) \tag{3.6}$$

has at least one solution, and any solution satisfies $\|u'\|_\infty < a$. Furthermore, the solution is unique if $b > 0$.

Proof. It is clear that $f(x,u) = b|u|^{p-2}u + e(x)$ satisfies Assumption (H_f). Let $\mathcal{M}$ be the fixed point operator (3.5) with $N_f(u) = b|u|^{p-2}u + e$, and let $(\lambda,u) \in [0,1]\times C_\#$ be such that $u = \mathcal{M}(\lambda,u)$. Then $u \in C^1$ and $\|u'\|_\infty < a$, so that, by (4.1),

$$\|\widetilde{u}\|_\infty < Ta. \tag{3.7}$$

Taking $x = 0$ in $u = \mathcal{M}(\lambda,u)$, we get, for $\lambda \in [0,1]$,

$$u(0) = Pu + QN_f(u)$$

which is equivalent to

$$\int_0^T [b|u|^{p-2}u + e(x)]\,dx = 0. \tag{3.8}$$

We claim that there exists $\rho > 0$ such that for any $\lambda \in [0,1]$ and any u such that $u = \mathcal{M}(\lambda,u)$, one has $\|\overline{u}\| < \rho$. If it were not the case, there would exist a sequence (λ_k) in $[0,1]$ and a nonzero sequence (u_k) in $C_\#$ such that $|\overline{u_k}| \to \infty$ and, using (3.8),

$$\int_0^T [b|\overline{u_k} + \widetilde{u}_k|^{p-2}(\overline{u_k} + \widetilde{u}_k) + e(x)]\,dx = 0 \quad (k \in \mathbb{N}),$$

and hence such that

$$b\int_0^T \left[\left|\frac{\overline{u_k}}{|\overline{u_k}|} + \frac{\widetilde{u}_k}{|\overline{u_k}|}\right|^{p-2}\left(\frac{\overline{u_k}}{|\overline{u_k}|} + \frac{\widetilde{u}_k}{|\overline{u_k}|}\right)\right] dx = -\frac{T\overline{e}}{|\overline{u_k}|^{p-1}} \quad (k \in \mathbb{N}). \tag{3.9}$$

Because of (3.7), $\widetilde{u}_k/|\overline{u_k}| \to 0$ uniformy on $[0,T]$, and, going if necessary to a subsequence, we can assume that $\overline{u_k}/|\overline{u_k}| \to v$ for some $v \in \mathbb{R}^n$ with $|v| = 1$. Letting

$k \to \infty$ in (3.9), we get $bv = 0$, a contradiction. Consequently, for any $\lambda \in [0,1]$, and any possible fixed point u of $\mathcal{M}(\lambda,\cdot)$, we have

$$\|u\|_\infty < \rho + Ta := R \tag{3.10}$$

Hence, the Leray-Schauder degree $d_{LS}[I - \mathcal{M}(\lambda,\cdot), B_R, 0]$ is well defined and independent of $\lambda \in [0,1]$. Using the reduction property of degree, and denoting by d_B the Brouwer degree in $\mathbb{R}^n$ [8, 10], we obtain

$$\begin{aligned} & d_{LS}[I - \mathcal{M}(1,\cdot), B_R, 0] = d_{LS}[I - \mathcal{M}(0,\cdot), B_R, 0] \\ = \; & d_{LS}[I - P - QN_f, B_R, 0] = d_B[-QN_f|_{\mathbb{R}^n}, B_R \cap \mathbb{R}^n, 0]. \end{aligned}$$

But, for any $c \in \mathbb{R}^n$,

$$QN_f(c) = b|c|^{p-2}c + \overline{e},$$

and it is easy to see that

$$d_B[-QN_f|_{\mathbb{R}^n}, B_R \cap \mathbb{R}^n, 0] = (-\ sgn\ b)^n.$$

Hence $\mathcal{M}(1,\cdot)$ has a fixed point u, and (3.6) has a solution, with $\|u'\|_\infty < a$.

For uniqueness, if we assume that $b > 0$ and that (3.6) has two solutions u and v, then, substracting the equations, taking the inner product by $u - v$ of the result and integrating over $[0,T]$, we obtain

$$0 \geq -\langle \phi(u') - \phi(v'), u' - v' \rangle = b \int_0^T \langle |u|^{p-2}u - |v|^{p-2}v, u - v \rangle \, dx \geq 0,$$

and hence

$$\int_0^T \langle |u|^{p-2}u - |v|^{p-2}v, u - v \rangle \, dx = 0. \tag{3.11}$$

