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NONLINEAR SCHRÖDINGER EQUATIONS ON $\mathbb{R}$: GLOBAL BIFURCATION, ORBITAL STABILITY AND NONLINEAR WAVEGUIDES

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It is a great pleasure to dedicate this paper to Professor Jeff Webb.

ABSTRACT. Global bifurcation and stability results are surveyed for standing waves of one-dimensional nonlinear Schrödinger equations in inhomogeneous media. A new bifurcation result for the asymptotically linear case is presented and applications to nonlinear waveguides are discussed.

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1. INTRODUCTION

We review some results about the nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} + f(x,\psi) = 0 \tag{NLS}$$

for $\psi = \psi(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{C}$. We consider situations where the nonlinearity f depends explicitly on x. General assumptions on f are: the restriction $f\,|_{\mathbb{R}\times\mathbb{R}_+}$ is a real-valued Carathéodory function; $f(x, \mathrm{e}^{i\theta}s) = \mathrm{e}^{i\theta} f(x,s)$ for almost every $x \in \mathbb{R}$, all $\theta \in \mathbb{R}$ and all $s \geq 0$; some global growth conditions for the Cauchy problem associated with (NLS) to be well-posed, that will be stated in due course.

We are particularly interested in the issues of bifurcation and stability of *standing waves* $\psi_\lambda(t,x) = \mathrm{e}^{i\lambda t}u(x)$ with $\lambda \in \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}$. A natural space to seek solutions of (NLS) is $C\big([0,T), H^1(\mathbb{R},\mathbb{C})\big) \cap C^1\big((0,T), H^{-1}(\mathbb{R},\mathbb{C})\big)$, for some $T > 0$. The standing wave ψ_λ is then a solution of (NLS) if and only if $u \in H^1(\mathbb{R},\mathbb{R})$ solves the *stationary equation*

$$u'' + f(x,u) = \lambda u, \quad x \in \mathbb{R}. \tag{SNLS}$$

A *solution* of (SNLS) is a couple $(\lambda, u_\lambda) \in \mathbb{R} \times H^1(\mathbb{R},\mathbb{R})$, satisfying (SNLS) in the sense of distributions. (Considering weak solutions is useful in the first place

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for technical reasons but in fact, under our assumptions, weak solutions are classical solutions.) In this paper we shall focus on the few situations where the existence of a global branch of solutions is known. An important problem is then to obtain conditions ensuring stability of the corresponding standing waves of (NLS) along the whole branch of solutions.

The bifurcation analysis of the stationary equation (SNLS) yields a great deal of information concerning the stability of the corresponding standing waves of (NLS). These are very closely related issues and, in particular, the monotonicity of the L^2 norm of solutions to (SNLS), $|u_\lambda|_{L^2}$, as a function of the bifurcation parameter λ, is of paramount importance in that respect. To exploit most conveniently the dependence on λ, it is desirable to obtain solutions of (SNLS) as smooth functions of λ. It is then possible to discuss the sign of $\frac{\mathrm{d}}{\mathrm{d}\lambda}|u_\lambda|_{L^2}$. Under appropriate conditions on the spectrum of the linearization of (NLS) at the standing wave $\mathrm{e}^{i\lambda t}u_\lambda$, this yields a stability criterion known as the *Vakhitov-Kolokolov condition*, that first appeared in [31] in the context of nonlinear waveguides. A general theory of stability for Hamiltonian systems with symmetries was later developed in [11], involving this criterion. We base our stability analysis on this work.

The nonlinear Schrödinger equation has numerous applications in physics and engineering science, including cold quantum gases, plasma physics and water waves. It plays a major role in *nonlinear optics*, a field that has evolved very closely to the mathematical theory of (NLS). Under some approximations, (NLS) arises as a model of *self-trapping* of an electromagnetic field in a nonlinear waveguide. The relevance of (NLS) and (SNLS) in this context was already put forth in the 60's, see [1, 2]. The physical phenomena responsible for the self-trapping effect are discussed in [1, 26], and linear stability of standing waves was already investigated in [31]. The basic principle of self-trapping can be heuristically understood as follows. The refractive index of a dielectric medium is modulated by an applied electric field through various microscopic phenomena. According to Fermat's principle, the light beam (electromagnetic waves in the optical regime) bends towards regions with higher refractive index. Hence the beam can be focused by the influence of its own electric field on the waveguide. The balance between spreading and focusing gives rise to localized waves known as *solitons*. There is a huge literature on optical solitons and a good review can be found in [18]. Solitons arising in various physical contexts present remarkable stability properties and play a crucial role in many engineered systems. The deep mathematical reasons for the stability of solitons are discussed in [27] in a broad context.

The standing waves described above are solitons and the mathematical results surveyed here apply to *planar self-focusing waveguides*. They yield existence and stability of so-called *guided TE modes*, which are particular solutions of Maxwell equations in a dielectric medium. We refer the reader to [21, 24, 7] for the description of the waveguides and the physical consequences of our results. Let us just mention here that, in this context, the variable t in (NLS) corresponds to the direction of propagation of the light beam, whereas x plays the role of the transverse direction. The paper [21] was one of the earliest rigorous works on guided waves in general stratified dielectrics. It provides sharp existence results for guided waves in many different configurations. In particular, in addition to the dependence on the electric field, the refractive index is allowed to vary accross the medium, either continuously (graded profile) or discontinuously (step profile). Here, we shall not consider as general a setting as in [21], but will rather focus on [24] and [7].

The stability of optical solitons in nonlinear media has been studied by many authors. A survey can be found in [17]. The papers [24] and [7] are concerned with planar structures with a graded profile and a self-focusing nonlinear response. The peculiarity of these works is that (nonlinear) stability of solitons of *arbitrary power* is obtained, as well as a detailed analysis of the asymptotic regimes. In [24], the existence of guided TE modes and their stability are proved, in a situation where the medium is inhomogeneous, even in the absence of electric field. The analysis is carried out along a global branch of solutions to (SNLS), bifurcating from an eigenvalue of the linearized operator. Two different asymptotic behaviours are observed as the power of the beam becomes large, depending on whether the nonlinear response is saturable or not. The mathematical results in [24] are the continuation of the papers [12, 13, 22]. In [7], we consider the case where the medium is homogeneous (constant refractive index) when the electric field is switched off. In this configuration, the branch of solutions bifurcates from the essential spectrum. This requires a different strategy that is not appropriate to deal with a saturable nonlinear response.

