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SOME REMARKS ON MATHER'S THEOREM AND AUBRY-MATHER SETS

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Dedicated to Professor J.R.L. Webb
on the occasion of his retirement

ABSTRACT. We illustrate Mather's theorem and its applications to the classification of Aubry-Mather sets. We discuss the equivalence of various definitions of Aubry-Mather set available in the literature.

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1. INTRODUCTION AND PRELIMINARIES

The purpose of this paper is to present and discuss some notions and results in Aubry-Mather theory. This research area was originated by the seminal papers by Mather [24] and Aubry-LeDaeron [1].

In his 1982 paper [24], Mather has described in terms of the "rotation number" the qualitative properties of the orbits of an area-preserving "twist" homeomorphism of the annulus. The definitions of twist map, of rotation number and Mather's theorem can be found in Definition 2.1, formula (1.1), Definition 2.14 and Theorem 2.2 below.

Independently of Mather in 1983 Aubry-LeDaeron [1], while treating the problem of the construction of one-dimensional crystals (Frenkel-Kontorova model), obtained substantially the same result.

Shortly after these pioneering papers, the abstract situation described in [1],[24] has been identified with the existence of an "Aubry-Mather" set.

A complete survey on the many contributions obtained after [1],[24] is beyond the aims of this paper. We shall limit ourselves to describing some of the earlier results

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in Aubry-Mather theory (cf. [12], [13], [24], [29]) and some more recent achievements [15], [16], [17], [34], [35]. For a more complete understanding of Aubry-Mather theory we refer, among the many abstract results, to the contributions of Bangert [2], Forni-Mather [11], Mather [25],[26],[27],[28], Moser [29], [30], [31], Rogel [37] and references therein; as for the applicability of Aubry-Mather theory to the qualitative theory of differential equations, we give some hint in Section 3 and refer to the forthcoming paper [6].

A (possibly interesting) aspect of our paper is that we give a systematic exposition of the above quoted results and develop complete (sometimes new) proofs, which are omitted or only sketched in the literature. Another original feature of our paper is that Theorems 2.16 and 2.19 below are deduced from Mather's original result (Theorem 2.2). Moreover, since the definition itself of Aubry-Mather set has been introduced in different ways by different authors, we prove the equivalence of such definitions (cf. Theorem 2.21).

In the rest of this Section we recall some preliminary concepts and results which are necessary for the rest of the paper.

Let us start with by fixing some notation and by recalling some definitions. We will deal with annuli and cylinders. Precisely, we set $A := S^1 \times [a, b]$, $\tilde{A} := \mathbb{R} \times [a, b]$ and $C := S^1 \times \mathbb{R}$, $\tilde{C} := \mathbb{R} \times \mathbb{R}$. The projections of A (or C) over the first and second component will be denoted by π_1, π_2, respectively (with a minor abuse of notation, we shall occasionally use the same symbol $\pi_i, i = 1, 2$ for projections of $\tilde{A}, \tilde{C}$ as well. By π_A we indicate the projection of $\tilde{A}$ on A. For a homeomorphism $\bar{F} : A \to A$, $\bar{F}(\theta, r) = (\bar{f}(\theta, r), \bar{g}(\theta, r))$ (or $\bar{F} : C \to C$), we denote by $F : \tilde{A} \to \tilde{A}, F(x, y) = (f(x, y), g(x, y))$ (or $\bar{F} : \tilde{C} \to \tilde{C}$) any lift of $\bar{F}$.

We now briefly recall the classification of S^1-homeomorphisms (Proposition 1.1 below). The details can be found in [33]. Let

$$\mathcal{P}_{\pm} = \{H : \mathbb{R} \to \mathbb{R} : H \text{ is an homeomorphism s.t. } H(x + 1) = H(x) \pm 1,\ \forall x \in \mathbb{R}\}.$$

It is important to recall that given a homeomorphism $h : S^1 \to S^1$ there exist a homeomorphism $H : \mathbb{R} \to \mathbb{R}$ and $k \in \mathbb{Z}$ such that (being π the projection of $\mathbb{R}$ on S^1) $H(\theta + 1) = H(\theta) \pm 1$ and $\pi \circ H = h \circ \pi$. If $H_1(\theta + 1) = H_1(\theta) + k_1$ and $H_2(\theta + 1) = H_2(\theta) + k_2$ for some $H_i, k_i, i = 1, 2$ then $H_1(\theta) - H_2(\theta) \in \mathbb{Z}$, for all θ.

As for the possibility of ordering an S^1-triple $\theta_1 = x_1 + \mathbb{Z}, \theta_2 = x_2 + \mathbb{Z}, \theta_3 = x_3 + \mathbb{Z}$ ($x_i \in \mathbb{R}, i = 1, 2, 3$), we say that $\theta_1 \prec \theta_2 \prec \theta_3$ if $x_1 < x_2 < x_3 < x_1 + 1$; for a given $h : S^1 \to S^1$, we say that h is order preserving in S^1 if $\theta_1 \prec \theta_2 \prec \theta_3$ implies $h(\theta_1) \prec h(\theta_2) \prec h(\theta_3)$. It is important to remark that h is order preserving (order reversing, respectively) and continuous in S^1 if and only if $H \in \mathcal{P}_+$ ($H \in \mathcal{P}_-$, respectively).

