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A CLASS OF MAPS RELATED TO THE SEMILINEAR SPECTRUM AND APPLICATIONS

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Dedicated to Prof. J.R.L. Webb on the occasion of his retirement.

ABSTRACT. In this paper, the notion of (a, q)-L-stably solvable maps, where L is a closed Fredholm operator of index zero, is introduced. Closely related to the spectrum of semilinear operators, the class of (a, q)-L-stably solvable maps generalizes both (a, q)-stably solvable and the L-stably solvable maps that were defined previously. We prove properties for the new class of operators including the continuation principle and discuss eigenvalues. We also show its applications in the study of solvability of a nonlinear system.

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1. INTRODUCTION

In the extensive development of nonlinear spectral analysis, a basic idea is to define a nonlinear spectrum that shares the classical properties with the linear spectrum and has nontrivial applications [2, 3]. The spectrum of a closed linear operator L defined on a Banach space is the set $\sigma(L)$ of all scalar values not in the resolvent set $\rho(L)$. The resolvent set $\rho(L)$ contains all λ such that the operator $\lambda I - L$, where I is the identity, satisfies the three conditions: it is surjective, the inverse $(\lambda I - L)^{-1}$ exists and is continuous. Following a similar approach, three requirements for the resolvent set of a nonlinear operator N, $\rho(N)$, were given by Furi, Martelli and Vignoli [11]. One of them requires that $\lambda I - N$ is *stably-solvable*, which corresponds to the surjectivity condition of the linear spectrum [11].

The class of *stably-solvable* maps was first introduced in [10]. Later, the authors also introduced the class of 0-epi maps in [12]. Closely related to *stably-solvable* maps, the 0-epi maps are critical in the nonlinear spectrum of [9] and in the most recent work on a locally defined nonlinear spectrum of continuous operators [5]. Stable

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solvability implies surjectivity. In addition to some good properties, the continuation principle for stably-solvable operators can be used as a tool to study solvability of operator equations, and further of particular classes of differential equations. In the past decade, stably-solvable operators have been studied and applied in spectral analysis. In particular, the concept of stably solvable map has been generalized in two directions. On one hand, it was extended to the (a, q)-stably solvable maps by Appell, Giorgieri and Väth [1], where $a \geq 0, q \geq 0$. As a bigger class, the (a, q)-stably solvable maps share some similar properties as that of the stably-solvable maps. The special case, $(0, 0)$-stably solvable maps reduce to the original definition. On the other hand, the notion of L-stably solvable operators was introduced in [8]. The idea was in the development of a spectrum for the semilinear operator $\lambda L - N$, where L is a closed Fredholm linear operator of index zero, N is nonlinear. Semilinear spectra were also studied in [4, 6]. In [4], a semilinear spectrum that extends the Furi-Martelli-Vignoli nonlinear spectrum [11] was introduced. The recent work [6] defines an A-semilinear spectrum using the theory of A-proper maps. The L-stably solvable maps are essential in studying semilinear spectra. L-stable solvability ensures surjectivity of a semilinear operator. When L coincides with the identity I, the L-stably solvable operators reduce to the stably-solvable operators.

In this paper, we combine both generalizations of the *stably-solvable* operators to define a more general class of nonlinear maps, (a, q)-L-stably solvable maps. Both the (a, q)-stably solvable maps and the L-stably solvable maps are subsets of the new class corresponding to $L = I$ and $a = 0, q = 0$ respectively. Some properties on (a, q)-L-stably solvable maps are obtained. In particular, a new Continuation Principle is proved. As application, existence of a solution for a nonlinear algebraic system is studied.

2. PRELIMINARIES

First, we introduce the definitions and notations that will be used in the sequel. Let X and Y be two Banach spaces. For a continuous operator $F \in C(X, Y)$, the upper and lower *measure of noncompactness* are defined as the following [1]:

$$[F]_A = \inf\{k : \alpha(F(M)) \leq k\alpha(M)\},$$

$$[F]_a = \inf\{k : \alpha(F(M)) \geq k\alpha(M)\},$$

where $\alpha(M)$ denotes the (Kuratowski) measure of noncompactness of a bounded set $M \subset X$ [7] . The upper and lower quasi-norms:

$$[F]_Q = \limsup_{\|x\|\to\infty} \frac{\|F(x)\|}{\|x\|},$$

$$[F]_q = \liminf_{\|x\|\to\infty} \frac{\|F(x)\|}{\|x\|}.$$

The notion of *stably-solvable* operators is given in [10] and is basic in the definition of asymptotic spectrum given [11].

