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DIFFERENTIAL INVARIANTS FOR HYPERBOLIC SYSTEMS

HALIS YILMAZ

Department of Mathematics, University of Dicle
Diyarbakir, Turkey
E-mail: maths@halisyilmaz.com

Dedicated to Prof. J.R.L. Webb on the occasion of his retirement.

ABSTRACT. We show that $\mathbb{Z}^3 \times \mathbb{R}^4$ Toda Lattice Equations can be obtained from the Laplace-Darboux transformations of invariants for a four-dimensional hyperbolic system. We also present the relationship between the invariants of $\mathbb{L}$ and the invariants of $\mathbb{M}$ when $[\mathbb{L}, \mathbb{M}] = 0$, where $\mathbb{L}$ and $\mathbb{M}$ are $n \times n$ operator matrices.

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1. INTRODUCTION

Toda lattice equations are a class of important integrable systems. The hyperbolic two dimensional Toda equations first appeared in the work of Darboux [2]. The original Toda equations, established by Toda [7, 8], are a system of ordinary differential equations in the form

$$\frac{d^2w}{dt^2} = e^{w_{i-1}} - 2e^{w_i} + e^{w_{i+1}} \ (i = 1, 2, ..., n). \tag{1.1}$$

The purpose of this paper is to show that the Laplace maps produce $\mathbb{Z}^3 \times \mathbb{R}^4$ Toda lattice equations. It should be emphasized that differential invariants are the main tool for Laplace Transformations. For this reason we first discuss the gauge invariants.

Let us consider the second order-linear hyperbolic equation

$$u_{,xy} + au_{,x} + bu_{,y} + cu = 0, \tag{1.2}$$

where a, b and c are real functions of x and y. The above equation can be written in the form

$$Lu \equiv (\partial_x\partial_y + a\partial_x + b\partial_y + c)u = 0. \tag{1.3}$$

The gauge transformation $L^g = g^{-1}Lg$, where g is a function of x and y, gives us the Laplace invariants

$$h = a_{,x} + ab - c, \tag{1.4}$$

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$$k = b_{,y} + ab - c. \tag{1.5}$$

These invariants have been used by many researchers in integrability theory (see e.g. [3], [4], [6]).

We now concentrate on the Laplace Transformations of system (1.2) which we write in the form:

$$(\partial_x\partial_y + a\partial_x + b\partial_y + c)u = 0.$$

This system can also be expressed in the form

$$(\partial_x + b)(\partial_y + a)u - hu = 0, \tag{1.6}$$

where $h = a_{,x} + ab - c$.

The Laplace map σ_1 is defined by setting

$$u^{\sigma_1} = (\partial_y + a)u, \tag{1.7}$$

so that the Laplace invariants of equation (1.3) transform as

$$h^{\sigma_1} = 2h - k - (\ln h)_{,xy}\,, \tag{1.8}$$

$$k^{\sigma_1} = h. \tag{1.9}$$

Similarly, the system (1.3) can be written in the form

$$(\partial_y + a)(\partial_x + b)u - ku = 0.$$

Another Laplace map, σ_2, is defined by setting

$$u^{\sigma_2} = (\partial_x + b)u.$$

Thus, we obtain the following transformations of Laplace invariants for system (1.3)

$$h^{\sigma_2} = k,$$

$$k^{\sigma_2} = 2k - h - (\ln k)_{,xy}\,.$$

Definition 1.1. Let $\mathbb{L}$ be an $n \times n$ differential operator such that

$$\mathbb{L} = \begin{pmatrix} \partial_1 + h_{11} & h_{12} & \dots & h_{1n} \\ h_{21} & \partial_2 + h_{22} & \dots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \dots & \partial_n + h_{nn} \end{pmatrix},$$

where ∂_i stands for $\partial/\partial x_i$, the h_{ij} are functions of x_1, x_2,...,x_n, and g a diagonal $n \times n$ matrix such that g^{-1} exists. Then H is *invariant* under the gauge transformation

$$\mathbb{L}^g = g^{-1}\mathbb{L}g,$$

so long as $H^g = H$.