The paper is organized as follows. Section 2 is devoted to the existence results of [12] and [7], where bifurcation for (SNLS) occurs from the principal eigenvalue of the linearized operator and from the bottom of the essential spectrum, respectively. Section 3 deals with the stability analysis along the global branches of solutions, for the two different scenarios. In Section 4, we present a new bifurcation result for a nonlinearity f that is *asymptotically linear* in the sense that there exists a function $f_\infty : \mathbb{R} \to \mathbb{R}$ such that $f(x,s)/s \to f_\infty(x)$ as $s \to \infty$. This behaviour describes waveguides with a saturable nonlinear response. We also suppose that $f(x,s)/s$ tends to a constant as $s \to 0$, meaning that the medium is homogeneous when the field is switched off. We obtain *asymptotic global bifurcation* in the spirit of Rabinowitz, however in a situation with a lack of compactness and regularity.

Notation. Throughout the paper, we shall denote by H the real Hilbert space $H^1(\mathbb{R},\mathbb{R})$ and its usual norm by $\|\cdot\|$. The topological dual of H is denoted by H^*. The space $H^1(\mathbb{R},\mathbb{C})$ will be denoted by H^1 and its dual by H^{-1}. H^1 is endowed with the inner product $\langle u,v\rangle := \mathrm{Re}\int_{\mathbb{R}} u'\overline{v'} + u\overline{v}\,\mathrm{d}x$, $u,v\in H^1$, making it a real Hilbert space. We also denote the associated norm by $\|\cdot\|$. Finally, for $1\le p\le\infty$, we shall write $L^p(\mathbb{R},K)$, $K=\mathbb{R}$ or $\mathbb{C}$, for the usual Lebesgue spaces, with norms $|\cdot|_{L^p}$ and we will merely write L^p when no confusion is possible.

2. TWO BIFURCATION SCENARIOS

A first basic assumption on the nonlinearity f is the following:

(f1) $f\in C^1(\mathbb{R}^2)$ with $f(x,0)=0$ for all $x\in\mathbb{R}$, $\partial_2 f(\cdot,0)\in C^1(\mathbb{R})$, and we have $\partial_2 f(x,s)\to\partial_2 f(x,0)$ as $s\to 0$, uniformly in $x\in\mathbb{R}$.

In particular, $u\equiv 0$ is a solution of (SNLS) for any $\lambda\in\mathbb{R}$. A solution (λ,u) is said to be *trivial* if $u\equiv 0$ and we call $\{(\lambda,0):\lambda\in\mathbb{R}\}\subset\mathbb{R}\times H$ the *line of trivial solutions*. A natural starting point is to seek local bifurcation from the line of trivial solutions. Of primary interest is the linearization of (SNLS) at $u\equiv 0$,

$$u''+\partial_2 f(x,0)u=\lambda u. \tag{2.1}$$

We shall consider here two distinct situations:

either

(f1-a) (f1) holds and $\partial_2 f(0,0)>0=\lim_{x\to\infty}\partial_2 f(x,0)$,

or

(f1-b) (f1) holds and $\partial_2 f(x,0)=0$ for all $x\in\mathbb{R}$.

Formally, in case (f1-a), one expects to get bifurcation from an eigenvalue of (2.1), whereas if (f1-b) holds, the linearization is

$$u''=\lambda u, \tag{2.2}$$

having purely continuous spectrum. In this case, standard bifurcation theory cannot be applied and other methods must be used to start the branch off the line of trivial solutions. In the next two subsections, we shall present results pertaining to these two different situations. In both cases, a continuation argument is used to globally extend a first, local, result. The case (f1-a) was considered in [12], where the authors proved the existence of a global, C^1, branch of solutions to (SNLS). The case (f1-b) has been treated in [7] with similar results. We shall make somewhat stronger hypotheses and we will not discuss the results as thoroughly as in [12] and [7]. We first state below the hypotheses that are common to the two different problems, in addition to (f1). In particular, we impose on f symmetry and monotonicity conditions that arise naturally in the modelling of symmetric self-focusing waveguides [7, 24].

(f2) $f(-x,s)=f(x,s)$ for all $(x,s)\in\mathbb{R}^2$.

(f3) $f(\cdot,s)$ is non-increasing on $[0,\infty)$ for all $s\ge 0$ and strictly decreasing at one point $x_0>0$.

(f4) $f(x,\cdot)$ is strictly increasing in $s \geq 0$ with $\partial_2 f(x,s) > s^{-1}f(x,s) > 0$ for all $x \in \mathbb{R}$ and all $s > 0$.

Note that (f4) ensures that $s^{-1}f(x,s)$ is a strictly increasing function of $s \in (0,\infty)$ for each fixed $x \in \mathbb{R}$. In the context of waveguides, this behaviour describes a self-focusing nonlinear response.

2.1 Bifurcation from the principal eigenvalue.

Under hypotheses (f1-a), (f2) to (f4), the following global bifurcation theorem was proved in [12] (we slightly adapt it to fit our setting).

Theorem 2.1. [12, Theorem 2] *Suppose that f satisfies (f1-a) and (f2) to (f4). Then there exist $0 < \lambda_* < \lambda^* \leq \infty$ and a function $u \in C^1\big((\lambda_*,\lambda^*),H\big)$ such that, for all $\lambda \in (\lambda_*,\lambda^*)$, $(\lambda,u(\lambda))$ is a solution of* (SNLS) *with $u(\lambda)$ positive, even and strictly decreasing on $(0,\infty)$. Moreover*

$$\lim_{\lambda\to\lambda_*} \|u(\lambda)\| = 0 \quad \textit{and} \quad \lim_{\lambda\to\lambda^*} \|u(\lambda)\| = \infty.$$

Proof. The proof is in two steps. First establish local bifurcation from the principal eigenvalue of (2.1) using the Crandall-Rabinowitz theorem. Then prove that the branch can be globally extended using the implicit function theorem. The regularity of the branch is inherited from that of f in the process and the asymptotic behaviour follows by *a priori* estimates. □

Remark 2.2. The results in [12] were actually established in $H^2(0,\infty)$ by studying the restriction of (SNLS) to the half-line and so the space H in Theorem 2.1 can be replaced by $H^2(\mathbb{R})$ and the limits also hold in the H^2 norm. The behaviour of the L^∞ norm of the solutions was also studied, yielding $\frac{\mathrm{d}}{\mathrm{d}\lambda}|u(\lambda)|_{L^\infty} > 0$ for all $\lambda \in (\lambda_*,\lambda^*)$.