Definition 2.1. An operator $F : X \to Y$ is called *stably-solvable* if and only if for any given compact map $h : X \to Y$ with zero quasinorm ($[h]_Q = 0$), the equation $F(x) = h(x)$ has a solution.

In [1], a generalization of the *stably-solvable* maps is given as below.

Definition 2.2. Given $a \geq 0$ and $q \geq 0$, a map $F \in C(X, Y)$ is called (a, q)-stably solvable if for any $G \in C(X, Y)$ with $[G]_A \leq a$ and $[G]_Q \leq q$, the equation

$$F(x) = G(x)$$

has a solution $x \in X$.

The $(0, 0)$-stably solvable maps are the stably-solvable as in Definition 2.1. It is shown in [1] that the (a, q)-stably solvable maps share common properties as that of the stably-solvable maps [11].

On the other hand, the concept of L-stably solvable operators is introduced in [8] to define a spectrum for the semilinear operator $L - N$, where $N : X \to Y$ is a nonlinear map and

$$L : \operatorname{dom}(L) \subset X \to Y$$

is a closed Fredholm operator of index zero, $\ker(L) \neq \{0\}$ and $\operatorname{dom}(L)$ dense in X. The set up for the linear operator L is similar as that used for the coincidence degree theory [13]. Following the notations of [13], let $X = \ker(L) \oplus X_1$, $Y = Y_0 \oplus \operatorname{im}(L)$ and $P : X \to \ker(L)$, $Q : Y \to Y_0$ be the respective projections. Also, let L_p denote the invertible operator L restricted to $\operatorname{dom}(L) \cap X_1$ into $\operatorname{im}(L)$, write $K_P = L_P^{-1}$, $K_{PQ} = K_P(I - Q)$. Let $\Pi : Y \to Y/\operatorname{im}(L)$ be the quotient map. Let $\Lambda : Y/\operatorname{im}(L) \to \ker(L)$ be the linear isomorphism. For $\lambda \in \mathbb{C}$, let $f_\lambda : X \to X$ be defined as

$$f_\lambda(x) = \lambda(I - P)x - (\Lambda\Pi + K_{PQ})Nx. \tag{2.1}$$

Definition 2.3. [13] Let $\Omega \subset X$ be an open bounded subset with $\Omega \cap \operatorname{dom}(L) \neq \emptyset$. A nonlinear map $N : \overline{\Omega} \to Y$ is said to be *L-compact* if it satisfies the following two conditions:

a) $\Pi N : \overline{\Omega} \to \operatorname{coker}(L)$ is continuous and $\Pi N(\overline{\Omega})$ is bounded.
b) $K_{PQ}N : \overline{\Omega} \to X$ is compact.

Definition 2.4. [8] $\lambda L - N$ is said to be L-stably solvable if the equation

$$\lambda Lx - Nx = h(x)$$

has a solution $x \in \operatorname{dom}(L)$ for every continuous bounded L-compact map $h : X \to Y$ with quasinorm $[h]_Q = 0$.

The operator $\lambda L - N$ can be studied by converting to f_λ, as shown by the following lemma [8].

Lemma 2.1. *Let $y \in Y$ and $\lambda \in \mathbb{C}$. Then*

a) $\lambda Lx - Nx = y$ *if and only if* $f_\lambda(x) = (\Lambda\Pi + K_{PQ})y$.
b) $\lambda L - N : \mathrm{dom}(L) \to Y$ *is onto if and only if* $f_\lambda : X \to \mathrm{dom}(L)$ *is onto.*

3. THE (a, q)-L-STABLY SOLVABLE MAP

The following definition generalizes both Definitions 2.2 and 2.4.