Consequently, because of the monotonicity and the continuity of the integrated function,

$$\langle |u|^{p-2}u - |v|^{p-2}v, u - v \rangle = 0.$$

which easily implies that

$$\left(|u|^{p-1} - |v|^{p-1}\right)(|u| - |v|) = 0$$

and hence $|u| = |v|$. Introduced in (3.11), this gives

$$\int_0^T |u|^{p-2}|u - v|^2 \, dx = 0,$$

and consequently $u = v$. ∎

4. MINIMIZATION PROBLEM

Let $a > 0$, $\Phi : \overline{B_a} \to \mathbb{R}$ satisfy Assumption (H_Φ), and $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfy the condition :

(H_F) *$F(\cdot, u)$ is measurable on $[0,T]$ for every $u \in \mathbb{R}^n$, $F(x,\cdot)$ is continuously differentiable on $\mathbb{R}^n$ for a.e. $x \in [0,T]$, and $\nabla_u F$ satisfies the L^1-Carathéodory conditions.*

Let $Lip^n_{\#} := (Lip_{\#}[0,T])^n$ denote the space of functions $u : [0,T] \to \mathbb{R}^n$ such that $u(0) = u(T)$, which are Lipschitzian with Lipschitz constant

$$[u]_{0,1} := \sup_{x,y \in [0,T], x \neq y} \frac{|u(x) - u(y)|}{|x-y|} < +\infty.$$

With the norm

$$\|u\|_{0,1} := \|u\|_\infty + [u]_{0,1},$$

$Lip^n_{\#}$ is a Banach space. Any element of $Lip^n_{\#}$ is a.e. differentiable, u' corresponds to the distributional derivative of u, and $\|u'\|_\infty = [u]_{0,1}$.

Notice that if $u \in Lip^n_{\#}$, then $\widetilde{u}_j$ vanishes at some $y_j \in [0,T]$ $(j = 1, \ldots, n)$, and therefore, for all $x \in [0,T]$, we have

$$|\widetilde{u}_j(x)| = |\widetilde{u}_j(x) - \widetilde{u}_j(y_j)| \leq \int_0^T |u_j'(t)|\,dt \leq T[u_j]_{0,1} \quad (j = 1, \ldots, n),$$

and hence

$$|\widetilde{u}|_\infty \leq T[u]_{0,1}. \tag{4.1}$$

If K denotes the closed convex subset of $Lip^n_{\#}$ defined by

$$K := \{u \in Lip^n_{\#} : |u'(x)| \leq a \ for \ a.e. \ x \in [0,T]\},$$

then the action integral

$$\mathcal{I}(u) := \int_0^T \{\Phi(u') + F(x,u) + \langle h, u\rangle\}\,dx \tag{4.2}$$

is well defined on K for any $h \in L^n_1$. This happens for example when Φ is given by (2.3), in which case (4.2) can be seen as the action integral associated to a system of relativistic forced oscillators.

The following lemma, given in [6] for $n = 1$, is useful to prove the lower semicontinuity of $\mathcal{I}$. The proof in [6] applies *verbatim* to arbitrary n.

Lemma 4.1. *If assumption (H_Φ) holds, then, for any sequence $(u_j)_{j\in\mathbb{N}}$ in K which converges uniformly on $[0,T]$ to some $u \in K$, one has*

$$\liminf_{j\to\infty} \int_0^T \Phi(u_j')\,dx \geq \int_0^T \Phi(u')\,dx. \tag{4.3}$$

We now give a sufficient condition for the existence of a minimum to $\mathcal{I}$.

Theorem 4.2. *If assumptions (H_Φ) and (H_F) hold, then $\mathcal{I}$ has a minimum over K if and only if it has a minimizing subsequence (u_k) such that $(\overline{u_k})$ is bounded.*

Proof. The necessity is obvious. For sufficiency, let (u_k) in K be a minimizing sequence for $\mathcal{I}$ with $(\overline{u_k})$ bounded. By (4.1), (u_k) is bounded in uniform norm and equicontinuous. So we can assume, using Arzelá-Ascoli's theorem and going if necessary to a subsequence, that (u_k) converges uniformly in $[0,T]$ to some continuous $u^* \in C_\#$. From the relations

$$\frac{|u_k(x) - u_k(y)|}{|x-y|} \leq a \quad (x \neq y \in [0,T], k \in \mathbb{N})$$

we easily get that $u^* \in K$. Consequently, using Lemma 4.1, we have

$$\inf_K \mathcal{I} = \lim_{k\to\infty} \mathcal{I}(u_k) \geq \mathcal{I}(u^*)$$

so that u^* minimizes $\mathcal{I}$ over K. ■

The following lemma provides the variational inequality satisfied by a minimizer of $\mathcal{I}$.