The concept of rotation number is one of the most important in this paper. Indeed, it can be proved that for $H \in \mathcal{P}_+$ the limit

$$\tau(H) = \lim_{n\to+\infty} \frac{H^n(x)}{n} = \lim_{n\to+\infty} \frac{H^n(x) - x}{n} = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(H^{k+1}(x) - H^k(x)\right) \quad (1.1)$$

exists, is finite and independent of $x \in \mathbb{R}$. It is then natural to define the rotation number of a S^1-homeomorphism h by $\tau(h) = \tau(H) + \mathbb{Z}$.

The following proposition summarizes the classification of S^1-homeomorphisms. In what follows, we denote by $L_\omega(\theta)$ the omega-limit set of an orbit starting from $\theta \in S^1$.

Proposition 1.1. (Nitecki [33]**)** *Let h be a S^1-homeomorphism and let $\tau(h)$ be its rotation number.*

(1) *h has periodic orbits if and only if $\tau(h) \in \mathbb{Q}$; in this case, h is topologically conjugate to the rotation of angle $\tau(h)$ in S^1.*
(2) *If $\tau(h) \notin \mathbb{Q}$ then $L_\omega(\theta)$ is independent of θ. Moreover, one of the following alternatives holds:*
(2a) *$L_\omega(\theta) = S^1$; in this case h is topologically conjugate to the rotation of angle $\tau(h)$ in S^1;*
(2b) *$L_\omega(\theta)$ is a Cantor invariant set; in this case h is topologically semiconjugate to the rotation of angle $\tau(h)$ in S^1.*

The orbits found in case (1) are either periodic or asymptotic to a periodic orbit.

The orbits found in case $(2a)$ are called (usual) quasi-periodic. In this situation, every orbit is dense in S^1. The orbits found in case $(2b)$ are called generalized quasi-periodic. In this situation, L_ω is called a Denjoy minimal subset (cf. [33]).

Example 1.2. The simplest example of S^1-homeomorphism is the rotation of angle $\omega \pmod 1$, defined by

$$\bar{R}_\omega : S^1 \to S^1, \quad \bar{R}_\omega(\theta) = \theta + \omega \pmod 1, \quad \omega \in \mathbb{R}.$$

Given a topological space X and a map $f : X \to X$, a subset M of X is called minimal for f if it is closed, f-invariant and there is no proper subset of M with the same properties.

2. THREE VERSIONS OF MATHER'S THEOREM

We first introduce the crucial notion for this section.

Definition 2.1. A homeomorphism $\bar{F} : A \to A$ is a twist homeomorphism if it is orientation preserving and such that

$$\bar{F}(\theta, a) = (\bar{f}(\theta, a), a), \bar{F}(\theta, b) = (\bar{f}(\theta, b), b), \forall \theta \in S^1 \quad (\textit{boundary preserving condition}) \quad (2.1)$$

$$\forall x \in \mathbb{R} \quad \textit{the function} \quad f(x,\cdot) \quad \textit{is strictly monotone} \quad (\textit{twist condition}). \tag{2.2}$$

In case $\bar{F}$ is of class $\mathcal{C}^1$, then the twist condition writes as

$$\frac{\partial f}{\partial y}(x,y) \neq 0 \quad \forall (x,y) \in \tilde{A}. \tag{2.3}$$

Without loss of generality, in what follows we shall assume $\partial f/\partial y(x,y) > 0$ for all $(x,y) \in \tilde{A}$.

Note that if $\bar{F}$ is a twist homeomorphism, then its lift F satifies

$$F(x,a) = (f(x,a),a), \qquad F(x,b) = (f(x,b),b), \qquad \forall x \in \mathbb{R}. \tag{2.4}$$

Let us introduce two functions $F_a : \mathbb{R} \to \mathbb{R}$, $F_b : \mathbb{R} \to \mathbb{R}$ as follows:

$$F_a(x) = f(x,a) \quad F_b(x) = f(x,b). \tag{2.5}$$

Note that $F_a, F_b \in \mathcal{P}_+$; this follows from the fact that any lift of given a homeomorphism in A commutes with the traslation $T(x,y) = (x+1,y)$. Then, according to Section 1, we can introduce the twist interval $[\rho(F_a), \rho(F_b)]$, being $\rho(F_a), \rho(F_b)$ the rotation number of F_a, F_b, respectively.

Significant examples of twist maps can be found in [2], [11], [13].

Before the statement of Mather's theorem, more notation is needed. Indeed, let $\bar{F} : A \to A$ be an area-preserving twist homeomorphism and let $F : \tilde{A} \to \tilde{A}$ be a lift of $\bar{F}$. The twist condition guarantees that for any $x \in \mathbb{R}$ we have $F_b(x) > F_a(x)$; then there exists $x_1 \in (F_a(x), F_b(x))$. Being $f(x,\cdot)$ continuous and increasing, there exists a unique $y \in [a,b]$ s.t. $f(x,y) = x_1$; moreover, there exists a unique $y_1 \in [0,1]$ s.t. $g(x,y) = y_1$. In other words, if we set $B = \{(x,x_1) \in \mathbb{R}^2 : F_a(x) \leq x_1 \leq F_b(x)\}$, we know that there exist $u, u_1 : B \to [a,b]$ such that

$$\begin{cases} u(x,x_1) = y \\ u_1(x,x_1) = y_1 \end{cases}, \qquad F(x,y) = (x_1,y_1). \tag{2.6}$$

It is important to remark that, being F continuous, the functions u, u_1 are continuous as well. The twist condition guarantees that B is simply connected. We are now in position to state

Theorem 2.2 (Mather [24])**.** *Let $\bar{F}$ be an area-preserving twist homeomorphism and let $\omega \in [\rho(F_a), \rho(F_b)]$. Then there exists a non-decreasing function $\phi : \mathbb{R} \to \mathbb{R}$ s.t.*

$$\phi(t+1) = \phi(t) + 1, \tag{2.7}$$

$$F(\phi(t), \eta(t)) = (\phi(t+\omega), \eta(t+\omega)), \tag{2.8}$$

being $\eta(t) = u(\phi(t), \phi(t+\omega))$.