Definition 1.2. The Laplace transformations σ_i $(i = 1, \ldots, n)$ are found by solving factorization problems of the form

$$\mathbb{L}\mathbb{D}_i = \mathbb{D}_i{}^{\sigma_i}\mathbb{L}^{\sigma_i},$$

where $\mathbb{D}_i$ and $\mathbb{D}_i{}^{\sigma_i}$ denote $n \times n$ operator matrices of the form

$$\mathbb{D}_i = \begin{pmatrix} \partial_i + d_{11} & d_{12} & \ldots & d_{1n} \\ d_{21} & \partial_i + d_{22} & \ldots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \ldots & \partial_i + d_{nn} \end{pmatrix},$$

where the d_{ij} are functions of x_1, x_2,...,x_n.

We aim to reformulate the second order hyperbolic equation as a system. For this purpose let us consider the following 2×2 matrix

$$\mathbb{L} = \begin{pmatrix} \partial_1 + h_{11} & h_{12} \\ h_{21} & \partial_2 + h_{22} \end{pmatrix},$$

where the h_{ij} are functions of x_1 and x_2.

The gauge transformation on $\mathbb{L}$ is $g^{-1}\mathbb{L}g = \mathbb{L}^g$ where g is a diagonal 2×2 matrix. Therefore, we obtain the following invariants [1]:

$$(12) = h_{12}h_{21}, \tag{1.10}$$

$$[12] = h_{11,2} - h_{22,1} + \frac{1}{2}\left(\ln\frac{h_{12}}{h_{21}}\right)_{,12}. \tag{1.11}$$

These invariants were also obtained by Tsarev for a strictly hyperbolic 2×2 first-order system [9].

We choose the above notations in order to show that those invariants are either symmetric or antisymmetric. The round and square brackets denote symmetric and antisymmetric objects under the permutation $1\leftrightarrow2$ respectively. In other words, one may show easily that $(ij) = (ji)$ and $[ij] = -[ji]$, where $i, j \in \{1, 2\}$.

The Laplace transformations σ_1, σ_2 are found by solving a factorization problem of the form

$$\mathbb{L}\mathbb{D}_1 = \mathbb{D}_1{}^{\sigma_1}\mathbb{L}^{\sigma_1}, \tag{1.12}$$

$$\mathbb{L}\mathbb{D}_2 = \mathbb{D}_2{}^{\sigma_2}\mathbb{L}^{\sigma_2}, \tag{1.13}$$

where $\mathbb{D}^{\sigma_1}$ and $\mathbb{D}^{\sigma_2}$ denote 2×2 differential operator matrices depending only upon the functions h_{ij} and $h_{ij}{}^{\sigma_i}$, their derivatives and differential operators $I\partial_1$, $I\partial_2$ respectively. The pair of systems (1.12 - 1.13) give us Laplace maps of the gauge invariants (12) and [12] in the following form:

$$(12)^{\sigma_1} - (12) = -[12] - \frac{1}{2}\left(\ln(12)\right)_{,12}, \tag{1.14}$$

$$[12]^{\sigma_1} - [12] = \frac{1}{2}\left(\ln(12)(12)^{\sigma_1}\right)_{,12}, \tag{1.15}$$

$$(12)^{\sigma_2} - (12) = [12] - \frac{1}{2}\left(\ln(12)\right)_{,12}, \tag{1.16}$$

$$[12]^{\sigma_2} - [12] = -\frac{1}{2}\left(\ln(12)(12)^{\sigma_2}\right)_{,12}. \tag{1.17}$$

Equations (1.14) and (1.16) produce the $\mathbb{Z} \times \mathbb{R}^2$ Toda lattice equations

$$(12)_{n+1} - 2(12)_n + (12)_{n-1} = -\left(\ln(12)_n\right)_{,12},$$
$$[12]_n = \frac{1}{2}\Big((12)_{n-1} - (12)_{n+1}\Big),$$

where $n \in \mathbb{Z}$ and $x_1, x_2 \in \mathbb{R}^2$.