Lemma 4.3. *If u minimizes $\mathcal{I}$ overs K, then*

$$\int_0^T [\Phi(v') - \Phi(u') + \langle \nabla_u F(x,u) + h(x), v-u\rangle]\, dx \geq 0 \quad \textit{for all} \quad v \in K. \tag{4.4}$$

Proof. Let $v \in K$. By assumption, we have, for all $\lambda \in (0,1]$,

$$\mathcal{I}(u) \leq \mathcal{I}[u + \lambda(v-u)],$$

i.e.

$$\int_0^T \{\Phi[u' + \lambda(v'-u')] - \Phi(u') + F[x, u+\lambda(v-u)] - F(x,u) \\ + \lambda\langle h(x), v-u\rangle\}\, dx \geq 0.$$

Applying the convexity of Φ we deduce that

$$\int_0^T \big\{\Phi(v') - \Phi(u') + \lambda^{-1}[F(x, u+\lambda(v-u)) - F(x,u)] \\ + \langle h(x), v-u\rangle\big\}\, dx \geq 0.$$

By Lebesgue dominated convergence theorem, we obtain, when $\lambda \searrow 0$,

$$\int_0^T [\Phi(v') - \Phi(u') + \langle \nabla_u F(x,u), v-u\rangle + \langle h(x), v-u\rangle]\, dx \geq 0.$$

■

5. EXISTENCE THEOREM

To obtain further information about the minimizer u, let us consider the auxiliary problem

$$(\phi(u'))' = u + e(x), \quad u(0) = u(T),\ u'(0) = u'(T), \tag{5.1}$$

where ϕ satisfies Assumption (H_Φ) and $e \in L_1^n$.

We know by Proposition 3.2 that, for any $e \in L_1^n$, problem (5.1) has a unique classical solution $\widehat{u}_e$, and $\|\widehat{u}_e'\|_\infty < a$.

Lemma 5.1. *For any $e \in L_1^n$, $\widehat{u}_e \in K$ is such that*

$$\int_0^T [\Phi(v') - \Phi(\widehat{u}_e') + \langle \widehat{u}_e + e(x), v - \widehat{u}_e\rangle]\, dx \geq 0 \quad \text{for all} \quad v \in K.$$

Proof. Given $v \in K$, we have, using integration by parts and (5.1),

$$\int_0^T [\Phi(v') - \Phi(\widehat{u}_e')]\, dx \geq \int_0^T \langle \phi(\widehat{u}_e'), v' - \widehat{u}_e'\rangle\, dx$$
$$= -\int_0^T \langle (\phi(\widehat{u}_e'))', v - \widehat{u}_e\rangle\, dx = -\int_0^T \langle \widehat{u}_e + e(x), v - \widehat{u}_e\rangle\, dx.$$

■

We can now combine the results of the previous sections to relate the existence of at least one (classical) solution for the periodic boundary value problem

$$(\phi(u'))' = \nabla_u F(x,u) + h(x), \quad u(0) = u(T),\ u(0) = u'(T) \tag{5.2}$$

to the existence of a minimizer of $\mathcal{I}$ on K.

Theorem 5.2. *If assumptions (H_Φ) and (H_F) hold, then any minimizer of $\mathcal{I}$ on K is a solution of (5.2).*

Proof. Let u be a minimizer of $\mathcal{I}$ over K. By Lemma 4.3, u satisfies the variational inequality (4.4), which can be written

$$\int_0^T [\Phi(v') - \Phi(u') + \langle u, v-u\rangle + \langle \nabla_u F(x,u) + h(x) - u, v - u\rangle]\, dx \geq 0$$
$$\text{for all } v \in K,$$

so that u is a solution of the variational inequality

$$\int_0^T [\Phi(v') - \Phi(u') + \langle u + e_u(x), v - u\rangle]\, dx \geq 0 \quad \text{for all } v \in K, \tag{5.3}$$

where

$$e_u = \nabla_u F[\cdot, u(\cdot)] + h - u \in L_1^n.$$

Now, given any $w \in K$, the unique solution $\widehat{u}_{e_w}$ of problem (5.1) with $e = e_w$ satisfies, by Lemma 5.1,

$$\int_0^T [\Phi(v') - \Phi(\widehat{u}_{e_w}') + \langle \widehat{u}_{e_w} + e_w(x), v - \widehat{u}_{e_w}\rangle]\, dx \geq 0 \quad \text{for all } v \in K. \tag{5.4}$$

Chosing $v = \widehat{u}_{e_u}$ in (5.3), $w = v = u$ (u the minimizer of $\mathcal{I}$ over K) in (5.4), and adding the resulting inequalities, we obtain

$$\int_0^T |u - \widehat{u}_{e_u}|^2\, dx \leq 0. \tag{5.5}$$

It follows from (5.5) that $u = \widehat{u}_{e_u}$ and hence that $\|u'\|_\infty = \|\widehat{u}_{e_u}'\|_\infty < a$. Moreover u is a classical solution of (5.2), since $\widehat{u}_{e_u}$ is a classical solution of (5.1) with $e = e_u$. ■

6. COERCIVE POTENTIAL

As a first application, let us consider the case of a coercive potential F. The following result was first proved by Berger and Schechter [4, 5] for the classical problem (1.2).