Note that Theorem 2.2 provides no information about the continuity of ϕ. However, in [24] it is proved the following

Proposition 2.3 (Mather [24]). *If t is a point of continuity of ϕ then so are $t+\omega$ and $t-\omega$. Moreover, if $\omega \in \mathbb{R} \setminus \mathbb{Q}$ then ϕ is not constant on any interval.*

In what follows we review (on the lines of [13]), comment and analyze some notions that will be used for a better understanding of Theorem 2.2 and for the statement of another version of the same result.

Let $(x_0, y_0) \in \tilde{A}$ such that $\pi_A(x_0, y_0) = (\theta_0, r_0)$; we write

$$O(x_0, y_0) = \{F^k(x_0, y_0) + (l, 0), \text{ with } k, l \in \mathbb{Z}\}. \tag{2.9}$$

Definition 2.4. A point $(\theta_0, r_0) \in A$ is a Birkhoff point of period (p, q) if

(1) For any $(x_0, y_0) \in \tilde{A}$ such that $\pi_A(x_0, y_0) = (\theta_0, r_0)$ it results

$$F^q(x_0, y_0) = (x_0 + p, y_0);$$

(2) There exists a homeomorphism $\bar{G} : S^1 \to S^1$ s.t. $\bar{G}^n(\theta_0) = \theta_n$.

In this situation, $O(\theta_0, r_0) \subset A$ is said a Birkhoff periodic orbit of type (p, q).

Remark 2.5. Condition (2) means that for every $(\theta_{n_1}, r_{n_1}), (\theta_{n_2}, r_{n_2}), (\theta_{n_3}, r_{n_3}) \in O(\theta_0, r_0)$ if $\theta_{n_1} \prec \theta_{n_2} \prec \theta_{n_3}$ then $\pi_1\bar{F}(\theta_{n_1}, r_{n_1}) \prec \pi_1\bar{F}(\theta_{n_2}, r_{n_2}) \prec \pi_1\bar{F}(\theta_{n_3}, r_{n_3})$.

Example 2.6. Consider the following map $\bar{F} : A \to A$:

$$\begin{cases} \theta_1 = \theta_0 + \beta + \alpha\,(r) \\ r_1 = r_0, \end{cases} \tag{2.10}$$

where $\beta \in \mathbb{R}, r_0 \in [a, b]$ and $\alpha : [a, b] \to \mathbb{R}$ is a $\mathcal{C}^1$ function such that $\alpha'(r) > 0$ for all r. This is one of the simplest examples of twist map. Let $\bar{r}_0 \in [a, b]$ be such that

$$\beta + \alpha\,(\bar{r}_0) = \frac{p}{q} \in \mathbb{Q}, \quad p,\ q \text{ relatively prime;}$$

then for all $\theta_0 \in S^1$ the point $(\theta_0, \bar{r}_0)$ is a Birkhoff periodic point of period (p, q). It is trivial to prove that (1) in Definition 2.4 is satisfied; as for (2), it is immediate to check that $\bar{R}_{p/q}$ is a homeomorphism of S^1 such that $\bar{R}^n_{p/q}(\theta_0) = \theta_n$.

A more general (related) notion is

Definition 2.7. The orbit $O(\theta_0, r_0)$ of a twist map F is called an Aubry-Mather orbit if there exists a homeomorphism $\bar{G} : S^1 \to S^1$ s.t. $\bar{G}^n(\theta_0) = \theta_n$.

Moreover,

Definition 2.8. A $\bar{F}$-invariant closed set $E \subset A$ is called an Aubry-Mather set if

(1) There exist a closed set $K \subset S^1$ and a continuous function $\psi : K \to [a, b]$ s.t. $E = graph\psi$;
(2) The map $\pi_1 \circ \bar{F} \circ (Id \times \psi)$ is the restriction to K of a homeomorphism $\bar{G} : S^1 \to S^1$.

According to Remark 2.5, condition (2) in Definition 2.8 means that the restriction of $\bar{F}$ to an Aubry-Mather set is order-preserving in the first component.

Example 2.9. Consider again the map (2.10). All circles $r = constant = r_0$ are (star-shaped) invariant curves such that the restriction of $\bar{F}$ to each of them is topologically conjugate to $\bar{R}_{\beta+\alpha(r_0)}$. If $\beta + \alpha(r_0) = p/q \in \mathbb{Q}$, then each orbit on this circle is a Birkhoff periodic one (see Example 2.6); in case $\beta+\alpha(r) = \omega \notin \mathbb{Q}$, each orbit is dense in $S^1 \times \{r_0\}$. In both cases, it is trivial to check that the condition (1) in Definition 2.8 is satisfied, and that condition (2) holds true with $\bar{G} \equiv \bar{R}_{\beta+\alpha(r_0)}$.

