Communications in Applied Analysis **15** (2011), no. 2, 3 and 4, 413–434

SOME RECENT RESULTS ON THE SPECTRUM OF MULTI-POINT EIGENVALUE PROBLEMS FOR THE p-LAPLACIAN

FRANÇOIS GENOUD[1] AND BRYAN P. RYNNE[2]

[1,2]Department of Mathematics, Heriot-Watt University
Edinburgh, EH14 4AS, Scotland
E-mail: F.Genoud@hw.ac.uk
E-mail: B.P.Rynne@hw.ac.uk

It is our pleasure to dedicate this paper to Professor Jeff Webb.

ABSTRACT. In this paper we describe some recent results regarding the spectral properties of p-Laplacian problems subject to various forms of multi-point boundary conditions. In particular, we consider Dirichlet and Neumann-type boundary conditions, and mixtures of these conditions. We also consider certain types of nonlocal boundary conditions expressed in terms of Stieltjes integrals, which have been discussed recently and which generalize the previously considered multi-point conditions.

It is shown that, under suitable assumptions, the structure of the spectrum and the properties of the eigenfunctions for these boundary value problems with nonlocal boundary conditions are very similar to those of the classical linear Sturm-Liouville problem with separated boundary conditions. Results on global bifurcation, non-resonance conditions and existence of nodal solutions for related nonlinear problems are also presented.

In a final section we discuss the necessity of some of the hypotheses we impose, and outline some open problems.

AMS (MOS) Subject Classification. 34B10, 34B08, 34L05, 47J15, 47N20.

1. INTRODUCTION

For $p > 1$, define $\phi_p(s) := |s|^{p-1}\operatorname{sgn} s$, $s \in \mathbb{R}$. In this paper we consider the nonlinear eigenvalue problem consisting of the equation

$$-\phi_p(u')' = \lambda\phi_p(u) \quad \text{on } (-1,1), \tag{1.1}$$

with $\lambda \in \mathbb{R}$, together with the multi-point boundary conditions

$$u(\pm 1) = \sum_{i=1}^{m^\pm} \alpha_i^\pm u(\eta_i^\pm), \tag{1.2}$$

Received October 28, 2010

or

$$\phi_p(u'(\pm 1)) = \sum_{i=1}^{m^\pm} \alpha_i^\pm \phi_p(u'(\eta_i^\pm)), \tag{1.3}$$

where $m^\pm \geq 1$ are integers, and $\eta_i^\pm \in [-1,1]$, $1 \leq i \leq m^\pm$ (with $\eta_i^+ \neq 1$, $\eta_i^- \neq -1$). We also consider mixed pairs of these conditions, with one condition holding at each of the end points $x = \pm 1$ (for convenience, we say that the conditions (1.2) or (1.3) hold at $x = \pm 1$, even though they are of course nonlocal).

We must also impose conditions on the coefficients $\alpha_i^\pm$, $i = 1, \ldots, m^\pm$, in (1.2) and (1.3). To discuss these conditions efficiently it will be convenient to introduce some further notation. For any integer $m \geq 1$, let $\mathcal{A}^m$ denote the set of $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ satisfying

$$\sum_{i=1}^{m} |\alpha_i| < 1. \tag{1.4}$$

The notation $\alpha = 0$, $\alpha > 0$, will mean $\alpha_i = 0$, $\alpha_i > 0$, $i = 1, \ldots, m$, respectively. We will assume throughout the paper that the coefficients $\alpha^\pm = (\alpha_1^\pm, \ldots, \alpha_{m^\pm}^\pm)$ in (1.2) and (1.3) satisfy

$$\alpha^\pm \in \mathcal{A}^{m^\pm}. \tag{1.5}$$

In Section 7 we will present some counterexamples which show that our main result on the spectral properties of the above problem may fail when this assumption does not hold. When $\alpha^\pm = 0$ the boundary conditions (1.2) and (1.3) reduce to the standard, separated Dirichlet and Neumann boundary conditions at $x = \pm 1$. Thus, in the general case with $\alpha^\pm \neq 0$ we will term (1.2) and (1.3) *Dirichlet-type* and *Neumann-type* boundary conditions respectively.

Given a suitable pair of boundary conditions, a number λ is an *eigenvalue* of the corresponding boundary value problem if there exists a non-trivial solution u, which is called an *eigenfunction*, and the *spectrum* is the set of eigenvalues. For the separated conditions (the case $\alpha^\pm = 0$) the spectral properties of the problem are well known, both in the linear Sturm-Liouville case ($p = 2$, see [3]) and in the p-Laplacian case ($p \neq 2$, see [1]). However, until recently, the basic spectral properties of multi-point problems had not been obtained. In this paper we describe some recent results regarding these properties. The case $\alpha^\pm = 0$ will play a central role in the analysis, with the results for the case of general $\alpha^\pm \in \mathcal{A}^{m^\pm}$ being obtained by continuation from $\alpha^\pm = 0$. The conditions (1.5) will play a crucial role in ensuring that the continuation procedure works.

Boundary value problems with multi-point boundary conditions have been extensively studied recently, see for example, [2, 5, 7, 9, 11, 14, 15, 17, 21, 22, 23], and the references therein. Many of these papers consider the problem on the interval $(0,1)$, and impose a single point (Dirichlet or Neumann) boundary condition at $x = 0$, and a multi-point boundary condition at $x = 1$. In our notation this corresponds to $\alpha^- = 0$, and we have used the interval $(-1, 1)$ in order to simplify the notation for the multi-point boundary conditions. Problems with multi-point boundary conditions at both end-points are considered in [9, 14] (and many references therein — the bibliography in [14] is particularly extensive).