Theorem 6.1. *Assume that Assumptions (H_Φ) and (H_F) hold. Then, for all $h \in L_1^n$ such that*

$$F(x,u) + \langle h(x), u\rangle \to +\infty \text{ as } |u| \to \infty \text{ uniformly for a.e. } x \in [0,T], \tag{6.1}$$

problem (5.2) has at least one solution minimizing $\mathcal{I}$ on K.

Proof. By Theorems 4.2 and 5.2, it suffices to prove that $\mathcal{I}$ admits a minimizing sequence (u_k) in K such that $(\overline{u_k})$ is bounded. From Assumption (6.1), given $r > 0$, there exists $\rho > 0$ such that

$$F(x,u) + \langle h(x), u\rangle \geq T^{-1}r - \min \Phi \tag{6.2}$$

for every $u \in \mathbb{R}^n$ such that $|u| \geq \rho$ and a.e. $x \in [0,T]$. Hence, for any $u \in K$ such that $|\overline{u}| \geq R := \rho + Ta$, we have $|u(x)| \geq \rho$ for all $x \in [0,T]$, and, using (6.2), $\mathcal{I}(u) \geq r$. In other words, $\mathcal{I}(u) \to +\infty$ when $u \in K$ and $|\overline{u}| \to \infty$, which implies that $\mathcal{I}$ is bounded from below, and any minimizing sequence (u_k) in K is such that $(\overline{u_k})$ is bounded. ■

For the use in examples, let us define the continuous mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ by

$$S(u) := (\sin u_1, \sin u_2, \ldots, \sin u_n),$$

so that

$$S(u) = \nabla c(u), \quad \text{with} \quad c(u) := -\sum_{j=1}^n \cos u_j \quad (u \in \mathbb{R}^n).$$

Example 6.2. *Given any $b \in L_\infty^n$ such that essinf $b > 0$, problem*

$$\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)' = b(x)\frac{u}{\sqrt{1+|u|^2}} + S(u) + h(x), \quad u(0) = u(T),\ u'(0) = u'(T) \tag{6.3}$$

has at least one solution for all $h \in L_\infty^n$ such that $\|h\|_\infty <$ essinf b.

Indeed, $F(x,u) = b(x)\sqrt{1+|u|^2} + c(u)$ and, for a.e. $x \in [0,T]$ and all sufficiently large $|u|$,

$$F(x,u) + \langle h(x), u\rangle \geq |u|\left(\frac{\text{essinf } b}{2}\sqrt{1+|u|^{-2}} - n^{1/2}|u|^{-1} - \|h\|_\infty\right).$$

The right-hand member tends to $+\infty$ as $|u| \to \infty$.

Example 6.3. *Given any $b \in L_\infty^n$ such that essinf $b > 0$, problem*

$$\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)' = b(x)e^{|u|^2}u + S(u) + h(x),\ u(0) = u(T),\ u'(0) = u'(T) \tag{6.4}$$

has at least one solution for all $h \in L_\infty^n$.

Indeed, $F(x,u) = \frac{b(x)}{2}e^{|u|^2} + c(u)$ and, for all sufficiently large $|u|$,

$$F(x,u) + \langle h(x), u\rangle \geq \frac{\text{essinf } b}{4} e^{|u|^2} - (n^{1/2} + \|h\|_\infty)|u|.$$

The right-hand member tends to $+\infty$ as $|u| \to \infty$.

7. NONLINEARITY WITH POLYNOMIAL GROWTH

Let us consider the case of problem (5.2) with a nonlinearity $\nabla_u F$ having a polynomial growth in u of power $\alpha \geq 0$, and a potential F satisfying a semi-coercivity condition of the Ahmad-Lazer-Paul type [1]. For the classical problem (1.2) the case where $\alpha = 0$ was considered in [14], and the case where $\alpha \in [0,1)$ in [20].

Define the mapping $\overline{F} : \mathbb{R}^n \to \mathbb{R}$ by

$$\overline{F}(u) = \frac{1}{T}\int_0^T F(x,u)\,dx.$$

Theorem 7.1. *Assume that Assumptions* (H_Φ), (H_F) *hold, and that there exists* $\alpha \geq 0$, $g, k \in L^1$ *nonnegative such that, for a.e.* $x \in [0,T]$ *and all* $u \in \mathbb{R}^n$, *one has*

$$|\nabla_u F(x,u)| \leq g(x)|u|^\alpha + k(x). \tag{7.1}$$

Then, for all $h \in L_1^n$ *such that*

$$|u|^{-\alpha}\left[\overline{F}(u) - \langle \overline{h}, u\rangle\right] \to +\infty \quad as \quad |u| \to \infty, \tag{7.2}$$

problem (5.2) has at least one solution minimizing $\mathcal{I}$ *on* K.