Let us now discuss what happens if we consider a more complicated map, such as

$$\begin{cases} \theta_1 = \theta_0 + \beta + \alpha(r_0) + \psi_1(\theta_0, r_0) \\ r_1 = r_0 + \psi_2(\theta_0, r_0). \end{cases}$$

After the above definitions and examples, some reference to KAM theory is in order. Indeed, KAM theory guarantees the persistence of invariant curves for certain values of $\beta + \alpha(r)$, under strong regularity and “smallness” assumptions on α, ψ_1 and ψ_2 (see [32]). An interesting aspect of Aubry-Mather theory is that in case such hypotheses are dropped and we consider the framework of Mather’s Theorem 2.2, then some evidence of the invariant curves obtained by KAM theory is preserved; indeed, one obtains the existence of invariant Cantor sets which can be compared with Denjoy minimal sets for homeomorphisms of S^1. These sets are the most interesting example of Aubry-Mather sets. We point out that while Denjoy minimal sets appear for a “small” class of S^1-homeomorphisms, the appearance of Cantor sets arising from Mather’s theory is the rule rather than the exception for a large class of twist maps (we refer to [29] for inspiring comments on this topic).

In what follows, we give formal rigour to the informal discussion developed above.

It is easy to see that every closed invariant subset of an Aubry-Mather set is an Aubry-Mather set; in particular, every such set contains a minimal subset. Moreover, for any (θ_0, r_0) belonging to an Aubry-Mather set the orbit $O(\theta_0, r_0)$ is an Aubry-Mather orbit. The discussion of the validity of the converse of this fact is crucial for the sequel of the paper; indeed, we have

Theorem 2.10. *The closure of an Aubry-Mather orbit is an Aubry-Mather set.*

In [13] Theorem 2.10 is stated without proof. One of our contributions in this paper is the development of a detailed proof; for the sake of simplicity, we consider the case when F satisfies a Lipschitz-type condition (cf. Definition 2.11 below). The general case can be settled with some more work, on the lines of [12].

First we recall

Definition 2.11. A twist map $\bar{F} : A \to A$ is called a Lipschitz increasing twist map if, for a lift $F(x, y) = (f(x, y), g(x, y))$:

(1) F, F^{-1} are lipschitzian in the first variable;
(2) there exists $c \geq 0$ s.t. for every $x \in \mathbb{R}$ and for every $y_1, y_2 \in [a,b]$ if $y_1 < y_2$ then $f(x,y_2) - f(x,y_1) \geq c(y_2 - y_1)$.

Some of the ideas used in [13] for the study of Birkhoff orbits are of some help in the proof of the two lemmata below (we refer to [39] for more details).

Lemma 2.12. *Let $\bar{F}$ be a twist Lipschitz homeomorphism of A and let $E \subset A$ be an Aubry-Mather set for $\bar{F}$. Consider the function ψ in Definition* 2.8*. Then there exists $L > 0$ (depending only on $\bar{F}$) s.t.*

$$|\psi(\theta_1) - \psi(\theta_2)| \leq L|\theta_1 - \theta_2| \qquad \forall \ \theta_1, \theta_2 \in K.$$

Proof. Arguing in $\tilde{A}$, let (x_1, y_1) and (x_2, y_2) such that $\pi_A(x_i, y_i) = (\theta_i, \psi(\theta_i))$ for $i = 1, 2$. We would like to prove that there exists $L > 0$ such that

$$|y_1 - y_2| \leq L\,|x_1 - x_2|\,.$$

Let us assume, without loss of generality, that $x_2 < x_1$, and first consider the case $y_1 < y_2$. We define

$$(x'_1, y'_1) = F(x_1, y_1)\,, \quad (x'_2, y'_2) = F(x_2, y_2)\,, \quad (\widetilde{x}, \widetilde{y}) = F(x_1, y_2)\,.$$

Since $y_1 < y_2$, from condition (2) in Definition 2.11 we have

$$\widetilde{x} - x'_1 \geq c\,(y_2 - y_1)\,. \tag{2.11}$$

Since $\overline{F}$ is order preserving in the first coordinate on the Aubry-Mather set we have

$$x_1 - x_2 > 0 \Rightarrow x'_1 - x'_2 > 0 \Rightarrow \widetilde{x} - x'_2 > 0.$$

Hence, being l the Lipschitz constant of F, we can write

$$\widetilde{x} - x'_2 = |\widetilde{x} - x'_2| \leq \|F(x_1, y_2) - F(x_2, y_2)\| \leq l\,(x_1 - x_2)\,.$$

We deduce

$$x'_1 > x'_2 \geq \widetilde{x} - l\,(x_1 - x_2)\,. \tag{2.12}$$

From (2.11) and (2.12) it follows

$$y_2 - y_1 \leq \frac{1}{c}\,(\widetilde{x} - x'_1) < \frac{l}{c}\,(x_1 - x_2)\,.$$

The case when $y_2 < y_1$ can be repeated with minor changes, using F^{-1} instead of F. Going back to A, the proof is complete. □

An analogous result is valid for Aubry-Mather orbits; indeed we have

Lemma 2.13. *Let $\bar{F}$ be a twist Lipschitz homeomorphism and let $\Gamma \subset A$ be an Aubry-Mather orbit for $\bar{F}$. Then there exists $L > 0$ (depending only on $\bar{F}$) s.t.*

$$|r_n - r_m| \leq L|\theta_n - \theta_m| \qquad \forall \ n, m \in \mathbb{Z}.$$

Proof. It is sufficient to repeat the argument in the proof of Lemma 2.12 using r_n instead of $\psi(\theta_n)$. □

We are now ready (arguing as in [39]) for the

Proof of Theorem 2.10. Let

$$\Gamma = \left\{ \bar{F}^n (\theta_0, r_0) = (\theta_n, r_n), n \in \mathbb{Z} \right\}$$

be an Aubry-Mather orbit, and let $E = \overline{\Gamma}$. Let us prove that E satisfies Definition 2.8.