A detailed discussion was carried out in [12] about the values of the limit points λ_* and λ^*. First, the nonlinearity is written as

$$f(x,s) = V(x)s + h(x,s), \tag{2.3}$$

where $V(x) := \partial_2 f(x,0)$, and the hypotheses on f are translated in terms of the functions V and h (see hypotheses (H1)-(H3) in [12, p. 640]). In particular, assumption (f1-a) implies that $V \in C^1(\mathbb{R})$ is even with

$$V(0) > 0 = \lim_{|x|\to\infty} V(x),$$

and the linearized equation (2.1) now reads

$$u'' + V(x)u = \lambda u. \tag{2.4}$$

The bifurcation point on the line of trivial solutions, $(\lambda_*, 0)$, is thus given by the following variational characterization of the principal eigenvalue of (2.4):

$$\lambda_* = -\inf_{\substack{u\in H\setminus\{0\}\\ |u|_{L^2}=1}} \int_{\mathbb{R}} u'(x)^2 - V(x)u(x)^2 \, \mathrm{d}x.$$

To estimate the value of the other end point, additional assumptions are required. First, there exist constants $\sigma, C > 0$ and a function $h_0 \in C^1(\mathbb{R})$ such that $h(x,s)/s^{1+\sigma} \to h_0(x) \geq C$ as $s \to 0^+$. Then f is supposed to satisfy one of the following hypotheses:

(L1) We have $\lim_{s\to\infty} \lim_{x\to\infty} s^{-1} f(x,s) = \infty$. In particular, it follows by (f3)-(f4) that $\lim_{s\to\infty} s^{-1} f(x,s) = \infty$ for all $x \geq 0$.

(L2) There exists $W \in L^\infty(0,\infty)$ such that $\lim_{s\to\infty} s^{-1} f(x,s) = W(x)$ uniformly for $x \geq 0$ bounded. It follows from (f1), (f3) and (f4) that W is continuous and non-increasing with $W(x) > V(x)$ for all $x \geq 0$.

Under the hypotheses above and one of (L1) or (L2), it is proven in [12] that all positive solutions to (SNLS) lie on the curve described in Theorem 2.1 and that

$$\lambda^* = \infty \quad \text{if (L1) holds} \quad \text{whereas} \quad \lambda^* < \infty \quad \text{if (L2) holds.}$$

It is also shown that if (L1) holds then $|u(\lambda)|_{L^\infty} \to \infty$ as $\lambda \to \infty$. Further estimates of λ^* are given in terms of the difference $W - V$ if (L2) holds but we will not give more details here.

Remark 2.3. In the context of waveguides, both situations (L1) and (L2) are important. A first approximation (the so-called Kerr approximation) of the nonlinear response of the medium leads to a power-type nonlinearity, satisfying (L1). To obtain an accurate description of a dielectric medium at arbitrary high power (large $|u|_{L^2}$), a behaviour like (L2) is in order. This accounts for a saturation phenomenon in the medium, where the nonlinear refractive index tends to a finite value (related to λ^*) when the intensity of the electric field (related to u) becomes large.

2.2. Bifurcation from the essential spectrum.

We now turn to the situation where (f1-b) holds, and the linearization of (SNLS) at $u \equiv 0$ has purely continuous spectrum. This corresponds to $V \equiv 0$ in the notation (2.3). In [7], the nonlinearity has the form

$$f(x,s) = q(x)|s|^{p-1}s, \tag{2.5}$$

where $p > 1$ and $q : \mathbb{R} \to \mathbb{R}$ satisfies the following hypotheses:

(q1) $q \in C^1(\mathbb{R}, \mathbb{R})$.

(q2) $q > 0$ on $\mathbb{R}$, q is even and $q'(x) < 0$ for all $x \in (0,\infty)$.

(q3) There exists $b \in (0,1)$ such that $\lim_{|x|\to\infty} |x|^b q(x) = 1$.

(q4) Setting $r(x) = xq'(x) + bq(x)$, we have $\lim_{|x|\to\infty} |x|^b r(x) = 0$.

Theorem 2.4. [7, Theorems 1.1, 1.3 and 1.4] *Let f be given by* (2.5) *with $p > 1$ and q satisfying (q1) to (q4). There exists a function $u \in C^1\big((0,\infty), H\big)$ such that $u(\lambda)$ is the unique positive even solution of* (SNLS), *for all $\lambda \in (0,\infty)$. Furthermore,*

$$\lim_{\lambda\to 0} \|u(\lambda)\| = 0 \quad \textit{and} \quad \lim_{\lambda\to\infty} \|u(\lambda)\| = \infty.$$

Remark 2.5. A much more detailed analysis of the asymptotic behaviour of various norms of the solutions is given in [7]. In particular, the limits above also hold in the H^2 norm and we have $|u(\lambda)|_{L^\infty} \to \infty$ as $\lambda \to \infty$. Moreover, the results are established under weaker assumptions, q being allowed to have a singularity at the origin.

We thus get a global branch containing all positive even solutions to (SNLS), with very similar properties to those of the previous subsection in the case where (L1) holds. The structure of the proof is also similar, starting by a local bifurcation result from the line of trivial solutions, and then proceeding by analytic continuation. However, the first step is quite different in the present case since the linearization (2.2) has no eigenvalues. There exists a fairly extensive literature on bifurcation from points of the essential spectrum involving variational or topological methods, see [19, 29] for closely related problems. However, these methods do not usually provide any regularity with respect to the bifurcation parameter. The strategy used in [7] goes back to [20], where a semilinear elliptic problem on the half-line is considered. It has also proved useful in higher dimensions, see [10, 8]. The core idea is the change of variables

$$\lambda = k^2 \quad \text{for } k > 0, \quad u(x) = k^{(2-b)/(p-1)} v(y), \; y \equiv kx, \tag{2.6}$$

transforming

$$u'' - \lambda u + q(x)|u|^{p-1}u = 0 \tag{2.7}$$

into the auxiliary equation

$$v'' - v + k^{-b} q(k^{-1}y)|v|^{p-1}v = 0, \quad \text{for } k > 0. \tag{2.8}$$

The precise hypotheses on the behaviour of q at infinity come in when letting $k \to 0$, formally yielding the limit problem

$$v'' - v + |y|^{-b}|v|^{p-1}v = 0. \tag{2.9}$$

It turns out that (2.9) has a unique positive even solution v_0 (ground state) that can be obtained by constraint minimization on the Nehari manifold. Moreover this solution is non-degenerate and one can apply the implicit function theorem to the point $(0, v_0) \in \mathbb{R} \times H$, yielding a local branch of solutions to (2.8). Solutions of (2.7) are then obtained via (2.6), and we precisely control the behaviour of the solutions as $\lambda = k^2 \to 0$. Once the analytic continuation is established, a similar procedure can be carried out to study the behaviour of the global branch as $\lambda \to \infty$.