Proof. By Theorems 4.2 and 5.2, it suffices to prove that $\mathcal{I}$ admits a minimizing sequence (u_k) in K such that $(\overline{u_k})$ is bounded. Using the elementary inequality in $\mathbb{R}^n$ $|y+z|^\alpha \leq 2^\alpha(|y|^\alpha + |z|^\alpha)$, we have, for all $u \in K$,

$$\begin{aligned}
\mathcal{I}(u) &= \int_0^T [\Phi(u') + F(x,\overline{u}) + F(x,u) - F(x,\overline{u}) + \langle h(x), u\rangle]\,dx \\
&\geq T\min\Phi + \int_0^T [F(x,\overline{u}) + \langle h(x), \overline{u}\rangle]\,dx \\
&+ \int_0^T \langle \int_0^1 \nabla_u F(x, \overline{u} + s\widetilde{u})\,ds + \widetilde{h}(x), \widetilde{u}\rangle\,dx \\
&\geq T\min\Phi + T[\overline{F}(\overline{u}) + \langle \overline{h}, \overline{u}\rangle] \\
&- \int_0^T\int_0^1 [g(x)|\overline{u} + s\widetilde{u}|^\alpha + k(x)]\,|\widetilde{u}|\,ds\,dx - \|h\|_1\|\widetilde{u}\|_\infty \\
&\geq T\min\Phi + T[\overline{F}(\overline{u}) + \langle \overline{h}, \overline{u}\rangle] \\
&- \|g\|_1 2^\alpha[|\overline{u}|^\alpha + (Ta)^\alpha]Ta - (\|k\|_1 + \|h\|_1)Ta,
\end{aligned}$$

where we have used (4.1). Hence

$$\begin{aligned}
\mathcal{I}(u) &\geq T\min\Phi + |\overline{u}|^\alpha\left\{T|\overline{u}|^{-\alpha}[\overline{F}(\overline{u}) + \langle \overline{h}, \overline{u}\rangle] - \|g\|_1 2^\alpha Ta\right\} \\
&- [(2Ta)^\alpha\|g\|_1 + \|k\|_1 + \|h\|_1]Ta
\end{aligned} \tag{7.3}$$

As Assumption (7.2) implies the existence of some $\rho > 0$ such that the second term in the right-hand member of (7.3) is positive for $|\overline{u}| \geq \rho$, $\mathcal{I}$ is bounded from below on K.

Let (u_k) be a minimizing sequence of $\mathcal{I}$ in K. By Assumption (7.2), the right-hand member of (7.3) tends to $+\infty$ when $|\overline{u}| \to \infty$, so that $(\overline{u}_k)$ is bounded. ∎

Remark 7.2. In the classical case (1.2) [20], $\alpha \in [0,1)$ in (7.1) and Assumption (7.2) is replaced by the stronger condition

$$|u|^{-2\alpha}\left[\overline{F}(u) - \langle \overline{h}, u\rangle\right] \to +\infty \quad as \quad |u| \to \infty.$$

Example 7.3. *Given any $b \in L_1$ such that $\overline{b} > 0$, problem (6.3) has at least one solution for all $h \in L_1^n$ such that $|\overline{h}| < \overline{b}$.*

Indeed, we have in this case $F(x,u) = b(x)\sqrt{1+|u|^2} + c(u)$, $\alpha = 0$, and, for any $v \in \mathbb{R}^n \setminus \{0\}$,

$$\overline{b}\sqrt{1+|v|^2} + c(v) + \langle \overline{h}, v\rangle \geq |v|\left[\overline{b}\sqrt{1+|v|^{-2}} - n^{1/2}|v|^{-1} - |\overline{h}|\right],$$

with the right-hand member tending to $+\infty$ when $|v| \to \infty$. With respect to Example 6.2, the use of the boundedness condition allows weakening the conditions upon b and h from essinf $b > 0$ and $\|h\|_\infty <$ essinf b to $\overline{b} > 0$ and $|\overline{h}| < \overline{b}$.