In the following section we will examine the 4×4 matrix case for the gauge invariants and their Laplace transformations (the case $n = 3$ was studied by Athorne [1]).

2. DIFFERENTIAL INVARIANTS

Let us consider the following 4×4 differential matrix $\mathbb{L}$

$$\mathbb{L} = \begin{pmatrix} \partial_1 + h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & \partial_2 + h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & \partial_3 + h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & \partial_4 + h_{44} \end{pmatrix},$$

where the h_{ij} are functions of x_1, x_2, x_3 and x_4.

The gauge transformation on $\mathbb{L}$ is $g^{-1}\mathbb{L}g = \mathbb{L}^g$ where g is a diagonal 4×4 matrix such that its diagonal entries are g_1, g_2, g_3 and g_4 respectively. So $\mathbb{L}^g = g^{-1}\mathbb{L}g$ gives us the following invariants:

$$(12) = h_{12}h_{21}, \qquad (23) = h_{23}h_{32}, \qquad (31) = h_{31}h_{13},$$
$$(34) = h_{34}h_{43}, \qquad (41) = h_{41}h_{14}, \qquad (42) = h_{42}h_{24},$$

$$[12] = h_{11,2} - h_{22,1} + \frac{1}{2}\left(\ln\frac{h_{12}}{h_{21}}\right)_{,12},$$

$$[23] = h_{22,3} - h_{33,2} + \frac{1}{2}\left(\ln\frac{h_{23}}{h_{32}}\right)_{,23},$$

$$[31] = h_{33,1} - h_{11,3} + \frac{1}{2}\left(\ln\frac{h_{31}}{h_{13}}\right)_{,31},$$

$$[34] = h_{33,4} - h_{44,3} + \frac{1}{2}\left(\ln\frac{h_{34}}{h_{43}}\right)_{,34},$$

$$[41] = h_{44,1} - h_{11,4} + \frac{1}{2}\left(\ln\frac{h_{41}}{h_{14}}\right)_{,41},$$

$$[42] = h_{44,2} - h_{22,4} + \frac{1}{2}\left(\ln \frac{h_{42}}{h_{24}}\right)_{,42},$$

$$\begin{array}{lll}
(123) = h_{12}h_{23}h_{31}, & (132) = h_{13}h_{32}h_{21}, & (234) = h_{23}h_{34}h_{42}, \\
(432) = h_{43}h_{32}h_{24}, & (341) = h_{34}h_{41}h_{13}, & (143) = h_{14}h_{43}h_{31}, \\
(412) = h_{41}h_{12}h_{24}, & (142) = h_{14}h_{42}h_{21}, & \\
(1234) = h_{12}h_{23}h_{34}h_{41}, & (1243) = h_{12}h_{24}h_{43}h_{31}, & (1324) = h_{13}h_{32}h_{24}h_{41}, \\
(1342) = h_{13}h_{34}h_{42}h_{21}, & (1423) = h_{14}h_{42}h_{23}h_{31}, & (1432) = h_{14}h_{43}h_{32}h_{21}.
\end{array}$$

The round and square brackets, as before, denote symmetric and antisymmetric objects under permutations of the set $\{1, 2, 3, 4\}$. The invariants satisfy algebraic relations:

$$(123)(132) = (12)(23)(31), \tag{2.1}$$

$$(234)(432) = (23)(34)(42), \tag{2.2}$$

$$(341)(143) = (31)(34)(41), \tag{2.3}$$

$$(412)(142) = (12)(41)(42), \tag{2.4}$$

$$(1234)(1432) = (12)(23)(34)(41), \tag{2.5}$$

$$(1423)(1324) = (23)(31)(41)(42), \tag{2.6}$$

$$(1243)(1342) = (12)(31)(34)(42), \tag{2.7}$$

and differential algebraic relations:

$$[12]_{,3} + [23]_{,1} + [31]_{,2} = \frac{1}{2}\left(\ln \frac{(123)}{(132)}\right)_{,123}, \tag{2.8}$$

$$[23]_{,4} + [34]_{,2} + [42]_{,3} = \frac{1}{2}\left(\ln \frac{(234)}{(432)}\right)_{,234}, \tag{2.9}$$

$$[12]_{,4} + [41]_{,2} + [24]_{,1} = \frac{1}{2}\left(\ln \frac{(412)}{(142)}\right)_{,124}, \tag{2.10}$$

$$[34]_{,1} + [41]_{,3} + [13]_{,4} = \frac{1}{2}\left(\ln \frac{(341)}{(143)}\right)_{,134}, \tag{2.11}$$

$$[12]_{,34} + [23]_{,41} + [34]_{,12} + [41]_{,23} = \frac{1}{2}\left(\ln \frac{(1234)}{(1432)}\right)_{,1234}. \tag{2.12}$$

3. LAPLACE MAPS

Let us now focus on the Laplace maps σ_i $(i = 1, \ldots, 4)$ in the case $n = 4$.

$$\mathbb{L}\mathbb{D}_i = \mathbb{D}_i^{\sigma_i}\mathbb{L}^{\sigma_i},$$

where $\mathbb{D}_i{}^{\sigma_i}$ are 4×4 differential operator matrices. Taking the case $i=1$ we have

$$\left(\frac{(123)}{(12)}\right)^{\sigma_1} = \frac{(123)}{(31)},\ \left(\frac{(132)}{(31)}\right)^{\sigma_1} = \frac{(132)}{(12)},$$

$$\left(\frac{(412)}{(12)}\right)^{\sigma_1} = \frac{(412)}{(41)},\ \left(\frac{(142)}{(41)}\right)^{\sigma_1} = \frac{(142)}{(12)},$$

$$\left(\frac{(341)}{(31)}\right)^{\sigma_1} = \frac{(341)}{(41)},\ \left(\frac{(143)}{(41)}\right)^{\sigma_1} = \frac{(143)}{(31)},$$

$$\begin{aligned}
(12)^{\sigma_1} - (31) &= \left(\frac{(132)}{(12)}\right)_{,1}, \\
(31)^{\sigma_1} - (12) &= \left(\frac{(123)}{(31)}\right)_{,1}, \\
(12)^{\sigma_1} - (41) &= \left(\frac{(142)}{(12)}\right)_{,1}, \\
(41)^{\sigma_1} - (12) &= \left(\frac{(412)}{(41)}\right)_{,1}, \\
(31)^{\sigma_1} - (41) &= \left(\frac{(143)}{(31)}\right)_{,1}, \\
(41)^{\sigma_1} - (31) &= \left(\frac{(341)}{(41)}\right)_{,1},
\end{aligned}$$

$$\begin{aligned}
(12)^{\sigma_1} - (12) &= -[12] - \frac{1}{2}\left(\ln(12)\right)_{,12}, \\
&= -[12]^{\sigma_1} + \frac{1}{2}\left(\ln(12)^{\sigma_1}\right)_{,12}, \\
(23)^{\sigma_1} - (23) &= [23]^{\sigma_1} - [23] = 0, \\
(34)^{\sigma_1} - (34) &= [34]^{\sigma_1} - [34] = 0, \\
(41)^{\sigma_1} - (41) &= [41] - \frac{1}{2}\left(\ln(41)\right)_{,41}, \\
&= [41]^{\sigma_1} + \frac{1}{2}\left(\ln(41)^{\sigma_1}\right)_{,41}.
\end{aligned}$$

We also have

$$\begin{aligned}
(31)^{\sigma_1} - (31) &= [31] - \frac{1}{2}\left(\ln(31)\right)_{,31} \\
&= [31]^{\sigma_1} + \frac{1}{2}\left(\ln(31)^{\sigma_1}\right)_{,31}, \\
(42)^{\sigma_1} - (42) &= [42]^{\sigma_1} - [42] = 0.
\end{aligned}$$

Transformations for σ_2, σ_3 and σ_4 follow by cyclic permutations of all indices $1,2,3,4$ in the expressions for σ_1. Let I be one of the invariants $(12),(23),..,[12],[23]$ etc.