3. ORBITAL STABILITY OF STANDING WAVES

We shall now address the question of stability of standing wave solutions of (NLS). A solution of (NLS) is a function $\psi \in C\big([0,T),H^1\big)\cap C^1\big((0,T),H^{-1}\big)$ satisfying (NLS) for all $t\in(0,T)$, for some $T>0$. We say that the solution is global if $T=\infty$.

Theorems 2.1 and 2.4 provide standing wave solutions of (NLS). It is clear that such solutions are global. A first step towards their stability analysis is to study the well-posedness of the Cauchy problem associated with (NLS). Two conservation laws are of major importance. The *energy* and the *charge* are respectively defined by $E,Q:H^1\to\mathbb{R}$,

$$E(z)=\frac{1}{2}\int_{\mathbb{R}}|z'|^2\,\mathrm{d}x-\int_{\mathbb{R}}F(x,z)\,\mathrm{d}x,\quad Q(z)=\frac{1}{2}\int_{\mathbb{R}}|z|^2\,\mathrm{d}x,\quad \text{for } z\in H^1, \tag{3.1}$$

where $F(x,z)=\int_0^{|z|}f(x,s)\,\mathrm{d}s$ for all $z\in\mathbb{C}$.

Theorem 3.1. [3, Section 3.5] *Suppose f satisfies (f1-a) or (f1-b) and there exist $C>0$ and $\alpha\in[0,4)$ such that*

$$|f(x,s)|\le C(1+|s|^\alpha)|s|\quad \textit{for all } (x,s)\in\mathbb{R}^2. \tag{3.2}$$

Then for any $\psi_0\in H^1$, there exists a unique global solution ψ of (NLS) *with initial condition $\psi(0,\cdot)=\psi_0$. Furthermore, $E,Q\in C^2(H^1,\mathbb{R})$ and the quantities $E(\psi(t,\cdot))$ and $Q(\psi(t,\cdot))$ are independent of $t\ge0$.*

Remark 3.2. The assumption $\alpha<4$ can be dropped if one is content to obtain only local solutions. But stable solutions are required to be global.

Let us now precise the notion of stability we shall be concerned with. We study the stability of a given standing wave $\psi_\lambda(t,x)=\mathrm{e}^{i\lambda t}u_\lambda(x)$ with respect to small perturbations of the initial condition in H^1. Since $\psi_\lambda(t,x)$ is periodic in t, one cannot expect to prove asymptotic stability nor even Liapounov stability in the usual sense. This is due to the invariance of (NLS) with respect to the action of the group $\{\mathrm{e}^{i\theta}\}_{\theta\in\mathbb{R}}$ on H^1. Counterexamples are for instance given in [24, Section 6.1]. The appropriate notion of stability in the present context is that of orbital stability.

Definition 3.3. *We say that the standing wave ψ_λ is orbitally stable if*

$$\forall\,\varepsilon>0\ \exists\,\delta>0 \textit{ such that}$$

for any solution $z(t,x)$ of (NLS) with initial data $z(0,\cdot)\in H^1$ we have

$$\|z(0,\cdot)-u_\lambda\|_{H^1}\le\delta \quad\Longrightarrow\quad \inf_{\theta\in\mathbb{R}}\|z(t,\cdot)-\mathrm{e}^{i\theta}u_\lambda\|_{H^1}\le\varepsilon\quad \forall\,t\ge0.$$

We say that ψ_λ is orbitally unstable if it is not stable.

Remark 3.4. When the nonlinearity does not depend explicitly on x, (NLS) is invariant under the translations of $\mathbb{R}$ and the notion of stability must be weakened to accommodate this additional group of symmetries, see [4].

A general theory of orbital stability for infinite-dimensional Hamiltonian systems was established in [11]. (See also [25], where this issue was revisited in great detail, and applied to (NLS).) The stability of a standing wave ψ_λ is related to spectral properties of the linearization of (NLS) at ψ_λ, which is essentially the Hessian of the corresponding Hamiltonian system. When the spectral conditions are satisfied, it can be inferred from [11] that, in the present context, the standing wave

$$\psi_\lambda(t,x) = \mathrm{e}^{i\lambda t} u_\lambda(x) \text{ is orbitally stable/unstable if } \frac{\mathrm{d}}{\mathrm{d}\lambda}\int_{\mathbb{R}} u_\lambda(x)^2\,\mathrm{d}x > 0/<0.$$

A necessary condition of this form for stability seems to have first appeared in 1973 in [31] and is therefore often referred to as the *Vakhitov-Kolokolov condition*. We shall simply call it the V-K condition below.

3.1. The spectral conditions.

We briefly explain how to interpret (NLS) as a Hamiltonian system (see [25] for more details). We make the identification $H \subset L^2 \cong (L^2)^* \subset H^*$, where $H = H^1(\mathbb{R},\mathbb{R})$ and $L^2 = L^2(\mathbb{R},\mathbb{R})$. We denote by $\mathcal{J} : H \hookrightarrow H^*$ the inclusion.

The stability theory in [11] is concerned with Hamiltonian systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = JE'(\varphi(t)) \tag{3.3}$$

with $\phi : [0,\infty) \to X$, X a real Hilbert space, $J : D(J) \subset X^* \to X$ a skew-symmetric linear operator, and $E : X \to \mathbb{R}$ the Hamiltonian. In the present context we set

$$X := H \times H, \quad J := \begin{pmatrix} 0 & \mathcal{J}^{-1} \\ -\mathcal{J}^{-1} & 0 \end{pmatrix},$$

and we identify $H^1 = H^1(\mathbb{R},\mathbb{C})$ with X by $H^1 \ni z \leftrightarrow \phi = (\operatorname{Re} z, \operatorname{Im} z) \in X$. The energy functional $E : H^1 \to \mathbb{R}$ in (3.1) is then interpreted as a functional on X in the obvious way and we have $E \in C^2(X,\mathbb{R})$. Similarly, $Q \in C^2(X,\mathbb{R})$. The notion of solution introduced above for (NLS) is now translated in the following way. We say that ϕ is a solution of (3.3) if there exists $T \in (0,\infty]$ such that $\phi \in C([0,T),X) \cap C^1((0,T),X^*)$ and (3.3) is satisfied for all $t \in (0,T)$, where the right-hand side is interpreted as an element of X^* by the identification $X \subset L^2 \times L^2 \subset X^*$.