The linear ($p = 2$) Dirichlet-type problem (1.1), (1.2), with $\alpha^- = 0$, $\alpha^+ > 0$ was recently considered in [23], where it was shown that the eigenvalues have algebraic multiplicity 1. However, other standard spectral properties, such as the nodal properties of the eigenfunctions, were not obtained in [23]. The Dirichlet-type problem with $\alpha^- = 0$, $\alpha^+ > 0$ was also considered in [17] (with $p = 2$) and in [5] (with $p \neq 2$). These papers set up a suitable operator formulation of the problem, showed that this operator is invertible, and derived various spectral and degree-theoretic results (including nodal properties of the eigenfunctions). The results obtained are similar to the standard spectral theory of the linear Sturm-Liouville problem, although a slightly different method of counting the nodes/oscillation of the eigenfunctions is required in order to deal with the multi-point boundary conditions. The paper [18] then considered the full multi-point, Dirichlet-type problem (1.1), (1.2) (with $\alpha^{\pm} \neq 0$), and all the results of [5] and [17] were extended to this case. Next, the full multi-point, Neumann-type problem (1.1), (1.3), and the mixed case were considered in [19]. Again, similar results to those of [18] were obtained, although with some significant differences. Specifically, the Neumann-type problem has non-trivial solutions of the form $\lambda = 0$ and u constant, so in this case the associated Neumann-type p-Laplacian operator is not invertible, and the problem is said to be *resonant*. For this operator a 'pseudo-inverse' was constructed in [19] which, after a suitable reformulation of the problem, enabled the discussion to proceed in a similar manner to that of [18]. The mixed boundary condition problem is nonresonant, but required a more complicated nodal count method than either the Dirichlet-type or Neumann-type conditions. We will outline these nodal-counting methods below (see Section 2.4 and Theorem 4.1), but there is a much more detailed discussion and motivation in [19].

1.1. More general, nonlocal boundary conditions.

Non-local boundary conditions more general than the above multi-point conditions have also been considered recently by several authors in various contexts, see for example [21] and the references therein. For instance, the Dirichlet-type conditions (1.2) can be replaced by an integral condition of the form

$$u(\pm 1) = \int_{-1}^{1} u(y)\, d\mu_{A^{\pm}}(y), \tag{1.6}$$

where $A^{\pm}$ are functions of bounded variations and the corresponding measures $\mu_{A^{\pm}}$ satisfy

$$\int_{-1}^{1} d|\mu_{A^{\pm}}| < 1. \tag{1.7}$$

Here, the right-hand side in (1.6) is a Lebesgue-Stieltjes integral with respect to the signed measure $\mu_{A^\pm}$ generated by $A^\pm$, and in (1.7) the term $|\mu_{A^\pm}|$ denotes the total variation of $\mu_{A^\pm}$ (we refer the reader to [10, Section 19] and [12, Section 36] for the required measure and integration theory).

By choosing $A^\pm$ to be suitable step functions we see that the Dirichlet-type boundary conditions (1.2) can be regarded as a special case of the condition (1.6). Also, it is clear that the condition (1.7) generalizes (1.4). The Neumann-type boundary conditions can also be generalized to an integral formulation in a similar manner.

All the results described below (for either Dirichlet, Neumann or mixed-type conditions) can be extended to deal with such integral formulations of these boundary conditions. To show this we will present a proof of our main result, Theorem 4.1, which, although it is based on the proof given in [18] for the Dirichlet-type conditions (1.2), has been rearranged and rewritten so that it can more readily be extended to the conditions (1.6). Naturally, this proof for the conditions (1.2) avoids any measure theoretic complications associated with the conditions (1.6). Then, in Section 6 we will describe the measure theoretic details required to extend the basic proofs (of Theorem 4.1 and other results) to deal with (1.6).

1.2. Plan of the paper.

The paper is organized as follows. In Section 2 we set out certain preliminary material. In particular:

(a) we summarize some elementary properties of the p-Laplacian sine function $\sin_p$, which will play an important role in the analysis;
(b) we introduce various function spaces, using which we then define a 'multi-point, p-Laplacian operator' and state its main properties;
(c) we define certain function spaces S_k and T_k, which we will use to describe the different kinds of nodal properties we encounter.

In Section 3 we prove an existence and uniqueness result for a problem consisting of equation (1.1) together with a single, multi-point, boundary condition. This problem could be regarded as a multi-point analogue of the usual initial value problem for equation (1.1). As usual, the uniqueness result for this 'multi-point, initial value problem' then implies the simplicity of the eigenvalues (for the eigenvalue problem having a pair of boundary conditions). Section 4 is devoted to the main results about the spectral properties of the p-Laplacian operator. In Section 5 we define an associated nonlinear operator and state the value of its topological degree. By following the proof of the well-known global bifurcation theorem of Rabinowitz, this degree can be used to prove a global bifurcation theorem — this theorem will be stated, together with some results on existence of nodal solutions which can be proved using the global bifurcation result. There are numerous other applications of spectral properties to general nonlinear problems which are sufficiently well-known to need no further remarks here. Then, in Section 6 we describe the measure theoretic details required to deal with the integral boundary conditions. Finally, in Section 7 we describe some counterexamples to show the necessity of some of our hypotheses and constructions, and we also discuss some open problems.