Remark 7.4. Anticoercive case. Using Rabinowitz' saddle point theorem, it is proved in [14] that *the classical periodic problem (1.2) has at least one solution when Assumption (7.1) with $\alpha = 0$ holds and*

$$\overline{F}(u) + \langle \overline{h}, u\rangle \to -\infty \quad as \quad |u| \to \infty. \tag{7.4}$$

It is an open problem to know if a corresponding result holds for problem (5.2). Notice that in the case of (1.2), if we split $(H^1_{\#}[0,T])^n$ as the direct sum of the space $\overline{H}$ of constant mappings and the space $\widetilde{H}$ of mappings with mean value zero, the corresponding action integral is coercive on $\widetilde{H}$ and anticoercive on $\overline{H}$, giving the saddle point structure. In the relativistic case, $\int_0^T \Phi(u')\,dx$ is bounded on K so that, when (7.4) holds, $\mathcal{I}$ is anticoercive. But the existence of a maximum on K is not guaranteed, because $\mathcal{I}$ is not *upper semi-continuous* with respect to uniform convergence.

8. CONVEX POTENTIAL

The polynomial growth condition upon $\nabla_u F$ can be dropped and condition (7.2) can be taken with $\alpha = 0$ in the case of a potential F convex in u. The following theorem extends to system (5.2) a result of [14] for the classical system (1.2).

Theorem 8.1. *Assume that Assumptions (H_Φ), (H_F) hold, and that $F(x,\cdot)$ is convex for a.e. $x \in [0,T]$. Then, for all $h \in L_1^n$ such that*

$$\overline{F}(u) + \langle \overline{h}, u\rangle \to +\infty \quad as \quad |u| \to \infty, \tag{8.1}$$

problem (5.2) has at least one solution minimizing $\mathcal{I}$ on K.

Proof. Again, by Theorems 4.2 and 5.2, it suffices to prove that $\mathcal{I}$ admits a minimizing sequence (u_k) in K such that $(\overline{u_k})$ is bounded. By Assumption (8.1), the real function $\overline{F} + \langle \overline{h}, \cdot \rangle$ achieves a minimum at some point $\overline{v} \in \mathbb{R}^n$, for which

$$\nabla \overline{F}(\overline{v}) + \overline{h} = 0. \tag{8.2}$$

Now, by the convexity of $F(x, \cdot)$,

$$\begin{aligned}
\mathcal{I}(u) &= \int_0^T [\Phi(u') + F(x, \overline{v}) + \langle h(x), \overline{v} \rangle]\, dx \\
&+ \int_0^T [F(x, u) - F(x, \overline{v}) + \langle h(x), u - \overline{v} \rangle]\, dx \\
&\geq T \min \Phi + T[\overline{F}(\overline{v}) + \langle \overline{h}, \overline{v} \rangle] + \int_0^T \langle \nabla_u F(x, \overline{v}) + h(x), u - \overline{v} \rangle\, dx \\
&= T \min \Phi + T[\overline{F}(\overline{v}) + \langle \overline{h}, \overline{v} \rangle] + \int_0^T \langle \nabla_u F(x, \overline{v}) + h(x), \widetilde{u} \rangle\, dx \\
&\geq T \min \Phi + T[\overline{F}(\overline{v}) + \langle \overline{h}, \overline{v} \rangle] - aT \|\nabla_u F(\cdot, \overline{v}) + h\|_1. \qquad (8.3)
\end{aligned}$$

Consequently, $\mathcal{I}$ is bounded from below on K.

Let (u_k) be a minimizing sequence of $\mathcal{I}$ in K. Without loss of generality, we can assume that

$$\mathcal{I}(u_k) \leq \inf_K \mathcal{I} + 1.$$

The convexity of $F(x, \cdot)$ implies that, for a.e. $x \in [0, T]$ and all $k \in \mathbb{N}$, one has

$$F[x, \overline{u_k}/2] = F[x, (1/2)(u_k(x) - \widetilde{u_k}(x))] \leq \frac{1}{2} F[x, u_k(x)] + \frac{1}{2} F[x, -\widetilde{u_k}(x)].$$

Hence, using (4.1),

$$\begin{aligned}
1 + \inf_K \mathcal{I} &\geq \mathcal{I}(u_k) \geq T \min \Phi + 2 \int_0^T [F(x, \overline{u_k}/2) + \langle h(x), \overline{u_k}/2 \rangle]\, dx \\
&- \int_0^T [F(x, -\widetilde{u_k}) - \langle h(x), \widetilde{u_k} \rangle]\, dx \\
&\geq 2T[\overline{F}(\overline{u_k}/2) + \langle \overline{h}, \overline{u_k}/2 \rangle] + T \min \Phi \\
&- \max_{|v| \leq Ta} \int_0^T [F(x, v) + \langle h(x), v \rangle]\, dx.
\end{aligned}$$

Condition (8.1) implies that $(\overline{u_k})$ is bounded. ∎

Example 8.2. *Given any $b \in L^1$ such that $b(x) \geq 0$ for a.e. $x \in [0, T]$ and $\overline{b} > 0$, problem*

$$\left(\frac{u'}{\sqrt{1 - |u'|^2}} \right)' = b(x) e^{|u|^2} u + h(x), \quad u(0) = u(T),\ u'(0) = u'(T)$$

has at least one solution for all $h \in L^n_1$.