(1) Let us set

$$K = \overline{\{\theta_n / n \in \mathbb{Z}\}},$$

and ψ as follows: $\psi(\theta_n) = r_n$. By Lemma 2.13 ψ is lipschitzian, hence continuous, and well defined on a dense subset of K; thus we can extend it to be a well defined continuous function on K.

(2) From Definition 2.7 one has $\overline{G}(\theta_{n-1}) = \theta_n$. On the other hand

$$\pi_1 \circ \overline{F} \circ (Id \times \psi)(\theta_{n-1}) = \pi_1 \circ \overline{F}(\theta_{n-1}, r_{n-1}) = \pi_1(\theta_n, r_n) = \theta_n.$$

Therefore $\overline{G} = \pi_1 \circ \overline{F} \circ (Id \times \psi)$ on a dense subset of K. We can extend the composition to be continuous, and the claim is proved. □

The notion of rotation number is crucial for the classification (on the lines of the classification of S^1-homeomorphisms) of Aubry-Mather orbits and sets; in order to give this important definition, we use the homeomorphism $\bar{G}$ in Definition 2.7, Definition 2.8, respectively.

Definition 2.14. (1) The rotation number ρ of an Aubry-Mather orbit Γ is defined by

$$\rho(\Gamma) := \rho(\bar{G});$$

(2) the rotation number ρ of an Aubry-Mather set E is defined by

$$\rho(E) := \rho(\bar{G}).$$

Recalling (1.1), the above definition is independent of G in Definitions 2.7 and 2.8.

Note that $\rho(E) = \lim_{n \to +\infty} \dfrac{\pi_1 \circ F^n(x, \psi(x))}{n}$ (with ψ as in Definition 2.8) and that if one considers a minimal Aubry-Mather minimal set E_m of E then $\rho(E_m) = \rho(E)$.

The above definition, together with the classification of S^1-homeomorphisms, yields the following useful result (which can be substantially found in [12], [13], [29], [31, Sections 2.5-2.6].

Theorem 2.15. *Let E_m be a minimal Aubry-Mather set with rotation number ω. Then one of the following alternatives holds:*

(1) *If $\omega = p/q \in \mathbb{Q}$ then E_m is a Birkhoff periodic orbit of type (p, q).*

(2) *If $\omega \notin \mathbb{Q}$ then either E_m is an invariant curve, and $\bar{F}_{|E_m}$ is topologically conjugate to the rotation of angle ω on S^1, or E_m is a Cantor invariant set, and $\bar{F}_{|E_m}$ is topologically semi-conjugate to the rotation of angle ω on S^1.*

Proof. Let E be an Aubry-Mather set, and let E_m be a minimal Aubry-Mather subset of E. Since the annulus is compact, every minimal subset of E is the limit set of each orbit on E. Therefore an Aubry-Mather minimal subset is a limit set of an Aubry-Mather orbit. Let Γ be an Aubry-Mather orbit in E. The projection

$$\pi_1 : \Gamma \to S^1, \quad \pi_1(\theta_n, r_n) = \theta_n$$

is a bijective correspondence between Γ and an orbit of the homeomorphism $\bar{G}$ (see Definition 2.7) on S^1. Thus we can apply the classification of the omega-limit sets of S^1-homeomorphism (Proposition 1.1), which is obviously valid also for alpha-limits. Hence, there are only three possible types of Aubry-Mather minimal subsets, and they are those described in the thesis. □

In what follows, we shall use the concepts of (usual) quasi-periodic orbit and of generalized quasi-periodic orbit for a twist homeomorphism of an annulus $S^1 \times [a, b]$ and of the cylinder $S^1 \times \mathbb{R}$; these concepts are the same as their analogue for S^1-homeomorphisms introduced after Proposition 1.1.

We are now ready to state and prove a version of Mather's theorem (Theorem 2.16 below) in terms of Aubry-Mather sets. Theorem 2.16 has been proved by Katok in [12]; in this paper, we give a different original proof which (contrary to the one in [12]) is based on the application of Theorem 2.2.