2. PRELIMINARIES

2.1. The function $\sin_p$.

By definition, the function $\sin_p$ satisfies

$$-\phi_p(\sin_p'(x))' = (p-1)\phi_p(\sin_p(x)), \quad x \in \mathbb{R}, \tag{2.1}$$

$$\sin_p(0) = 0, \quad \sin_p'(0) = 1 \tag{2.2}$$

(this initial value problem has a unique solution, see [1, Lemma 3.1]). It is shown in [11] that $\sin_p$ is a $2\pi_p$-periodic, C^1 function on $\mathbb{R}$, where $\pi_p := 2(\pi/p)/\sin(\pi/p)$, and $\sin_p(n\pi_p) = 0$, $\sin_p'((n+\frac{1}{2})\pi_p) = 0$, $n \in \mathbb{Z}$. Moreover, for any $x \in \mathbb{R}$,

$$\sin_p(x+\pi_p) = -\sin_p(x), \tag{2.3}$$

$$|\sin_p x|^p + |\sin_p' x|^p = 1. \tag{2.4}$$

Thus the graph of $\sin_p$ resembles a sine wave, and indeed, $\sin_2$ is the usual sin function, with $\pi_2 = \pi$.

Remark 2.1. The notations $\sin_p$, π_p have been used in various senses. The ones used here are taken from [11].

2.2. Notation and function spaces.

We let $\mathcal{A} := \mathcal{A}^{m^-} \times \mathcal{A}^{m^+}$ and write $\boldsymbol{\alpha} := (\alpha^-, \alpha^+) \in \mathcal{A}$ and $\boldsymbol{\eta} := (\eta^-, \eta^+) \in \mathcal{E}$, where $\mathcal{E}$ denotes the cube in which the point $\boldsymbol{\eta}$ may lie, as described in the introduction. We also let $\mathbf{0} := (0,0)$. In most of the paper we will regard $\boldsymbol{\alpha}$, $\boldsymbol{\eta}$, p, as constant, and omit them from the notation. However, in some of the discussion it will be convenient to regard some, or all, of these as variable, and we will then indicate the dependence on these variables in the obvious manner. For example, eigenvalues will normally be denoted λ_k, but will occasionally be regarded as depending on, say, $\boldsymbol{\alpha}$, and we then write $\lambda_k(\boldsymbol{\alpha})$.

For any integer $n \geq 0$, let $C^n[-1,1]$ be the usual Banach space of real-valued, n-times continuously differentiable functions on $[-1,1]$, with the usual sup-type norm, denoted by $|\cdot|_n$ (all function spaces in this paper will be real). In order to define a suitable function space in which to search for solutions of (1.1), together with the boundary conditions, we denote by B.C. either the boundary conditions (1.2) or (1.3), or a mixture of these, and define the Banach space

$$X := \{u \in C^1[-1,1] : \phi_p(u') \in C^1[-1,1] \text{ and } u \text{ satisfies B.C.}\},$$
$$\|u\|_X := |u|_1 + |\phi_p(u')|_1, \quad u \in X.$$

When necessary, we shall distinguish between these cases by writing X_D, X_N, or X_M when B.C. stands for (1.2) or (1.3) or a mixture of these, respectively; when no confusion is possible, or for statements holding in all cases, we simply write X. We also let $Y := C^0[-1,1]$, with the norm $\|\cdot\|_Y := |\cdot|_0$.

For any continuous function $f : \mathbb{R} \to \mathbb{R}$ we use the notation $f : Y \to Y$ to denote the corresponding Nemitskii operator defined by $f(u)(x) := f(u(x))$, $x \in [-1,1]$, for $u \in Y$. The operator $f : Y \to Y$ is bounded and continuous. In addition, it can easily be seen that the operator $\phi_p : Y \to Y$ is invertible, with inverse $\phi_p^{-1} = \phi_{p^*}$, where $p^* := p/(p-1) > 1$.

2.3. Definition and invertibility of the multi-point, p-Laplacian operator.

We define $\Delta_p : X \to Y$ by

$$\Delta_p(u) := \phi_p(u')', \quad u \in X.$$

By the definition of the spaces X, Y, the operator Δ_p is well-defined and continuous. The following result is proved in [18, Theorem 3.1], and shows that with Dirichlet-type boundary conditions (and certain additional conditions) the operator $\Delta_p : X_D \to Y$ has a continuous inverse.

Theorem 2.2. *Suppose that one of the following conditions holds:*
(a) $\alpha^- \geq 0$ *or* $\alpha^+ \geq 0$; *(b)* $\max\{\eta_i^-\} \leq \min\{\eta_i^+\}$; *(c)* $p = 2$.
Then the operator $\Delta_p : X_D \to Y$ *is bijective, and the inverse operator* $\Delta_p^{-1} : Y \to X_D$ *is continuous. Also,* $\Delta_p^{-1} : Y \to C^1[-1,1]$ *is completely continuous.*

For the mixed case the following result is proved in [19, Theorem 9.1].