Indeed, $F(x,u) = \frac{b(x)}{2} e^{|u|^2}$ is convex in u for a.e. $x \in [0,T]$,

$$\overline{F}(u) + \langle \overline{h}, u \rangle \geq \frac{\overline{b}}{2} e^{|u|^2} - |\overline{h}||u|,$$

and the right-hand member tends to $+\infty$ as $|u| \to \infty$.

With respect to Example 6.3, the use of the convexity of F allows to replace the assumption essinf $b > 0$ by the weaker one $b(x) \geq 0$ and $\overline{b} > 0$. But the oscillatory term $S(u)$ has to be dropped.

Remark 8.3. More general classes of potentials than convex ones have been considered, like ones involving γ-quasisubadditive [17] or (λ, μ)-subconvex potentials [27], or potentials subquadratic in Rabinowitz sense [22]. We will not consider those classes here.

Remark 8.4. An easy consequence of Theorem 8.1 is that, *for any $b > 0$ and $h \in L_1^n$, problem (3.6) has a unique solution, which minimizes over K the functional*

$$\mathcal{I}_p(u) = \int_0^T [\Phi(u') + b\frac{|u|^p}{p} + \langle h(x), u \rangle]\, dx.$$

Indeed, $F(x,u) = b\frac{|u|^p}{p}$ is strictly convex and

$$b\frac{|u|^p}{p} - \langle \overline{h}, u \rangle \to +\infty \quad as \quad |u| \to \infty.$$

The uniqueness follows from the fact that $\mathcal{I}_p$ is strictly convex.

On the other hand, Theorem 3.2 implies also the *existence of a solution of problem (3.6) for any $b < 0$ and $h \in L_1^n$*. It is an open problem to find a variational proof of this result. In the classical case, the proof uses the dual least action principle [13].

9. PERIODIC POTENTIAL

We now consider the case of a potential F periodic with respect to each variable u_j and extend to system (5.2) a result of [13] (see also [14]) for the classical system (1.2), and a result of [6] for the scalar relativistic case.

Let $e_i \in \mathbb{R}^n$ be defined by $(e_i)_j = \delta_{ij}$ $(i,j = 1,\ldots,n)$.

Theorem 9.1. *Assume that Assumptions (H_Φ), (H_F) hold, and that there exists $T_i > 0$ such that*

$$F(x, u + T_i e_i) = F(x,u) \quad (i = 1,\ldots,n) \tag{9.1}$$

for a.e. $x \in [0,T]$ and all $u \in \mathbb{R}^n$. Then, for all $h \in L_1^n$ such that

$$\overline{h} = 0, \tag{9.2}$$

problem (5.2) has at least one solution minimizing $\mathcal{I}$ on K.

Proof. Conditions (9.1) and (9.2) imply that, for any $u \in K$, and $1 \leq i \leq n$, one has

$$\mathcal{I}(u + T_i e_i) = \int_0^T [\Phi(u') + F(x,u) + \langle h, u\rangle]\,dx + T\langle \overline{h}, T_i e_i\rangle = \mathcal{I}(u).$$

Hence it is equivalent to minimize $\mathcal{I}$ over

$$\widehat{K} := \{u \in K : 0 \leq \overline{u}_i \leq T_i \quad (i = 1, \ldots, n)\}.$$

One shows, like in the proof of Theorem 7.1 that $\mathcal{I}$ is bounded from below over $\widehat{K}$. Any minimizing sequence (u_k) in $\widehat{K}$ is obviously such that $(\overline{u_k})$ is bounded. ∎

Example 9.2. *Given any $A \in \mathbb{R}$, problem*

$$\left(\frac{u'}{\sqrt{1 - |u'|^2}}\right)' + AS(u) = h(x), \quad u(0) = u(T),\ u'(0) = u'(T)$$

has at least one solution for all $h \in L_1^n$ such that $\overline{h} = 0$.

This corresponds to $F(x,u) = -Ac(u)$, so that $F(x, u + 2\pi e_i) = F(x,u)$ for all $x \in [0,T]$, $u \in \mathbb{R}^n$ and $1 \leq i \leq n$.

Remark 9.3. It has been proved independently in [7, 11, 15] (see also [14]), using Lusternik-Schnirelman arguments, that *when Assumptions (H_F), (9.1) and (9.2) hold, the classical problem (1.2) has at least $n+1$ geometrically distinct solutions, i.e. solutions whose i^{th} component do not differ by a multiple of T_i $(i = 1, \ldots, n)$.* It is an open problem to know if such a result holds for (5.2).