Definition 3.1. Given $a \geq 0, q \geq 0$, a map $\lambda L - N$ is called (a, q)-L-stably solvable if for any $h \in C(X, Y)$ with $[K_{PQ}h]_A \leq a$ and $[h]_Q \leq q$, the equation $\lambda L - N = h(x)$ has a solution $x \in X$.

Theorems 3.3 and 3.4 show that the new class of operators share some common properties as that of (a, q)-stably solvable and L-stably solvable operators. The proof is based on the following lemma (Lemma 4.3, [8]).

Lemma 3.2. *Let $\Phi : Y/\mathrm{im}(L) \to Y_0$ be the natural linear isomorphism and $J = \Phi\Lambda^{-1}$. Then $\Lambda\Pi + K_{PQ} : Y \to \mathrm{dom}(L)$ is a linear isomorphism and $L + JP : \mathrm{dom}(L) \to Y$ is invertible with the continuous inverse $(L + JP)^{-1} = \Lambda\Pi + K_{PQ}$.*

Theorem 3.3. *Let $H = L + JP$ and f_λ be defined by (2.1). If $[f_\lambda]_q > 0$, then the following two statements are equivalent:*

a) $\lambda Lx - Nx = h(x)$ *has a solution for any bounded continuous L-compact map $h : X \to Y$ with bounded support.*
b) $\lambda L - N$ *is $(0, q)$-L-stably solvable for any $q < [Hf_\lambda]_q$.*

Proof: b) obviously implies a). In the following, we show that a) also implies b). Suppose $h : X \to Y$ is continuous, bounded L-compact map with $[h]_Q \leq q$. For $x \in X$, let

$$\sigma_n(x) = \begin{cases} 1 & \text{if } \|x\| \leq n, \\ 0 & \text{if } \|x\| > 2n, \end{cases}$$

then $\sigma_n (n = 1, 2, \ldots)$ can be extended to continuous functionals defined on X with $0 \leq \sigma_n \leq 1$. Let $h_n(x) = \sigma_n(x)h(x)$, $h_n (n = 1, 2, \ldots)$ are bounded L-compact maps with bounded support. Assume x_n $(n = 1, 2, \ldots)$ are solutions of the equation

$$\lambda L(x) - N(x) = h_n(x),$$

then $\{x_n\}$ must be bounded. Otherwise, by Lemma 2.1 we can obtain

$$f_\lambda(x_n) = (\Lambda\Pi + K_{PQ})\sigma_n(x_n)h(x_n).$$

Hence

$$\begin{aligned}\frac{\|f_\lambda(x_n)\|}{\|x_n\|} &= \|\sigma_n(x_n)H^{-1}\|\frac{\|h(x_n)\|}{\|x_n\|}\\ &\le q\|H^{-1}\| < [f_\lambda]_q.\end{aligned}$$

Note that the last step uses the fact $[Hf_\lambda]_q \le \dfrac{[f_\lambda]_q}{\|H^{-1}\|}$. This contradiction shows that $\{x_n\}$ is bounded. Assume that $M > 0$ such that $\|x_n\| < M$ for all n. For $n > M$, $\sigma_n(x_n) = 1$, therefore $(\lambda L - N)(x_n) = h(x_n)$, $\lambda L - N$ is $(0, q)$-L-stably solvable. □

Theorem 3.4. *Assume a) or b) of Theorem 3.3 hold. In addition, assume that* $[(\lambda L - N)]_a > 0$. *Then* $\lambda L - N$ *is* (a, q)-L-*stably salvable for any* $a < [(\lambda L - N)]_a$ *and* $q < [Hf_\lambda]_q$.

The proof of Theorem 3.4 follows directly from Corollary 1 of [1] and Theorem 3.3. The following lemma [8] is essential for the proof of the *Continuation Principle* given in Theorem 3.6.