Theorem 2.3. *The operator* $\Delta_p : X_M \to Y$ *is bijective, and the inverse operator* $\Delta_p^{-1} : Y \to X_M$ *is continuous. Also,* $\Delta_p^{-1} : Y \to C^1[-1,1]$ *is completely continuous.*

Remark 2.4. Unlike Theorem 2.2, no additional conditions are required in Theorem 2.3 for the invertibility of Δ_p in the mixed case; it is not clear if the additional conditions used in Theorem 2.2 in the Dirichlet-type case are actually necessary.

Remark 2.5. In the case of Neumann-type conditions the operator $\Delta_p : X_N \to Y$ is not invertible, since $\Delta_p(u_c) = 0$ for constant functions u_c. However, it is possible to construct closed, codimension-1 subspaces $\overline{X} \subset X_N$ and $\overline{Y} \subset Y$ such that the restriction $\Delta_p|_{\overline{X}} : \overline{X} \to \overline{Y}$ is invertible (roughly speaking, the idea is to take the quotients of X and Y with respect to the space of constant functions). It is then possible to define a 'pseudoinverse' for $\Delta_p : X_N \to Y$. This procedure is described in detail in [19, Section 3].

2.4. Nodal properties and function spaces.

In order to describe the nodal/oscillation properties of the eigenfunctions precisely, we now define certain sets of functions. These sets will play a crucial role in the investigation of the properties of the spectrum.

For any C^1 function u, if $u(x_0) = 0$ then x_0 is a *simple* zero of u if $u'(x_0) \neq 0$. For any integer $k \geq 0$ and any $\nu \in \{\pm\}$, we define the following sets.

S_k^ν is the set of $u \in X_N$ satisfying:

S-(a) $u(\pm 1) \neq 0$ and $\nu u(-1) > 0$;
S-(b) u has only simple zeros in $(-1, 1)$, and has exactly k such zeros.

T_k^ν is the set of $u \in X_D$ satisfying:

T-(a) $u'(\pm 1) \neq 0$ and $\nu u'(-1) > 0$;
T-(b) $\phi_p(u')$ has only simple zeros in $(-1, 1)$, and has exactly k such zeros;
T-(c) u has a zero strictly between each consecutive zero of u'.

We also define $S_k := S_k^+ \cup S_k^-$, $T_k := T_k^+ \cup T_k^-$.

Remark 2.6. (a) S_k, T_k, $k \geq 0$, are open, disjoint sets in X_N, X_D, respectively;
(b) if $u \in S_k$ then u has exactly k 'interior' or 'nodal' zeros in $(-1, 1)$;
(c) if $u \in T_k$ then u has at least $k - 1$ and at most k zeros in $(-1, 1)$; if we define a *bump* of u to be a zero of u', then by definition u has exactly k bumps.

Eigenfunction oscillation properties of Sturm-Liouville problems with general, separated boundary conditions are usually described using sets similar to the sets S_k, $k \geq 0$, which count interior (nodal) zeros, see [16, Section 2] for example. In the multi-point setting it was shown in [19] that the sets S_k are suitable for describing the oscillation properties in the case of Neumann-type conditions (see also Theorem 4.1 below, which states our main results on the properties of the spectrum). However, for Dirichlet-type conditions it is found that counting nodal zeros fails to adequately describe the oscillation properties, see Example 7.1 below (in the linear case $p = 2$). Instead, it is necessary to use the sets T_k in this case (see [17] and [18], and Theorem 4.1 below).

The mixed case is more complicated, and certain sets R_k, which count a mixture of nodes and bumps, were introduced in [19] to deal with this case — for brevity we will omit any further discussion of this case here. A detailed discussion of all these sets, and the reasons why they are suitable for the corresponding boundary value problems, is given in [19].

3. PROBLEMS WITH A SINGLE BOUNDARY CONDITION

In this section we consider the problem

$$-\phi_p(u')' = \lambda\phi_p(u) \quad \text{on } \mathbb{R}, \tag{3.1}$$

$$u(\eta_0) = \sum_{i=1}^{m} \alpha_i u(\eta_i), \tag{3.2}$$

with a single, multi-point, Dirichlet-type boundary condition. This problem could be regarded as a 'multi-point, initial value problem', and the following theorem is analogous to the usual ODE existence and uniqueness result for initial value problems (a similar result also holds for Neumann-type boundary conditions, see Theorem 4.1 in [19]).

Theorem 3.1. *For fixed* $\lambda > 0$, $m \geq 1$, $\alpha \in \mathcal{A}^m$, $\eta_0 \in \mathbb{R}$ *and* $\eta \in \mathbb{R}^m$, *there exists a function* $u(\lambda, \alpha, \eta_0, \eta) \in C^1(\mathbb{R})$ *such that any solution of* (3.1), (3.2), *has the form* $u = Cu(\lambda, \alpha, \eta_0, \eta)$, *for some* $C \in \mathbb{R}$.