With the above setting, it is easily verified that $\phi(t) = (\operatorname{Re} z(t), \operatorname{Im} z(t))$ is a solution of (3.3) if and only if z is a solution of (NLS). Standing waves are particular solutions of the form $\phi(t) = T(\lambda t)\varphi$, $\varphi \in X$, where

$$T(\theta) := \begin{pmatrix} (\cos\theta)I & -(\sin\theta)I \\ (\sin\theta)I & (\cos\theta)I \end{pmatrix}, \quad \theta \in \mathbb{R}, \quad I : H \to H \text{ the identity},$$

defines a 1-parameter group of isometries $\{T(\theta) : X \to X\}_{\theta\in\mathbb{R}}$, corresponding to the group $\{e^{i\theta}\}_{\theta\in\mathbb{R}}$ acting on H^1 in the (NLS) formalism. These symmetry groups leave (3.3) and (NLS) invariant, respectively. Now the stationary equation (SNLS) is equivalent to

$$E'(\varphi) + \lambda Q'(\varphi) = 0 \tag{3.4}$$

for functions $\varphi \in X$ of the form $\varphi = (u, 0)$. Hence the results of Section 2 provide smooth global branches of solutions $\{(\lambda, \varphi_\lambda) : \lambda \in \mathcal{I}\} \subset \mathbb{R} \times X$ of (3.4), with $\varphi_\lambda = (u(\lambda), 0)$ and $\mathcal{I} = (\lambda_*, \lambda^*)$ if Theorem 2.1 applies, $\mathcal{I} = (0, \infty)$ for Theorem 2.4.

We now introduce the bounded linear operator $H_\lambda : X \to X^*$,

$$H_\lambda := E''(\varphi_\lambda) + \lambda Q''(\varphi_\lambda), \quad \lambda \in \mathcal{I}.$$

We define the *spectrum* of H_λ as the following subset of $\mathbb{R}$:

$$S(H_\lambda) := \big\{\xi \in \mathbb{R} : H_\lambda - \xi\widetilde{R} : X \to X^* \text{ is not an isomorphism}\big\},$$

where $\widetilde{R} = \operatorname{diag}(R, R)$ and $R := -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + 1 : H \to H^*$ is the Riesz isomorphism. It turns out that, under the hypotheses of either Theorem 2.1 or Theorem 2.4, the operator $\widetilde{R}^{-1}H_\lambda : X \to X$ is bounded and self-adjoint, and its spectrum coincides with $S(H_\lambda)$. Furthermore, H_λ is explicitly given by

$$H_\lambda = \begin{pmatrix} -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} - \partial_2 f(x, u(\lambda)) + \lambda & 0 \\ 0 & -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} - \dfrac{f(x, u(\lambda))}{u(\lambda)} + \lambda \end{pmatrix}, \quad \lambda \in \mathcal{I}.$$

The spectral conditions required for the stability analysis are the following:

(S1) there is $\nu_\lambda < 0$ such that $S(H_\lambda) \cap (-\infty, 0) = \{\nu_\lambda\}$ and $\dim\ker(H_\lambda - \nu_\lambda\widetilde{R}) = 1$;

(S2) $\ker H_\lambda = \operatorname{span}\{(0, u(\lambda))\}$;

(S3) $S(H_\lambda) \setminus \{\nu_\lambda, 0\}$ is bounded away from zero in $\mathbb{R}$.

3.2. The stability results.

Under appropriate assumptions, we establish the orbital stability of the standing waves of (NLS) given by Theorem 2.1 and Theorem 2.4. Let us already emphasize here that the situations in which stability holds along a global branch are rather scarce. We shall mention when local results hold under less restrictive assumptions.

Under the hypotheses of Theorem 2.1, a thorough stability analysis has been carried out in a series of papers [13, 22, 24], with applications to nonlinear waveguides. We state here the results in their most general form [24, Section 7].

A convenient way to describe our assumptions is to suppose that the function h in (2.3) has the form $h(x, s) = g(x, s^2)s$, i.e., to write f as

$$f(x, s) = V(x)s + g(x, s^2)s. \tag{3.5}$$

This particular structure arises naturally in the modelling of nonlinear waveguides, see [24]. We now make the following hypotheses.

(A1) $V \in C^1(\mathbb{R})$ is even with $V'(x) \leq 0$ and $V(0) > 0 = \lim_{|x|\to\infty} V(x)$.

(A2) $g \in C(\mathbb{R} \times [0,\infty)) \cap C^1(\mathbb{R} \times (0,\infty))$ with $g(x,0) = 0$ and $h \in C^1(\mathbb{R}^2)$ where $h(x,s) = g(x,s^2)s$. Furthermore, for all $K > 0$, g and $\partial_2 h$ are bounded and uniformly continuous on $\mathbb{R} \times [0,K]$.

(A3) For all $x \geq 0$ and $s > 0$, $g(-x,s) = g(x,s)$, $\partial_1 g(x,s) \leq 0$ and $\partial_2 g(x,s) > 0$.

It is not difficult to see that under (A1)-(A3), defining f by (3.5), the hypotheses of Theorem 2.1 are satisfied. In fact, (A1)-(A3) do not impose any stronger regularity but merely ensures that both the linear and the nonlinear part in the decomposition (3.5) have good monotonicity properties.

The following hypotheses are needed to prove stability.

(A4) There is $\beta \in [0,2)$ and $C > 0$ such that

$$|g(x,s)| \leq C(1 + |s|^\beta) \quad \text{for all } x \in \mathbb{R} \text{ and } s \geq 0.$$

(A5) For fixed $s > 0$, $\partial_2 g(x,s)$ is non-increasing in $x \in [0,\infty)$ and, for fixed $x \in \mathbb{R}$, $s\partial_2 g(x,s)$ is non-decreasing in $s \in [0,\infty)$.