Theorem 2.16 (Katok [12], Moser [29])**.** *Let $\bar{F} : A \to A$ be an area-preserving twist homeomorphism of the annulus. Then, for every $\omega \in [\rho(F_a), \rho(F_b)]$ there exists an Aubry-Mather set E_ω having rotation number ω. If $\omega \in \mathbb{Q}, \omega = p/q$, then E_ω contains Birkhoff periodic points of type (p, q). If $\omega \in \mathbb{R} \setminus \mathbb{Q}$ there are two possibilities: either E_ω is an invariant curve, and every orbit in E_ω is an usual quasi-periodic orbit, or E_ω contains a Cantor invariant set and every orbit in E_ω is a generalized quasi-periodic orbit.*

Proof. Theorem 2.2 is applicable. Let ϕ, η, ω as in Theorem 2.2; consider

$$M = \overline{\{(\phi(t), \eta(t))\, / t \text{ is a continuity point of } \phi\}} \subset \tilde{A},$$

and the projection E of M on the annulus A. We claim that E is an Aubry-Mather set with rotation number ω. By Theorem 2.2 and Proposition 2.3, the set E is closed and invariant. Arguing in $\tilde{A}$, we can easily verify the conditions (1) and (2) in Definition 2.8. Indeed, we first notice that from (2.7) the map ϕ is well defined on S^1; setting

$$N = \overline{\{\phi(t)\, / t \text{ is a continuity point of } \phi\}},$$

it turns out that $M = \operatorname{graph}\psi$, where ψ is defined by

$$\psi(\phi(t)) = u(\phi(t), \phi(t+\omega)) = \eta(t),$$

and continuous. As for (2), let $G : N \to \mathbb{R}$

$$G|_N(\phi(t)) = \pi_1 \circ F \circ (Id \times \psi)(\phi(t)) = \pi_1 \circ F(\phi(t), \eta(t)) =$$
$$\pi_1(\phi(t+\omega), \eta(t+\omega)) = \phi(t+\omega).$$

Arguing as in the proof of Addendum 2 in [24], if ϕ is constant on (α, β), then ϕ is constant also in $(\alpha+\omega, \beta+\omega)$. Hence $G|_N$ is well defined:

$$\phi(t_1) = \phi(t_2) \Rightarrow G|_N(\phi(t_1)) = G_N(\phi(t_2)).$$

The monotonicity of ϕ implies that $G|_N$ is an homeomorphism of N, so that we can reguard it as the restriction to N of an homeomorphism of $\mathbb{R}$.

Now, as in [24] (section 2), it follows

$$\rho(E) = \lim_{n\to\infty} \frac{\phi(t+n\omega)}{n} = \omega.$$

Thus we have proved the first part of Theorem 2.16. The second part of the thesis of the theorem follows from the classification of Aubry-Mather sets in Theorem 2.15. □

Remark 2.17. Observe that the two possibilities in case $\omega \in \mathbb{R} \setminus \mathbb{Q}$ can be described in terms of the continuity properties of ϕ. More precisely,
(1) Assume that ϕ is continuous. Define $\Phi = (\phi, \eta)$; since, by Proposition 2.3, the function ϕ is not constant of any interval it follows that Φ is an homeomorphism between $\mathbb{R}$ and M such that

$$F \circ \Phi(t) = \Phi \circ T_\omega(t),$$

where $T_\omega(t) = t + \omega$. Hence, $\bar{\Phi}$ is an homeomorphism between S^1 and E and $\bar{F}|_E$ is topologically conjugate to the rotation R_ω of angle ω on S^1. Summing up, E is an invariant curve and each orbit on E is an usual quasi-periodic one.
(2) If ϕ is not continuous on $\mathbb{R}$, we can exclude the previous cases. Indeed, E cannot contain Birkhoff periodic points, otherwise it should be $\omega \in \mathbb{Q}$. If E contains an arc, mapping $\bar{F}$ on this arc repeatedly, from the irrationality of ω we could fill up the graphic of an invariant closed curve. In this case from the monotonicity of ϕ and (2.7) we could deduce that ϕ is continuous.

We now state and prove another version of Mather's theorem; this result (Theorem 2.19 below) is the one which is frequently used in the applications to ODEs (cf. [3], [5], [34], [35]). Roughly speaking, we shall work with a cylinder instead of an annulus; for this reason, it is important to give the following

Definition 2.18. Let $\bar{F} : C \to C$ be a homeomorphism and let $F : \tilde{C} \to \tilde{C}$ be a lift of $\bar{F}$. We say that $\bar{F}$ is boundary preserving if

$$\lim_{y\to\pm\infty} \pi_2 \circ F(x, y) = \pm\infty. \tag{2.13}$$

In what follows the definition of twist homeomorphism will be understood with condition (2.1) replaced by (2.13).

The notions of Aubry-Mather set and of its rotation number in this context can be introduced in the same way as in the case of homeomorphisms of the annulus; however, the fact that the bounded interval $[a, b]$ is now replaced by the whole $\mathbb{R}$ requires some caution and comments. Indeed, one first has to define

$$\alpha_{\pm}(x) = \lim_{y \to \pm\infty} (\pi_1 \circ F(x, y) - x).$$

In case $\alpha_{\pm}$ are constant (which is the case in applications to ODEs, cf. page 125 in [35] or Lemma 4.11 in [7]) then we set $\rho_{\pm}(F) = \alpha_{\pm}(x)$. This notational choice is coherent with the notion of rotation number we used up to now; indeed, if $\alpha_{\pm}(x)$ is independent of x the restriction of $\bar{F}$ to the boundary components of C is a constant rotation of angle $\alpha_{\pm}(x)$ (see also (2.14)). Then we can state

Theorem 2.19. *Let $\bar{F} : \tilde{C} \to \tilde{C}$ be an area-preserving twist homeomorphism. Assume that $\rho_{\pm}(F) \in \mathbb{R} \cup \{\pm\infty\}$ are well-defined. Then, for every $\omega \in (\rho_-(F), \rho_+(F))$ there exists an Aubry-Mather set E_ω having rotation number ω. If $\omega \in \mathbb{Q}, \omega = p/q$ then E_ω contains periodic points of type (p, q). If $\omega \in \mathbb{R} \setminus \mathbb{Q}$ there are two possibilities: either E_ω is an invariant curve, and every orbit in E_ω is an usual quasi-periodic orbit, or E_ω contains a Cantor invariant set and every orbit in E_ω is a generalized quasi-periodic orbit.*

Theorem 2.19 is stated in [35] and its proof is only sketched; below we give a detailed proof based on the application of Theorem 2.16, and refer to [39] for more details. We observe that in [35] it is given a supplementary information (crucial for the applications) on E_ω when ω approaches $\rho_{\pm}(F)$; this aspect is quoted in (3.2) and will developed in detail in [6].