Proof. Regarding λ as fixed, for any $\theta \in \mathbb{R}$ we define a function $w(\theta) \in C^1(\mathbb{R})$ by

$$s := (\lambda/(p-1))^{1/p}, \quad w(\theta)(x) := \sin_p(sx + \theta), \quad x \in \mathbb{R}. \tag{3.3}$$

The discussion in Section 2 shows that any solution of (3.1) must have the form $u = Cw(\theta)$, for suitable C, $\theta \in \mathbb{R}$. Also, the function $w(\theta)$ satisfies (3.2) iff

$$\Gamma(\theta, \alpha) := \sin_p(s\eta_0 + \theta) - \sum_{i=1}^{m} \alpha_i \sin_p(s\eta_i + \theta) = 0, \quad \theta \in \mathbb{R} \tag{3.4}$$

(it will be convenient to regard $\alpha \in \mathcal{A}^m$ as variable here). Thus, it suffices to consider the set of solutions of (3.4). The function $\Gamma : \mathbb{R} \times \mathcal{A}^m \to \mathbb{R}$ is C^1, and we will denote the partial derivative of Γ with respect to θ by Γ_θ.

Lemma 3.2. *If* $\Gamma(\theta, \alpha) = 0$, *for some* $(\theta, \alpha) \in \mathbb{R} \times \mathcal{A}^m$, *then* $\Gamma_\theta(\theta, \alpha) \neq 0$.

Proof. Suppose, on the contrary, that

$$\Gamma(\theta, \alpha) = \Gamma_\theta(\theta, \alpha) = 0, \tag{3.5}$$

for some $(\theta, \alpha) \in \mathbb{R} \times \mathcal{A}^m$. Regarding (θ, α) as fixed, and writing

$$S(\eta) := \sin_p(s\eta + \theta), \quad S^\dagger(\eta) := \sin_p'(s\eta + \theta), \quad \eta \in \mathbb{R},$$

the condition (3.5) becomes

$$S(\eta_0) = \sum_{i=1}^{m} \alpha_i S(\eta_i), \quad S^\dagger(\eta_0) = \sum_{i=1}^{m} \alpha_i S^\dagger(\eta_i). \tag{3.6}$$

Now, by (2.4) and (3.6),

$$
\begin{aligned}
1 &= \Big(\sum_{i=1}^{m} \alpha_i S(\eta_i)\Big)\phi_p(S(\eta_0)) + \Big(\sum_{i=1}^{m} \alpha_i S^\dagger(\eta_i)\Big)\phi_p(S^\dagger(\eta_0)) \\
&\le \sum_{i=1}^{m} |\alpha_i|\Big(|S(\eta_i)||S(\eta_0)|^{p-1} + |S^\dagger(\eta_i)||S^\dagger(\eta_0)|^{p-1}\Big) \\
&\le \sum_{i=1}^{m} |\alpha_i|\Big(|S(\eta_i)|^p + |S^\dagger(\eta_i)|^p\Big)^{1/p}\Big(|S(\eta_0)|^p + |S^\dagger(\eta_0)|^p\Big)^{(p-1)/p} \\
&< 1,
\end{aligned}
$$

since α satisfies (1.4). This shows that (3.5) cannot hold. □

It follows immediately from the definition of Γ that:

(a) for any $\alpha \in \mathcal{A}^m$, the function $\Gamma(\cdot, \alpha)$ is π_p-antiperiodic;
(b) when $\alpha = 0$, $\Gamma(\cdot, 0)$ has exactly 1 zero in $[0, \pi_p)$ and this zero is simple.

Since $\mathcal{A}^m$ is connected, it now follows from Lemma 3.2, the implicit function theorem, and continuity, that property (b) holds for all $\alpha \in \mathcal{A}^m$. This proves the theorem. □

4. EIGENVALUES OF THE p-LAPLACIAN

We now consider the eigenvalue problem

$$
-\Delta_p(u) = \lambda\phi_p(u), \quad u \in X, \tag{4.1}
$$

for the operator $-\Delta_p : X \to Y$ with either $X = X_D$ or $X = X_N$. We say that λ is an *eigenvalue* of $-\Delta_p$ if (4.1) has a non-trivial solution u, which will be termed an *eigenfunction*. If λ is an eigenvalue of $-\Delta_p$, with corresponding eigenfunction u, then tu is also an eigenfunction for all non-zero $t \in \mathbb{R}$, and we say that λ is *simple* if every eigenfunction corresponding to λ is of this form.

Theorem 4.1.

(a) Dirichlet case. *The spectrum of $-\Delta_p$ consists of a sequence of simple eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$. For each $k = 1, 2, \dots$, λ_k has an eigenfunction $u_k \in T_k$.*

(b) Neumann case. *The spectrum of $-\Delta_p$ consists of a sequence of simple eigenvalues $0 = \lambda_0 < \lambda_1 < \dots$. For each $k = 0, 1, \dots$, λ_k has an eigenfunction $u_k \in S_k$.*

In each case, $\lim_{k\to\infty} \lambda_k = \infty$.

Remark 4.2. The result in the mixed case is the same as in the Dirichlet case, with the sets T_k replaced by the sets R_k defined in [19], see [19, Theorem 9.4].

Proof. We give the proof of the Dirichlet case, the proof of the Neumann case is quite similar (see [19, Section 5]). Let (λ, u) be a non-trivial solution of (4.1). We will state a sequence of results regarding λ and u, which culminate in the proof of the theorem.

Lemma 4.3. $|u(\pm 1)| < |u|_0$. *Hence,* $\max |u|$ *is attained in* $(-1, 1)$.

Proof. By (1.2) and (1.4),

$$|u(\pm 1)| \le \sum_{i=1}^{m^\pm} |\alpha_i^\pm||u(\eta_i^\pm)| \le |u|_0 \sum_{i=1}^{m^\pm} |\alpha_i^\pm| < |u|_0. \tag{4.2}$$

□

Corollary 4.4. $\lambda > 0$.