(A6) Defining $Q(x,s) := \dfrac{2g(x,s) + x\partial_1 g(x,s)}{s\partial_2 g(x,s)} - 1$, we have that, for each fixed $s > 0$, $Q(x,s)$ is a non-negative, non-increasing function of $x \in [0,\infty)$ and, for fixed $x \in \mathbb{R}$, $Q(x,s)$ is a non-decreasing function of $s \in [0,\infty)$.

Remark 3.5. By (A5), $s\partial_2 g(x,s) \geq \partial_2 g(x,1) > 0$ for all $s \geq 1$ and so $g(x,s) \geq \partial_2 g(x,1)\ln s$ for all $s \geq 1$. Hence, $\lim_{s\to\infty} g(x,s) = \infty$ for all $x \in \mathbb{R}$, showing that for f defined by (3.5), the hypothesis (L2) of Section 2 can not occur. Therefore, Theorem 3.7 below does not apply to the asymptotically linear case. As far as we know, no global stability results are available in this case.

Example 3.6. A nonlinearity satisfying (A5) is given by

$$g(x,s) = (1+x^2)^{-b/2} s^\sigma,$$

with $\sigma > 0$ and $b \geq 0$. Moreover, in this case the function Q in (A6) is given by

$$Q(x,s) = \frac{1}{\sigma}\left\{2 - \sigma - b + \frac{b}{1+x^2}\right\}.$$

Hence, (A6) is satisfied if and only if $0 < \sigma \leq 2$ and $0 \leq b \leq 2 - \sigma$.

Theorem 3.7. [24, Theorem 7.3] *Suppose f is given by* (3.5)*, where V and g satisfy (A1) to (A6). Then the standing wave $\psi_\lambda(t,x) = \mathrm{e}^{i\lambda t}u(\lambda)(x)$ of* (NLS)*, with $u \in C^1\big((\lambda_*,\infty), H\big)$ given by Theorem 2.1, is orbitally stable for all $\lambda \in (\lambda_*,\infty)$.*

Proof. The proof follows from the arguments in [13] and [22]. The spectral conditions (S1)-(S3) are verified in [22, Section 4] in the case where $g(x, s) \equiv g(s)$ is independent of x. The proof is easily generalized under assumptions (A1)-(A3).

In [13], the V-K condition for stability is proved to hold at any $\lambda \in (\lambda_*, \lambda^*)$ under the hypotheses above. The proof goes by continuation, on studying the sign of

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \int_{\mathbb{R}} u(\lambda)^2 \, \mathrm{d}x = 2 \int_{\mathbb{R}} u(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} u(\lambda) \, \mathrm{d}x \tag{3.6}$$

along the branch of solutions. Since bifurcation from the line of trivial solutions occurs at $\lambda = \lambda_*$, one already knows that this quantity must be positive at some value of $\lambda > \lambda_*$. It is then enough to prove that it cannot vanish. This is done by using sign and monotonicity properties of $v(\lambda) := \frac{\mathrm{d}}{\mathrm{d}\lambda} u(\lambda)$. These properties are obtained from the equation satisfied by $v(\lambda)$,

$$v'' + \partial_2 f(x, u(\lambda))v = \lambda v + u(\lambda), \tag{3.7}$$

first by perturbation for λ close to λ_*, and then by continuation to the whole interval (λ_*, ∞). Integral identities involving (SNLS) and (3.7) are then used to show that (3.6) does not vanish. The assumptions (A5) and (A6) are crucial in this last step. □

Remark 3.8. Consider f given by (3.5) with any V satisfying (A1) and g as in Example 3.6.

(i) For $0 < \sigma < 2$ and $0 \leq b \leq 2 - \sigma$ all the hypotheses of Theorem 3.7 are met. This case is important for waveguides since the Kerr nonlinearity is $\sigma = 1$.

(ii) If $\sigma = 2$, (A6) holds and the V-K condition is satisfied everywhere but the condition (A4) ensuring global existence for the Cauchy problem fails. We may have blow up of solutions in finite time.

(iii) If $\sigma > 2$, and for $b = 0$, it was shown in [13] that $\int_{\mathbb{R}} u(\lambda)^2 \, \mathrm{d}x \to 0$ as $\lambda \to \infty$. Hence, (3.6) cannot be positive everywhere.

3.3. The null linear potential case.

Let us now turn to the case where the hypothesis (f1-b) holds and the nonlinearity has the form (2.5). This amounts to setting $V \equiv 0$ and

$$g(x, s) = q(x) s^{(p-1)/2}, \quad p > 1, \tag{3.8}$$

in (3.5). In addition to the hypotheses of Theorem 2.4, we only need to suppose that

(q5) the function $x \dfrac{q'(x)}{q(x)}$ is non-increasing on $(0, \infty)$.

Remark 3.9. The function g in Example 3.6 is of the form (3.8) with

$$q(x) = (1 + x^2)^{-b/2}.$$

It is easy to verify that the hypotheses (q1)-(q5) are satisfied in this case.

Theorem 3.10. [7, Theorem 1.7] *Let f be given by (2.5), suppose that q satisfies (q1) to (q5) and that $1 < p < 5 - 2b$. Then the standing wave $\psi_\lambda(t,x) = \mathrm{e}^{i\lambda t}u(\lambda)(x)$ is orbitally stable for all $\lambda \in (0,\infty)$, where $u \in C^1\big((0,\infty), H\big)$ is given by Theorem 2.4.*

Proof. The proof is similar to that of Theorem 3.7. In [7], we were able to state sharp stability/instability conditions near $\lambda = 0$ (see [7, Theorem 1.5]), namely, for $\lambda > 0$ sufficiently small, the standing waves are stable if $p < 5 - 2b$ and unstable if $p > 5 - 2b$. The procedure outlined in Section 2, using the change of variables (2.6), yields very precise informations on the behaviour of $u(\lambda)$ as $\lambda \to 0$. We proved that, for $\lambda > 0$ small, (3.6) is positive if $p < 5 - 2b$ and negative if $p > 5 - 2b$. For $p < 5-2b$, a continuation argument similar to [13] then allowed us to show that (3.6) remains positive for all $\lambda \in (0,\infty)$. The spectral conditions are handled in the same way as in [22]. They hold for any $p > 1$, and only require hypotheses (q1)-(q4). □

Remark 3.11. In the context of (3.8), the condition for global well-posedness of the initial value problem is $p < 5$. Hence, for $\lambda > 0$ small enough, it follows by Theorem 1.5 of [7] that instability occurs for $5-2b < p < 5$ even though the solutions of (NLS) are global in time. It would be interesting to further investigate the nature of instability in this regime. Since the V-K condition is violated, we expect to have some form of *focusing instability*, as described in [17, Section IV].