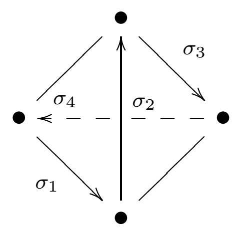

FIGURE 1. Laplace tetrahedron

One can verify that

$$I^{\sigma_i\sigma_j} = I^{\sigma_j\sigma_i},$$

where $i, j = 1, 2, 3, 4$ and $i \neq j$. This relation must be verified for all choices of invariant I. We also have

$$I^{\sigma_1\sigma_2\sigma_3\sigma_4} = I.$$

Hence the maps $\sigma_1, \sigma_2, \sigma_3$ and σ_4 obey the following group relations:

$$\sigma_1\sigma_2 = \sigma_2\sigma_1,\ \sigma_2\sigma_3 = \sigma_3\sigma_2,\ \sigma_3\sigma_1 = \sigma_1\sigma_3,$$
$$\sigma_1\sigma_4 = \sigma_4\sigma_1,\ \sigma_2\sigma_4 = \sigma_4\sigma_2,\ \sigma_3\sigma_4 = \sigma_4\sigma_3,\ \sigma_1\sigma_2\sigma_3\sigma_4 = \mathrm{id},$$

where id denotes the identity. The relations of the Laplace maps are shown in figure 1. We now wish to obtain Toda lattice equations for the case $n = 4$. By doing some simple algebraic calculations we have the following equations:

$$\begin{aligned}
(12)^{\sigma_1} - 2(12) + (12)^{\sigma_2} &= -\left(\ln(12)\right)_{,12},\\
(23)^{\sigma_2} - 2(23) + (23)^{\sigma_3} &= -\left(\ln(23)\right)_{,23},\\
(31)^{\sigma_1} - 2(31) + (31)^{\sigma_3} &= -\left(\ln(31)\right)_{,31},\\
(34)^{\sigma_3} - 2(34) + (34)^{\sigma_4} &= -\left(\ln(34)\right)_{,34},\\
(41)^{\sigma_1} - 2(41) + (41)^{\sigma_4} &= -\left(\ln(41)\right)_{,41},\\
(42)^{\sigma_2} - 2(42) + (42)^{\sigma_4} &= -\left(\ln(42)\right)_{,42}.
\end{aligned}$$

Since $\sigma_1\sigma_2\sigma_3\sigma_4 = \mathrm{id}$, we may write σ_4 as $\sigma_4 = \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$. So by substituting $(12)_{n,m,p} = (12)^{\sigma_1^n\sigma_2^m\sigma_3^p}$ and $[12]_{n,m,p} = [12]^{\sigma_1^n\sigma_2^m\sigma_3^p}$ in the above expressions, we obtain the $\mathbb{Z}^3 \times \mathbb{R}^4$ Toda lattice equations:

$$\begin{aligned}
(12)_{n+1,m,p} - 2(12)_{n,m,p} + (12)_{n,m+1,p} &= -\left(\ln(12)_{n,m,p}\right)_{,12},\\
(23)_{n,m+1,p} - 2(23)_{n,m,p} + (23)_{n,m,p+1} &= -\left(\ln(23)_{n,m,p}\right)_{,23},\\
(31)_{n+1,m,p} - 2(31)_{n,m,p} + (31)_{n,m,p+1} &= -\left(\ln(31)_{n,m,p}\right)_{,31},\\
(34)_{n,m,p+1} - 2(34)_{n,m,p} + (34)_{n-1,m-1,p-1} &= -\left(\ln(34)_{n,m,p}\right)_{,34},\\
(41)_{n+1,m,p} - 2(41)_{n,m,p} + (41)_{n-1,m-1,p-1} &= -\left(\ln(41)_{n,m,p}\right)_{,41},\\
(42)_{n,m+1,p} - 2(42)_{n,m,p} + (42)_{n-1,m-1,p-1} &= -\left(\ln(42)_{n,m,p}\right)_{,42},
\end{aligned}$$