Proof. If $\lambda \le 0$ then it can be shown, using the differential equation (1.1), that $\max |u|$ must occur at an end point ± 1, which contradicts Lemma 4.3. □

By Corollary 4.4, any eigenvalue and eigenfunction of $-\Delta_p$ must have the form

$$\lambda = (p-1)s^p, \quad w(s,\theta)(x) := \sin_p(sx + \theta), \quad x \in [-1, 1], \tag{4.3}$$

for suitable $s > 0$, $\theta \in \mathbb{R}$ (up to a scaling of the eigenfunction). This, together with Lemma 4.3, also proves the following result.

Corollary 4.5. *If u is an eigenfunction of $-\Delta_p$ then $u'(-1)u'(1) \ne 0$, and $u \in T_k$, for some $k \ge 1$. Also, $u \notin \partial T_l$ for any $l \ge 1$.*

Substituting (4.3) into (1.2) shows that $\lambda = (p-1)s^p$ is an eigenvalue of $-\Delta_p(\boldsymbol{\alpha})$ (we will now regard $\boldsymbol{\alpha}$ as variable) if and only if the pair of equations

$$\Gamma^\pm(s, \theta, \boldsymbol{\alpha}) := \sin_p(\pm s + \theta) - \sum_{i=1}^{m^\pm} \alpha_i^\pm \sin_p(s\eta_i^\pm + \theta) = 0 \tag{4.4}$$

holds for some $\theta \in \mathbb{R}$. Thus, to prove Theorem 4.1 it suffices to consider the set of solutions of (4.4). In fact, the proof will be by continuation away from the separated boundary condition case, corresponding to $\boldsymbol{\alpha} = \mathbf{0}$, where the required information on the zeros of $\Gamma^\pm$ is obvious. Corollary 4.5 ensures the preservation of the nodal properties of the eigenfunctions during this process. For reference, we state the required results when $\boldsymbol{\alpha} = \mathbf{0}$ in the following lemma.

Lemma 4.6. *If $\boldsymbol{\alpha} = \mathbf{0}$ then the points*

$$s_k^0 = \theta_k^0 = \frac{k}{2}\pi_p, \qquad k = 1, 2, \dots,$$

satisfy (4.4). The spectrum of $-\Delta_p(\mathbf{0})$ is the set $\{\lambda_k^0 = (p-1)(s_k^0)^p : k \ge 1\}$. For each $k \ge 1$ the eigenfunction $w(s_k^0, \theta_k^0) \in T_k$.

Of course, by periodicity with respect to θ, there are other zeros of (4.4) than those given by Lemma 4.6, but these do not yield distinct solutions of the eigenvalue problem. In fact, to remove these additional zeros and to make the domain of θ, compact, from now on we will regard θ as lying in the circle obtained from the interval $[0, 2\pi_p]$ by identifying the points 0 and $2\pi_p$ (which we denote by S^1_p); we then regard the domains of the functions $\Gamma^{\pm}$ as $(0,\infty) \times S^1_p \times \mathcal{A}$. Also, it is clear that these functions are C^1. The following proposition provides some information on the signs of the partial derivatives Γ^ν_s, Γ^ν_θ at the zeros of Γ^ν.

Proposition 4.7. *For either* $\nu \in \{\pm\}$,

$$\Gamma^\nu(s,\theta,\boldsymbol{\alpha}) = 0 \implies \nu\Gamma^\nu_s(s,\theta,\boldsymbol{\alpha})\,\Gamma^\nu_\theta(s,\theta,\boldsymbol{\alpha}) > 0. \tag{4.5}$$

Proof. By a similar proof to that of Lemma 3.2 it can be shown that

$$\Gamma^\nu(s,\theta,\boldsymbol{\alpha}) = 0 \implies \Gamma^\nu_s(s,\theta,\boldsymbol{\alpha})\,\Gamma^\nu_\theta(s,\theta,\boldsymbol{\alpha}) \neq 0. \tag{4.6}$$

Now let $(s^0,\theta^0,\boldsymbol{\alpha}^0)$ be an arbitrary zero of Γ^ν, which we regard as fixed, and consider the equation

$$G^0(\theta,t) := \Gamma^\nu(s^0,\theta,t\boldsymbol{\alpha}^0) = 0, \quad (\theta,\ t) \in S^1_p \times [0,1]. \tag{4.7}$$

Clearly, by (4.6),

$$G^0(\theta^0,1) = 0, \quad \text{and} \quad G^0(\theta,t) = 0 \implies G^0_\theta(\theta,t) \neq 0, \tag{4.8}$$

so, by the implicit function theorem, we can solve (4.7) for θ as a function of t in a neighbourhood of $(\theta^0,1)$. Moreover, by (4.8) and the compactness of S^1_p, it can be shown that the domain of this solution function contains the interval $[0,1]$, that is, we have a C^1 solution function $t \to \theta(t) : [0,1] \to S^1_p$, such that

$$\theta(1) = \theta^0, \quad \Gamma^\nu(s^0,\theta(t),t\boldsymbol{\alpha}^0) = 0, \quad t \in [0,1]$$

(see the proof of part (b) of Lemma 4.8 below for a similar argument).