Remark 3.12. We are not aware of any global stability result for other nonlinearities in the case of bifurcation from the essential spectrum. Our proof relies on the homogeneity of (2.5). However, local results are available for more general nonlinearities. See for instance [6], where perturbations of the case (2.5) are considered (the results there are established for $x \in \mathbb{R}^N$ with $N \geq 3$ but they also hold if $N = 1$.)

4. THE ASYMPTOTICALLY LINEAR CASE

We suppose that the nonlinearity f behaves like in hypothesis (L2) of Section 2. In this case, Theorem 2.1 shows that asymptotic bifurcation occurs at the point $\lambda^* < \infty$, in the sense that $\|u(\lambda)\| \to \infty$ as $\lambda \to \lambda^*$. The proof of Theorem 2.4 does not allow for a similar result. The method based on the scaling (2.6) strongly relies upon the homogeneity of the power law nonlinearity (2.5). It is natural to ask whether a global branch of solutions bifurcating from the bottom of the essential spectrum of the linearized operator exists in the asymptotically linear case. We only have partial answers so far. Theorems 2.1 and 2.4 were both obtained by first starting the branch off the line of trivial solutions and then using a continuation argument. Our approach here is reversed. We will show that, under appropriate assumptions, a global branch of solutions can be obtained, "bifurcating from infinity". The method is topological in nature and does not yield as much regularity as in Theorems 2.1 and 2.4. Moreover, the continuation of the branch down to the line of trivial solutions is still work in progress.

We suppose that f has the form

$$f(x,s) = g(x,s^2)s, \tag{4.1}$$

where $g(x,s)$ is an even function of $x \in \mathbb{R}$, for all $s \geq 0$. Under the hypotheses below, it is well known that seeking positive even solutions decaying at infinity to the stationary equation (SNLS) is then equivalent to solving

$$\begin{cases} u''(x) + g(x,u(x)^2)u(x) = \lambda u(x) & \text{for } x > 0, \\ u(x) > 0 & \text{for } x \geq 0, \\ u'(0) = \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{N}$$

A natural space to tackle this problem is

$$X := \{u \in H^2(0,\infty) : u'(0) = 0\} \quad \text{with} \quad \|\cdot\|_X := \|\cdot\|_{H^2}.$$

We suppose that there exists $g_\infty \in C(\mathbb{R}_+)$ such that $g(x,s) \to g_\infty(x)$ as $s \to \infty$, for all $x \geq 0$. Rewriting the equation as

$$u'' + g_\infty(x)u + h(x,u)u = \lambda u \quad \text{with} \quad h(x,u) := g(x,u^2) - g_\infty(x)$$

and performing the inversion $u \mapsto v := u/\|u\|_X^2$, we get

$$v'' + g_\infty(x)v + h(x, v/\|v\|_X^2)v = \lambda v. \tag{4.2}$$

The following linear problem plays a crucial role in our analysis:

$$v'' + g_\infty(x)v = \lambda v. \tag{4.3}$$

The strategy is then to get bifurcation from the principal eigenvalue $\lambda_\infty > 0$ of (4.3), under the assumption that $g_\infty(0) > g_\infty(\infty)$. Bifurcation of positive solutions from $(\lambda_\infty, 0) \in \mathbb{R} \times X$ for (4.2) then leads to asymptotic bifurcation for (N). This approach to asymptotic bifurcation goes back to Rabinowitz [14] and Toland [28] in 1973, where they use Rabinowitz's global bifurcation theory based on the Leray-Schauder degree for compact operators. In the same spirit, we get an unbounded connected set of solutions (λ, u) of (N), with $\|u\| \to \infty$ as $\lambda \to \lambda_\infty$. However, the present situation is more difficult for the equation is set on an unbounded interval, which has to major consequences. First, the problem is not asymptotically linear in the strict sense (the inverted problem (4.2) is not Fréchet differentiable at $v = 0$) and we need to use a truncation procedure. Moreover, since the truncated problem is also defined on the half-line, we cannot use standard global bifurcation theory and we have to appeal to a recent theorem of Stuart and Zhou [23, Theorem A.1] based on a topological degree for compact perturbations of C^1 Fredholm maps.

The precise hypotheses we make on g are the following.

(g1) $g \in C(\mathbb{R}_+^2)$ and $g(x, \cdot) \in C^1(0, \infty)$ for all $x \geq 0$.

(g2) There exists $M \in \mathbb{R}$ such that $0 < g(x, s) \leq M < \infty$ for all $(x, s) \in \mathbb{R}_+^2$.

(g3) For each fixed $s \geq 0$, $g(x, s)$ is non-increasing in $x \geq 0$ and strictly decreasing at one point $x_0 > 0$.

(g4) For each fixed $x \geq 0$, $g(x, s)$ is strictly increasing in $s \geq 0$.

(g5) There is $g_\infty \in C(\mathbb{R}_+)$ such that $g(x, s) \to g_\infty(x)$ as $s \to \infty$ uniformly in $x \geq 0$, with

$$0 < g_\infty(\infty) < g_\infty(0).$$

(g6) There is $g_0 \in \mathbb{R}$ such that $g(x, 0) \equiv g_0$ and $g(x, s) \to g_0$ as $s \to 0$ uniformly in $x \geq 0$, with

$$0 \leq g_0 < g_\infty(\infty).$$

(g7) $\partial_2 g(\cdot, s) \in L^\infty(0, \infty)$ for all $s \geq 0$ and $\{\partial_2 g(x, \cdot)\}_{x \geq 0}$ is equicontinuous on $(0, \infty)$.

Remark 4.1. Setting $\lambda_0 := g_\infty(\infty) > 0$, (g5) implies that the principal eigenvalue λ_∞ of (4.3) satisfies $\lambda_\infty > \lambda_0$.

Example 4.2. The function

$$g(x, s) = G(x) \frac{s}{1 + s}$$

satisfies (g1)-(g7) provided $G \in C(\mathbb{R}_+)$ is non-increasing with $G(0) > G(\infty) > 0$. It appears in the modelling of waveguides with photorefractive materials (see [5]).