where $(n, m, p) \in \mathbb{Z}^3$ label the vertices of a regular tetrahedral lattice and $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.

4. RELATIONS FOR COMMUTING OPERATORS

As an example of the use of invariants let us answer the natural question: *What is the relationship between the invariants of* $\mathbb{L}$ *and the invariants of* $\mathbb{M}$ *when* $[\mathbb{L}, \mathbb{M}] = 0$*, where* $\mathbb{L}$ *and* $\mathbb{M}$ *are* $n \times n$ *operator matrices?* We start with 2×2 case before moving on to the 3×3 case.

The case $n = 2 \times 2$ **:** Let us choose $\mathbb{L}$ and $\mathbb{M}$ as

$$\mathbb{L} = \begin{pmatrix} \partial_x + h_{11} & h_{12} \\ h_{21} & \partial_y + h_{22} \end{pmatrix}, \; \mathbb{M} = \begin{pmatrix} \partial_s + k_{11} & k_{12} \\ k_{21} & \partial_t + k_{22} \end{pmatrix}.$$

The gauge transformations on $\mathbb{L}$ and $\mathbb{M}$ are

$$\mathbb{L}^g = g^{-1}\mathbb{L}g \text{ and } \mathbb{M}^g = g^{-1}\mathbb{M}g,$$

where g is a diagonal 2×2 matrix. So the above transformations give us the following gauge invariants for $\mathbb{L}$ and $\mathbb{M}$ respectively:

$$\begin{aligned} (12)^L &= h_{12}h_{21}, \\ [12]^L &= h_{11,y} - h_{22,x} + \frac{1}{2}\left(\ln \frac{h_{12}}{h_{21}}\right)_{,xy}, \\ (12)^M &= k_{12}k_{21}, \\ [12]^M &= k_{11,t} - k_{22,s} + \frac{1}{2}\left(\ln \frac{k_{12}}{k_{21}}\right)_{,st}. \end{aligned}$$

If $[\mathbb{L}, \mathbb{M}] = 0$ then we have the following invariants relations:

$$\begin{aligned} (12)^L \left\{(12)^M_{,x} + (12)^M_{,y}\right\}^2 &= (12)^M \left\{(12)^L_{,s} + (12)^L_{,t}\right\}^2, \\ [12]^M_{,xy} &= [12]^L_{,st}. \end{aligned}$$

These relations, which are attractively symmetric under the interchange of $\{\mathbb{L}, x, y\}$ with $\{\mathbb{M}, s, t\}$, reflect the invariant geometry of the commutation condition $[\mathbb{L}, \mathbb{M}] = 0$.

The case $n = 3 \times 3$ **:** In this case $\mathbb{L}$ and $\mathbb{M}$ are 3×3 matrices

$$\mathbb{L} = \begin{pmatrix} \partial_x + h_{11} & h_{12} & h_{13} \\ h_{21} & \partial_y + h_{22} & h_{23} \\ h_{31} & h_{32} & \partial_z + h_{33} \end{pmatrix}, \; \mathbb{M} = \begin{pmatrix} \partial_s + k_{11} & k_{12} & k_{13} \\ k_{21} & \partial_t + k_{22} & k_{23} \\ k_{31} & k_{32} & \partial_u + k_{33} \end{pmatrix}.$$