Proof. Consider the homeomorphism

$$\Psi : \tilde{C} \to S := \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad \Psi(x, y) = (x, \arctan y) \text{ for every } (x, y) \in \mathbb{R}^2.$$

Let $G : S \to S$ be defined by $G := \Psi \circ F \circ \Psi^{-1}$, i.e.

$$G(u, v) = (f(u, \tan v), \arctan g(u, \tan v)).$$

Note that, by the monotonicity of $\tan(\cdot)$ in $(-\pi/2, \pi/2)$, the map G is a monotone twist homeomorphism of S. Moreover, we have

$$\lim_{v \to \pm(\pi/2)^{\mp}} \pi_2 \circ G(u, v) = \pm\frac{\pi}{2};$$

thus we can extend G to a boundary preserving homeomorphism defined on $\bar{S} = \mathbb{R} \times [-\pi/2, \pi/2]$, which can be regarded as a lift of a boundary preserving homeomorphism

$\bar{G}$ defined on $A^* := S^1 \times [-\pi/2, \pi/2]$. The twist interval of $\bar{G}$ is the same as the one of $\bar{F}$: indeed for every $u \in \mathbb{R}$

$$\begin{aligned}\lim_{v\to\pm(\pi/2)^{\mp}} \pi_1 \circ G(u,v) &= \lim_{v\to\pm(\pi/2)^{\mp}} f(x, \tan v) = \\ &= \lim_{y\to\pm\infty} (f(u,y)-u)+u = u+\rho_{\pm}(F).\end{aligned} \tag{2.14}$$

This shows that the restriction of G to the boundaries $\mathbb{R} \times \{\pm\pi/2\}$ is a rotation of angle $\rho_{\pm}(F)$, which obviously has $\rho_{\pm}(F)$ as rotation number (cf. the comment before (3.1)). Now we can apply Theorem 2.16 on $\bar{G}$; observe that if E is an Aubry-Mather set of $\bar{G}$ on A^*, then $\Psi^{-1}(E)$ is an Aubry-Mather set of $\bar{F}$ on C with the same rotation number as E; this completes the proof. □

We end this Section by recalling that in the literature (cf. [29], [34], [37]) another definition of Aubry-Mather set is available. We show in what follows that this definition (Definition 2.20 below) is equivalent to Definition 2.8 which we used throughout the paper. Indeed, we have

Definition 2.20 (Moser [29], Pei [34], Rogel [37])**.** Let $\bar{F} : A \to A$ be a twist homeomorphism and let $F : \tilde{A} \to \tilde{A}$ be a lift of $\bar{F}$. A closed $\bar{F}$-invariant set $E \subset A$ is an Aubry-Mather set with rotation number ω if

(1) the set M s.t. $E = \pi_A(M)$ is of the form

$$M = \overline{\{(\phi(t), \eta(t)) : t \text{ is a continuity point of } \phi\}},$$

being $\phi : \mathbb{R} \to \mathbb{R}$ non-decreasing, $\eta : N \to [a,b]$ (with N a closed subset of $\mathbb{R}$) such that

$$\phi(t) = \phi(t) + 1, \qquad \eta(t+1) = \eta(t),$$

(2)

$$F(\phi(t), \eta(t)) = (\phi(t+\omega), \eta(t+\omega)).$$

Note that by our choice of Definition 2.8 we have been able to classify Aubry-Mather sets (Theorem 2.15) taking advantage of Proposition 1.1. On the other hand, if we had used Definition 2.20 the mere existence of an Aubry-Mather set could have been immediately obtained (cf. the statement of Theorem 2.2).

The proof of the following result, though probably known to experts, is not found in the literature (we refer to [39] for more details).

Theorem 2.21. *Definition* 2.8 *and Definition* 2.20 *are equivalent.*

Proof. If E satisfies Definition 2.20, arguing as in the first part of the proof of Theorem 2.16 it follows that E satisfies also Definition 2.8.