Theorem 4.3. *Under hypotheses (g1)-(g7), there exists a connected set* $\Sigma \subset \mathbb{R} \times X$ *with the following properties:*

(i) (λ, u) *is a solution of (N) with* $\lambda \in (\lambda_0, \lambda_\infty)$ *if and only if* $(\lambda, u) \in \Sigma$.

(ii) $\inf \mathrm{P}\,\Sigma = \lambda_0$ *and* $\sup \mathrm{P}\,\Sigma = \lambda_\infty$ *where* $\mathrm{P}\,(\lambda, u) := \lambda$ *for all* $(\lambda, u) \in \mathbb{R} \times X$.

(iii) For any $\lambda \in (\lambda_0, \lambda_\infty)$ *there is a unique solution* (λ, u_λ) *to (N) and so*

$$\Sigma = \{(\lambda, u_\lambda) \text{ solution of (N)} : \lambda_0 < \lambda < \lambda_\infty\}.$$

Furthermore, Σ *is a continuous curve in* $\mathbb{R} \times X$.

(iv) Σ *is bounded away from the line of trivial solutions in* $\mathbb{R} \times X$.

(v) If $\{(\lambda_n, u_n)\} \subset \Sigma$ *with* $\lambda_n \to \lambda > \lambda_0$ *we have*

$$\lim_{n\to\infty} |u_n|_{L^\infty} = \lim_{n\to\infty} |u_n|_{L^2} = \lim_{n\to\infty} \|u_n\|_X = \infty \quad \text{if and only if} \quad \lambda = \lambda_\infty.$$

Remark 4.4. Our formulation of Theorem 4.3 is slightly redundant, reminiscent of our method of proof. Note that we get a continuous curve of solutions and not only a connected set, as is usually provided by the topological approach to bifurcation. This extra regularity follows by a uniqueness result for positive solutions of ordinary differential equations of the type (N) due to Toland [30], which is a consequence of the monotonicity properties of the nonlinear term.

Proof. The full proof of Theorem 4.3 will appear in [9], we briefly sketch it here. We first formulate (4.2) as an operator equation by defining $L(\lambda), H : X \to Y$,

$$L(\lambda)v = v'' + g_\infty(x)v - \lambda v \quad \text{and} \quad H(v) = h(x, v/\|v\|_X^2)v \quad \text{for } v \neq 0, \quad H(0) = 0,$$

for all $\lambda \in \mathbb{R}$. Then (4.2) is equivalent to

$$L(\lambda)v + H(v) = 0. \tag{4.4}$$

The global bifurcation theorem of Stuart and Zhou, Theorem A.1 in [23], applies to equations of this form where $L(\lambda)$ is a C^1 family of Fredholm operators of index zero satisfying a transversality condition, and H is compact with $H'(0) = 0$. The hypotheses on $L(\lambda)$ are easily verified, for $\lambda > \lambda_0$. However, H is neither compact, nor differentiable at $v = 0$. To handle these difficulties, we truncate the problem as

$$L(\lambda)v + H_n(v) = 0, \tag{4.5}$$

where $H_n : X \to Y$ is defined by

$$H_n(v)(x) = \chi_{[0,n]}(x)H(v)(x) \quad \text{for all } v \in X,\ n \in \mathbb{N}. \tag{4.6}$$

Here, $\chi_{[0,n]}$ denotes the characteristic function of the interval $[0, n]$, for all $n \in \mathbb{N}$. Some work is then required to show that the compactness and zero derivative assumptions are satisfied by H_n. The global bifurcation theorem can hence be applied to the approximate problem (4.5), yielding for each $n \in \mathbb{N}$ a connected set $\mathcal{C}_n$ of positive solutions to (4.5) bifurcating from the point $(\lambda_\infty, 0) \in \mathbb{R} \times X$. Furthermore, by *a priori* estimates, the collection $\{\mathcal{C}_n\}$ is uniformly bounded in $\mathbb{R} \times X$ and $\inf \mathrm{P}\,\mathcal{C}_n$ can be made as close to λ_0 as desired by choosing n large enough. A fairly standard limiting procedure then yields the existence of a connected set $\mathcal{C}$ of solutions to (4.2), bifurcating from $(\lambda_\infty, 0)$, bounded in $\mathbb{R} \times X$ with $\inf \mathrm{P}\,\mathcal{C} = \lambda_0$ and $\sup \mathrm{P}\,\mathcal{C} = \lambda_\infty$. Furthermore, the only point where $\mathcal{C}$ approaches the line of trivial solutions is $(\lambda_\infty, 0)$. By the preceding constructions, the set Σ in Theorem 4.3 and properties (i), (ii) and (iv) are readily obtained by inversion, on setting

$$\Sigma = \left\{ \left(\lambda, \frac{v}{\|v\|_X^2} \right) : (\lambda, v) \in \mathcal{C} \setminus \{(\lambda_\infty, 0)\} \right\}.$$

Finally, the continuous parametrization in (iii) follows by a uniqueness result for (N), and (v) from the properties of $\mathcal{C}$ and elementary manipulations of (N). □

Remark 4.5.

(i) In terms of waveguides, the limiting value $\lambda = \lambda_\infty$ corresponds to the saturation value of the wavenumber as both the power of the beam ($\sim |u|_{L^2}^2$) and the intensity of the applied electric field ($\sim |u|_{L^\infty}$) become large. See [7, Section 6] for more details.

(ii) To study the behaviour of (N) for small solutions, it is convenient to write the equation as

$$u'' + [g(x,u^2) - g_0]u = [\lambda - g_0]u. \tag{4.7}$$

The linearization of (4.7) at $u = 0$ is

$$u'' = [\lambda - g_0]u. \tag{4.8}$$

We expect that the branch of solutions given by Theorem 4.3 can be extended to meet the line of trivial solutions at the point $(g_0, 0) \in \mathbb{R} \times X$. However, the range $\lambda \in (g_0, \lambda_0]$ is not accessible by our proof of Theorem 4.3. We are currently studying it by different methods and the desired continuation should soon be available.

(iii) By even extension to $\mathbb{R}$, solutions of (N) become solutions of the stationary equation (SNLS), providing standing waves of (NLS). To carry out a stability analysis as in Section 3, one still has to show that the branch of solutions is in fact a C^1 curve.

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