Lemma 3.5. *Let* $B_r = \{y \in Y : \|y\| \le r\}$, π *be the radial retraction,* $\pi : Y \to B_r$. *Let* $h : X \to Y$ *be a continuous* L-*compact map. Then* $\pi h : X \to Y$ *is* L-*compact.*

Theorem 3.6. (Continuation Principle) *Let* $\lambda L - N : X \to Y$ *be* (a, q)-L-*stably solvable, and suppose that* $h : X \times [0, 1] \to Y$ *satisfies* $[h(x, 0)]_q < 1$ *and*

$$\alpha(h(M \times [0, 1])) \le a\alpha(M), \quad M \subset X \;\; \text{bounded.}$$

Let

$$S = \{x : x \in X, \lambda L(x) - N(x) = h(x, t), t \in [0, 1]\},$$

and assume that $(\lambda L - N)(S)$ *is bounded. Then the equation*

$$\lambda Lx - Nx = h(x, 1) \tag{3.1}$$

has a solution.

Proof: Suppose that $r > 0$ is such that $(\lambda L - N)(S)$ is contained in the interior of B_r. Let $\varphi : Y \to [0, 1]$ be continuous such that $\varphi(y) = 1$ for $y \in \overline{(\lambda L - N)(S)}$ and $\varphi(y) = 0$ for all $\|y\| \ge r$. Let $\pi : Y \to B_r$ be the radial retraction and denote

$$h_1(x) = h(x, \varphi(\lambda L - N)(x)).$$

Let Ω be an open bounded subset of X. By the assumption $\alpha(h(\Omega \times [0, 1])) \le a\alpha(\Omega)$ and Lemma 3.5, we have

$$\alpha(K_{PQ}\pi h_1(\Omega)) \le \alpha(K_{PQ}h_1(\Omega)) \le a\alpha(\Omega).$$

So $[K_{PQ}\pi h_1]_A \le a$. Next, since

$$[\pi h_1]_Q = \limsup_{\|x\|\to\infty} \frac{\|\pi h(x, \varphi(\lambda L - N)(x))\|}{\|x\|} = 0,$$

there exists x_0 such that

$$\lambda Lx_0 - Nx_0 = \pi h(x_0, \varphi(\lambda L - N)(x_0)). \tag{3.2}$$

If $Lx_0 - Nx_0 \geq r$, then $\varphi(\lambda L - N)(x_0) = 0$, $\|h(x_0, 0)\| < r$, so $\|\pi h(x_0, 0)\| < r$. Which contradicts to (3.2). Hence, $Lx_0 - Nx_0 < r$ and which implies

$$\|\pi h(x_0, \varphi(\lambda L - N)(x_0))\| < r.$$

Hence

$$\pi h(x_0, \varphi(\lambda L - N)(x_0)) = h(x_0, \varphi(\lambda L - N)(x_0)),$$

$$\lambda Lx_0 - Nx_0 = h(x_0, \varphi(\lambda L - N)(x_0)). \tag{3.3}$$

Equation (3.3) ensures that $x_0 \in S$, so $\varphi(\lambda L - N)(x_0) = 1$. Therefore, we have

$$\lambda Lx_0 - Nx_0 = h(x_0, 1).$$

The proof is completed. □

4. APPLICATION TO A NONLINEAR ALGEBRAIC SYSTEM

The results of Section 3 can be applied to the study of the following nonlinear algebraic system:

$$x = \lambda AF(x), \tag{4.1}$$

where $\lambda > 0$ is a parameter, x and $F(x)$ denote the column vectors:

$$\operatorname{col}(x_1, x_2, \ldots, x_n) \quad \text{and} \quad \operatorname{col}(f_1(x_1), f_2(x_2), \cdots, f_n(x_n))$$

respectively with $f_k : R \to R$, $k \in \{1, 2, ..., n\} = [1, n]$ and n is a positive integer. $A = (a_{ij})_{n\times n}$ is an $n \times n$ matrix and all its entries are positive numbers.

System (4.1) can be rewritten by a summability formula as the form

$$x_i = \lambda \sum_{j=1}^{n} a_{ij} f_j(x_j), \; i \in [1, n], \tag{4.2}$$

which can be seen as the analogue of the Hammerstein integral equation:

$$\psi(x) = \lambda \int_G K(x, y) f(\psi(y)) \, dy. \tag{4.3}$$

The importance of equation (4.3) is well known and it has been studied since 1930. However, much less is known for the nonlinear problems (4.1) or (4.2). In fact, many interesting problems in various areas such as difference equations, boundary value problems, dynamical networks [14], stochastic process and numerical analysis etc. can be converted to system (4.1) [15]. Applying the Continuation Principle for (a, q)-L-stably solvable maps, we prove existence of a solution and surjectivity for system (4.1). We will use $r(A)$ to denote the spectral radius of the linear operator A.