Next, by the definition of Γ^ν in (4.4), it is clear that the inequality (4.5) holds at $(s,\theta,\boldsymbol{\alpha}) = (s^0,\theta(0),\mathbf{0})$ and hence, by (4.6) and continuity, (4.5) holds at $(s^0,\theta(t),t\boldsymbol{\alpha}^0)$, $t \in [0,1]$. In particular, putting $t = 1$ shows that (4.5) holds at $(s^0,\theta^0,\boldsymbol{\alpha}^0)$, which completes the proof of Proposition 4.7. □

We now return to the pair of equations (4.4). To solve these using the implicit function theorem we define the determinant

$$D(s,\theta,\boldsymbol{\alpha}) := \begin{vmatrix} \Gamma^-_s(s,\theta,\boldsymbol{\alpha}) & \Gamma^-_\theta(s,\theta,\boldsymbol{\alpha}) \\ \Gamma^+_s(s,\theta,\boldsymbol{\alpha}) & \Gamma^+_\theta(s,\theta,\boldsymbol{\alpha}) \end{vmatrix}, \quad (s,\theta,\boldsymbol{\alpha}) \in (0,\infty) \times S^1_p \times \mathcal{A}.$$

It now follows from the sign properties of $\Gamma^\pm_s$, $\Gamma^\pm_\theta$ proved in Proposition 4.7 that

$$\Gamma^+(s,\theta,\boldsymbol{\alpha}) = \Gamma^-(s,\theta,\boldsymbol{\alpha}) = 0 \implies D(s,\theta,\boldsymbol{\alpha}) \neq 0, \tag{4.9}$$

and hence we can solve (4.4) for (s, θ), as functions of $\boldsymbol{\alpha}$, in a neighbourhood of an arbitrary solution of (4.4).

Now suppose that $(s^0, \theta^0, \boldsymbol{\alpha}^0) \in (0, \infty) \times S_p^1 \times \mathcal{A}$ is an arbitrary solution of (4.4). By (4.9) and the implicit function theorem, there exists a maximal open interval N^0 containing 1 and a C^1 solution function

$$t \to (s(t), \theta(t)) : N^0 \to (0, \infty) \times S_p^1,$$

such that

$$(s(1), \theta(1)) = (s^0, \theta^0), \quad \Gamma^{\pm}(s(t), \theta(t), t\boldsymbol{\alpha}^0) = 0, \quad t \in N^0.$$

Furthermore, by Corollary 4.5 and continuity, there exists an integer $k^0 \geq 1$ such that

$$u(t) := w(s(t), \theta(t)) \in T_{k^0}, \quad t \in N^0. \tag{4.10}$$

Lemma 4.8. (*a*) *There exists constants* C^0, $\delta^0 > 0$ *such that* $\delta^0 \leq s(t) \leq C^0$, $t \in N^0$; (*b*) $0 \in N^0$.

Proof. (a) From the form of $w(s, \theta)$, there exists $C^0 > 0$ such that, for any $\theta \in S_p^1$, if $s \geq C^0$ then $w(s, \theta) \notin T_{k^0}$, so by (4.10), $s(t) \leq C^0$, $t \in N^0$. Now suppose that there is a sequence $t_n \in N^0$, $n = 1, 2, \ldots$, with $s(t_n) \to 0$. By Lemma 4.3, for each $n \geq 1$, $|w(s(t_n), \theta(t_n))(x_n)| = 1$ at some $x_n \in (-1, 1)$, so we may suppose that $|w(s(t_n), \theta(t_n)) - \epsilon|_0 \to 0$, where $\epsilon = 1$ or $\epsilon = -1$. However, this contradicts (1.2) and $\boldsymbol{\alpha} \in \mathcal{A}$, so a suitable $\delta^0 > 0$ must also exist.

(b) Suppose that $0 \notin N^0$, and let $\bar{t} = \inf\{t \in N^0\}$. By part (a) of the lemma there exists a decreasing sequence $t_n \in N^0$, $n = 1, 2, \ldots$, and a point $(\bar{s}, \bar{\theta}) \in (0, \infty) \times S_p^1$, such that

$$\lim_{n\to\infty} t_n = \bar{t}, \quad \lim_{n\to\infty} (s(t_n), \theta(t_n)) = (\bar{s}, \bar{\theta}).$$

Clearly, the point $(\bar{s}, \bar{\theta}, \bar{t}\boldsymbol{\alpha})$ satisfies (4.4) so, by the above results, the solution function extends to an open neighbourhood of $\bar{t}$, which contradicts the choice of $\bar{t}$ and the maximality of the interval N^0. □

For any given $\boldsymbol{\alpha} \in \mathcal{A}$, the above arguments have shown that:

(a) any solution $(s, \theta, \boldsymbol{\alpha}) \in (0, \infty) \times S_p^1 \times \mathcal{A}$ of (4.4) can be continuously connected to exactly one of the solutions $\{(s_k^0, \theta_k^0, \mathbf{0}) : k \geq 1\}$.

Similar arguments show that:

(b) any of the solutions $\{(s_k^0, \theta_k^0, \mathbf{0}) : k \geq 1\}$ can be continuously connected to exactly one solution, say $(s_k(\boldsymbol{\alpha}), \theta_k(\boldsymbol{\alpha}), \boldsymbol{\alpha}) \in (0, \infty) \times S_p^1 \times \mathcal{A}$, of (4.4).