REFERENCES

[1] S. Ahmad, A.C. Lazer, J.L. Paul, Elementary critical point theory and perturbations of elliptic boundary value problems, *Indiana Univ. Math. J.* **25** (1976), 933–944.

[2] C. Bereanu, J. Mawhin, Existence and multiplicity results for some nonlinear problems with singular ϕ-laplacian, *J. Differential Equations* **243** (2007), 536–557.

[3] C. Bereanu, J. Mawhin, Boundary value problems for some nonlinear systems with singular ϕ-Laplacian, *J. Fixed Point Theory Appl.* **4** (2008), 57–75.

[4] M.S. Berger, *Nonlinearity and Functional Analysis*, Academic Press, New York, 1977.

[5] M.S. Berger, M. Schechter, On the solvability of semi-linear gradient operator equations, *Adv. in Math.* **25** (1977), 97–132.

[6] H. Brezis, J. Mawhin, Periodic solutions of the forced relativistic pendulum, *Differential Integral Equations* **23** (2010), 801–810.

[7] K.C. Chang, On the periodic nonlinearity and the multiplicity of solutions, *Nonlinear Anal.* **13** (1989), 527–537.

[8] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[9] J. Ma, C.L. Tang, Periodic solutions for some nonautonomous second-order systems, *J. Math. Anal. Appl.* **275** (2002), 482–494.

[10] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, CBMS Series No. 40, American Math. Soc., Providence, 1979.

[11] J. Mawhin, Forced second order conservative systems with periodic nonlinearity, *Ann. Inst. Henri-Poincaré Anal. Non Linéaire* **6** suppl. (1989), 415–434.

[12] J. Mawhin, Periodic solutions of the forced pendulum: classical vs relativistic, *Le Matematiche* **65** (2010), 97–107.

[13] J. Mawhin, M. Willem, Variational methods and boundary value problems for vector second order differential equations and applications to the pendulum equation, in: *Nonlinear Analysis and Optimization*, Lect. Notes Math. No. 1107, Springer, Berlin, 1984, 181–192.

[14] J. Mawhin, M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, New York, 1989.

[15] P. Rabinowitz, On a class of functionals invariant under a Z_n action, *Trans. Amer. Math. Soc.* **310** (1988), 303–311.

[16] M. Schechter, Periodic non-autonomous second-order dynamical systems, *J. Differential Equations* **223** (2006), 290–302.

[17] C.L. Tang, Periodic solutions of non-autonomous second order systems with γ-quasisubadditive potential, *J. Math. Anal. Appl.* **189** (1995), 671–675.

[18] C.L. Tang, Periodic solutions of non-autonomous second order systems, *J. Math. Anal. Appl.* **202** (1996), 465–469.

[19] C.L. Tang, Some existence results for periodic solutions of non-autonomous second order systems. *Acad. Roy. Belg. Bull. Cl. Sci.* (6) **8** (1997),13–19.

[20] C.L. Tang, Periodic solutions for nonautonomous second order systems with sublinear nonlinearity. *Proc. Amer. Math. Soc.* **126** (1998), 3263–3270.

[21] C.L. Tang, X.P. Wu, Periodic solutions for second order systems with not uniformly coercive potential, *J. Math. Anal. Appl.* **259** (2001), 386–397.

[22] C.L. Tang, X.P. Wu, Notes on periodic solutions of subquadratic second order systems, *J. Math. Anal. Appl.* **285** (2003), 8–16.

[23] C.L. Tang, X.P. Wu, A note on periodic solutions of nonautonomous second-order systems, *Proc. Amer. Math. Soc.* **132** (2004), 1295–1303.

[24] Y. Tian, G.S. Zhang, W.G. Ge, Periodic solutions for a quasilinear non-autonomous second-order system, *J. Appl. Math. Comput.* **22** (2006), 263–271.

[25] Z.Y. Wang, J.H. Zhang, Periodic solutions of non-autonomous second order systems with p-Laplacian, *Electronic J. Differential Equations* **2009-17** (2009), 1–12.

[26] X.P. Wu, Periodic solutions for nonautonomous second-order systems with bounded nonlinearity, *J. Math. Anal. Appl.* **230** (1999), 135–141.

[27] X.P. Wu, C.L. Tang, Periodic solutions of a class of nonautonomous second order systems, *J. Math. Anal. Appl.* **236** (1999), 227–235.

[28] F.K. Zhao, X. Wu, Existence and multiplicity of periodic solution for non-autonomous second-order systems with linear nonlinearity, *Nonlinear Anal.* **60** (2005), 325–335.