Theorem 4.7. *Let* $\lambda < \dfrac{1}{r(A)[F]_Q}$. *Then the operator*

$$x - \lambda AF(x) : \mathbb{R}^n \to \mathbb{R}^n$$

is onto. In particular, system (4.1) *has a solution.*

Proof: Let $G = \lambda AF$, it can be seen that $[G]_A < 1$ and $[G]_Q < 1$. By Proposition 1 of [1], the map $I - G$ is onto. □

Theorem 4.8. *For any* $\lambda < \dfrac{1}{\|A\|[F]_Q}$, *there exists a solution for system* (4.1).

Proof: It is known that the identity map I in a Banach space is (a, q)-stably solvable for $a, q \in [0, 1)$ (Example 1, [1]). Let $H(x, t) : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ be defined as

$$H(x, t) = \lambda t AF(x),$$

then $H(x, 0) = 0$ for any $M \subset \mathbb{R}^n$ and $\alpha(H(M \times [0, 1])) = 0$. Let

$$S = \{x : x \in \mathbb{R}^n, x = \lambda t AF(x), t \in [0, 1]\}.$$

We claim that S is bounded. Otherwise there are $x_n \in \mathbb{R}^n$ such that $\|x_n\| \to \infty$,

$$\lim_{n\to\infty} \frac{\lambda t_n AF(x_n)}{\|x_n\|} \le |\lambda|\|A\| \lim_{n\to\infty} \frac{\|F(x_n)\|}{\|x_n\|} \le |\lambda|\|A\|[F]_Q < 1.$$

This contradicts with $\dfrac{\lambda t_n AF(x_n)}{\|x_n\|} = 1$. By Proposition 4 of [1] (which is a special case of Theorem 3.6), the equation

$$x = H(x, 1) = \lambda AF(x)$$

has a solution. □

Theorem 4.9. *Assume that* $r(A)[F]_q < 1$ *and*

$$\lambda > \max\left\{\frac{1}{r(A)[F]_q}, \ \frac{1}{1 - r(A)[F]_q}\right\}.$$

Then system (4.1) *has a solution.*

Proof: Let $\mu = \frac{1}{\lambda}$, $G(x) = \mu x$ and

$$H(x, t) = AF(x) + \mu(1 - t)x.$$

Then $G(x)$ is (μ, μ)-stably solvable and not (a, q)-stably solvable for any $a, q > \mu$. (Corollary 3, [1]). Since

$$H(x, 0) = AF(x) + \mu x$$

and

$$\lambda > \frac{1}{1 - r(A)[F]_q},$$

we have

$$[H(x,0)]_q \le r(A)[F]_q + \mu < 1.$$

Also $\alpha(H(M \times [0,1])) = 0$ for bounded $M \subset \mathbb{R}^n$. Let

$$S = \{x : G(x) = H(x,t)\},$$

we show that S is bounded. In fact, assume that there exist $\{x_n\}_{n=1}^{\infty}$, $\|x_n\| \to \infty$ as $n \to \infty$ and $t_n \in [0,1]$ such that

$$\mu x_n = AF(x_n) + \mu(1-t_n)x_n,$$

then

$$AF(x_n) = \mu t_n x_n,$$
$$\frac{\lambda \|AF(x_n)\|}{\|x_n\|} = t_n \to t_0 \in [0,1].$$

However, from the assumptions, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{\lambda\|AF(x_n)\|}{\|x_n\|} &= \lambda r(A) \lim_{n\to\infty} \frac{\|F(x_n)\|}{\|x_n\|} \\ &\ge \lambda r(A)[F]_q > 1, \end{aligned}$$

which is a contradiction. So, S is bounded and the equation $\mu x = AF(x)$, which is equivalent to system (4.1), has a solution. □

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