Assume now that E satisfies Definition 2.8 and let N be the subset of $\mathbb{R}$ such that its projection on S^1 is K. One can choose for N a parametrization of the form

$$N = \overline{\{\phi(t) : t \text{ is a continuity point of } \phi\}},$$

with ϕ monotone increasing and $\phi(t+1) = \phi(t) + 1$. Since $E = \operatorname{graph}\psi$ (cf. (1) in Definition 2.8), we can consider a lift $\psi^* : N \to [a,b]$ of ψ and define

$$\eta : \{t \in \mathbb{R} : t \text{ is a continuity point of } \phi\} \to [a,b], \quad \eta(t) = \psi^*(\phi(t)).$$

Then we have

$$\eta(t+1) = \psi^*(\phi(t+1)) = \psi^*(\phi(t)+1) = \psi^*(\phi(t)) = \eta(t).$$

Extending the function η on the whole set $\overline{\{t \in \mathbb{R} : t \text{ is a continuity point of } \phi\}}$ as a continuous function, we can take

$$M = \overline{\{(\phi(t), \eta(t)) : t \text{ is a continuity point of } \phi\}}$$

and thus $E = \pi_A(M)$. Notice now that from (2) of Definition 2.8

$$f(\phi(t), \eta(t)) = \pi_1 \circ F(\phi(t), \eta(t)) = G(\phi(t)),$$

where G is a lift of an homeomorphism on S^1 with rotation number ω. Hence G is at least topologically semi-conjugate to a rotation R_ω, and $G \circ \phi = \phi \circ R_\omega$; thus

$$f(\phi(t), \eta(t)) = \phi(t+\omega) \text{ and } g(\phi(t), \eta(t)) = \eta(t+\omega). \qquad \square$$

3. FINAL COMMENTS

We wish to devote some, by far not exhausting, remarks on the applications of Aubry-Mather theory to ordinary differential equations. More details and comments shall appear in [6].

Applications are based on the use of Theorem 2.16 or, more frequently, of Theorem 2.19 to the Poincaré map $\bar{P}$ associated to a planar system of first order equations (usually arising from a second order scalar equation). In order to accomplish this program, suitable action-angle variables are introduced; in case the first order system is Hamiltonian, it follows that $\bar{P}$ is an area-preserving homeomorphism. The verification of the twist property requires more technical efforts. Indeed, first one has to develop careful asymptotic (as, roughly speaking, the radial coordinate goes to infinity) estimates on $\bar{P}$ in a subset C_1 of C (resp. in A_1 of A). Then, in order to apply Mather's theorem, it is necessary to extend $\bar{P}|_{C_1}$ (resp. $\bar{P}|_{A_1}$) to a monotone twist homeomorphism on C (on A, resp.); this is a non-trivial step, since the extension has to be area-preserving (cf. [7],[34],[39]). Finally, if for $D \subset \tilde{C}$ we set

$$\begin{aligned} \pi_2^-(D) &= \inf\{\bar{y} : (\bar{x}, \bar{y}) \in D\}, \\ \pi_2^+(D) &= \sup\{\bar{y} : (\bar{x}, \bar{y}) \in D\}, \end{aligned} \tag{3.1}$$

one has to prove that

$$\lim_{\omega \to \rho_\pm(F)} \pi_2^\mp(E_\omega) = \pm\infty. \tag{3.2}$$

In this framework, we quote the results of Pei [34], [35], who considered a second order Duffing equation with different growth rates for the nonlinearities. He obtained the existence of subharmonic solutions and quasi-periodic solutions (usual or generalized ones) via Theorem 2.19. Similar results can be found in [5] for a second order asymmetric perturbed oscillator at resonance. In the same spirit, Denzler [8], Jiang-Pei [17], Pei [15] opened the way to generalizations to some first order Hamiltonian systems.

Having in mind applications to non necessarily Hamiltonian systems (when the area-preserving condition is not guaranteed) a variant of Mather's theorem (in the version we stated as Theorem 2.19) has been given by Chow-Pei [7, Theorem 3.1]. More precisely, in [7] it is considered a map $\bar{F} : C \to C$ satisfying a reversibility condition with respect to an involution ψ of the cylinder s.t.

$(R1)$ the lift Ψ of ψ satisfies

$$\Psi(x+1, y) = \Psi(x, y) - (1, 0),$$

and F is reversible with respect to Ψ.

$(R2)$ the fixed point set L of Ψ is a $\mathcal{C}^1$-closed curve in the plane, $\pi_1(L)$ is bounded and $\pi_2(L) = \mathbb{R}$.

The existence of Aubry-Mather sets was then shown in [7] in case of a second order equation; analogue results were subsequently given, among others, in [4] and [22].

In a forthcoming paper [6] we shall state and prove the analogue of Theorem 2.16 for a reversible twist map of the annulus. In this way, a complete survey of abstract results on the existence and classification of Aubry-Mather sets will be accomplished. Moreover, in [6] we shall give a new application to a class of reversible first order systems arising from [10].

Notice that the careful asymptotic estimates of the Poincaré map required for the obtention of the above quoted results in many cases are the same as those used in the study of the boundedness of solutions by means of Moser twist theorem for area-preserving twist maps. We refer e.g. to the works [3], [18], [19], [20], [36]. In the framework of reversible systems the reader may consult [14], [21], [22], [23], [38] and references therein.

We end this section by briefly recalling two aspects of Aubry-Mather theory that, though very important, were beyond the aims of this paper.

J. Moser [29], [30] has pointed out and developed the underlying variational principle in Mather's proof. Indeed, instead of the regularization technique in [24] he used the concept of generating function and explained the monotone twist property in terms of the Legendre condition (cf. Chapter 3 in [31]). For extensions and further developments of Aubry-Mather theory in this direction, we refer (among others) to [8], [27], [31] and references therein. The variational formulation is also the key to generalizations to higher dimensions [28].

Finally, it is important to point out that the setting of Aubry-Mather theory as developed by Mather [28] has recently given raise to important applications in the study of the existence of C^1 critical subsolutions of the Hamilton-Jacobi equation. We refer, among many others, to the contributions by Fathi-Siconolfi in [9].

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