So the gauge transformation on $\mathbb{L}$ is $g^{-1}\mathbb{L}g = \mathbb{L}^g$ which gives the following invariants for $\mathbb{L}$, where g is a diagonal 3×3 matrix:

$$\begin{aligned} &(12)^L = h_{12}h_{21}, \qquad (23)^L = h_{23}h_{32}, \qquad (31)^L = h_{31}h_{13}, \\ &(123)^L = h_{12}h_{23}h_{31}, \qquad (132)^L = h_{13}h_{32}h_{21}, \end{aligned}$$

$$[12]^L = h_{11,y} - h_{22,x} + \frac{1}{2}\left(\ln\frac{h_{12}}{h_{21}}\right)_{,xy},$$
$$[23]^L = h_{22,z} - h_{33,y} + \frac{1}{2}\left(\ln\frac{h_{23}}{h_{32}}\right)_{,yz},$$
$$[31]^L = h_{33,x} - h_{11,z} + \frac{1}{2}\left(\ln\frac{h_{31}}{h_{13}}\right)_{,xz}.$$

Similarly if we apply the gauge transformation on $\mathbb{M}$, $g^{-1}\mathbb{M}g = \mathbb{M}^g$, we get the following invariants for $\mathbb{M}$, where g is a diagonal 3×3 matrix:

$$(12)^M = k_{12}k_{21}, \quad (23)^M = k_{23}k_{32}, \quad (31)^M = k_{31}k_{13},$$
$$(123)^M = k_{12}k_{23}k_{31}, \quad (132)^M = k_{13}k_{32}k_{21},$$

$$[12]^M = k_{11,t} - k_{22,s} + \frac{1}{2}\left(\ln\frac{k_{12}}{k_{21}}\right)_{,st},$$
$$[23]^M = k_{22,u} - k_{33,t} + \frac{1}{2}\left(\ln\frac{k_{23}}{k_{32}}\right)_{,tu},$$
$$[31]^M = k_{33,s} - k_{11,u} + \frac{1}{2}\left(\ln\frac{k_{31}}{k_{13}}\right)_{,su}.$$

Let us now focus on the condition $[\mathbb{L},\mathbb{M}] = 0$, which gives the following invariant relations:

$$[12]^M_{,xy} = [12]^L_{,st},$$
$$[23]^M_{,yz} = [23]^L_{,tu},$$
$$[31]^M_{,xz} = [31]^L_{,su},$$

$$(12)^L\left\{(12)^M_{,x} + (12)^M_{,y}\right\}^2 = (12)^M\left\{(12)^L_{,s} + (12)^L_{,t}\right\}^2,$$
$$(23)^L\left\{(23)^M_{,y} + (23)^M_{,z}\right\}^2 = (23)^M\left\{(23)^L_{,t} + (23)^L_{,u}\right\}^2,$$
$$(31)^L\left\{(31)^M_{,x} + (31)^M_{,z}\right\}^2 = (31)^M\left\{(31)^L_{,s} + (31)^L_{,u}\right\}^2,$$

$$\frac{(123)^M}{(132)^M} = \frac{(123)^L}{(132)^L}.$$

5. **CONCLUSIONS**

In this paper we have discussed the differential invariants and their Laplace maps for both scalar and matrix cases. We have presented the relationship between the invariants of $\mathbb{L}$ and the invariants of $\mathbb{M}$ for 2×2 and 3×3 cases when $[\mathbb{L},\mathbb{M}] = 0$.

In section 2, we have obtained the complete system of invariants for a 4×4 hyperbolic system. It should be known that the cyclic quantities $(ij) = h_{ij}h_{ji}$ and $(ijk) = h_{ij}h_{jk}h_{ki}$ were obtained by Tsarev [9] for any $n\times n$ strictly hyperbolic first-order linear system.

We also have shown that the laplace maps produce $\mathbb{Z}^3 \times \mathbb{R}^4$ Toda Lattice Equations. It should be noted that the Laplace maps are an important tool for constructing exact solutions of both linear [5, 9, 10, 13] and nonlinear [11, 12] partial differential equations.

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