Furthermore, for each $k \geq 1$, the solution $(s_k(\boldsymbol{\alpha}), \theta_k(\boldsymbol{\alpha}), \boldsymbol{\alpha})$ yields a simple eigenvalue $\lambda_k(\boldsymbol{\alpha})$ of $-\Delta_p(\boldsymbol{\alpha})$, with corresponding eigenfunction $u_k(\boldsymbol{\alpha}) := w(s_k(\boldsymbol{\alpha}), \theta_k(\boldsymbol{\alpha})) \in T_k$, and there is no other eigenvalue $\widetilde{\lambda} \neq \lambda_k(\boldsymbol{\alpha})$ having an eigenfunction $\widetilde{u} \in T_k$.

Next, Lemma 4.6 shows that $s_k(\mathbf{0}) < s_{k+1}(\mathbf{0})$, and Theorem 3.1 shows that $s_k(\boldsymbol{\alpha}) \neq s_{k+1}(\boldsymbol{\alpha})$, for $\boldsymbol{\alpha} \in \mathcal{A}$, so it follows from the continuation construction that $s_k(\boldsymbol{\alpha}) < s_{k+1}(\boldsymbol{\alpha})$, for all $\boldsymbol{\alpha} \in \mathcal{A}$.

Finally, the fact that $u_k \in T_k$, together with the properties of $\sin_p$, shows that $\lim_{k\to\infty} \lambda_k = \infty$. This concludes the proof of Theorem 4.1. □

We end this section with the following result on continuous dependence of the eigenvalues on the parameters. This result is of interest in its own right, but is also crucial in, for example, the proof of Theorem 5.2 below.

Corollary 4.9. *For each $k \geq 1$ the eigenvalue λ_k depends continuously on the variables $(\boldsymbol{\alpha}, \boldsymbol{\eta}, p)$ in $\mathcal{A} \times \mathcal{E} \times (1, \infty)$.*

Proof. This follows from the implicit function theorem and the construction of the eigenvalues in the proof of Theorem 4.1, using the fact that the functions $\Gamma^{\pm}$ there are C^1 functions of the variables $(\boldsymbol{\alpha}, \boldsymbol{\eta})$. These functions are not C^1 functions of p, but equation (2.5) in [13] shows that they are continuous with respect to p; the continuous dependence of the eigenvalues on p then follows from the form of the implicit function theorem in [4, Theorem 15.1]. □

5. SOME BIFURCATION THEORY

In this section we discuss a nonlinear eigenvalue problem related to (4.1). We first study the topological degree of a nonlinear operator naturally associated to (4.1). Then we present a Rabinowitz-type global bifurcation result, a non-resonance condition and an existence result for nodal solutions.

Remark 5.1. Throughout this section we consider the Dirichlet case (so $X = X_D$). In the mixed case the results and the proofs are essentially identical (using the sets R_k defined in [19], instead of the sets T_k) — see [19, Section 9]. In the Neumann case the results are also essentially identical (using the sets S_k), but the proofs are more delicate since Δ_p is not invertible — see [19, Section 7].

5.1. The topological degree of a nonlinear operator.

We assume that the hypotheses of Theorem 2.2 hold, so that the inverse operator Δ_p^{-1} exists (this assumption is not necessary in the mixed case, see Remark 2.4). The eigenvalue problem (4.1) is then equivalent to the equation

$$u + K_\lambda(u) = 0, \quad u \in Y, \tag{5.1}$$

where $K_\lambda := \Delta_p^{-1} \circ (\lambda \phi_p) : Y \to Y$. In particular, (5.1) has a non-trivial solution u if and only if λ is an eigenvalue of the operator $-\Delta_p$. Furthermore, the operator K_λ is completely continuous (by Theorem 2.2), and homogeneous (in the sense that

$K_\lambda(tu) = tK_\lambda(u)$, for any $t \in \mathbb{R}$ and $u \in Y$). Thus, if λ is not an eigenvalue of $-\Delta_p$ then the Leray-Schauder degree $\deg(I + K_\lambda, B_r, 0)$ is well defined for any $r > 0$, where B_r denotes the open ball in Y centered at 0 with radius r, see [24, Chapter 13].

Theorem 5.2. *Suppose that the hypotheses of Theorem* 2.2 *hold. Then, for any* $r > 0$,

$$\deg(I + K_\lambda, B_r, 0) = \begin{cases} 1, & \text{if } \lambda < \lambda_1, \\ (-1)^k, & \text{if } \lambda \in (\lambda_k, \lambda_{k+1}),\ k \geq 1. \end{cases}$$

Proof. When $p = 2$ the operator K_λ is simply the linear operator $\lambda\Delta_2^{-1} : Y \to Y$. By [18, Lemma 5.13], all the characteristic values of $-\Delta_2^{-1}$ have algebraic multiplicity 1, so in this case the result follows from the Leray-Schauder index theorem (see, for example, [24, Proposition 14.5]). We can now prove the general result by continuation with respect to p, varying p from the known, linear case $p = 2$ to general $p \neq 2$ (this idea was first used in [8]). Hence, to explicitly indicate the dependence of the operator K_λ on p we write $K_{\lambda,p}$, and we denote the eigenvalues of $-\Delta_p$ by $\lambda_k(p)$, $k \geq 1$.

Now fix $k \geq 1$, $p > 2$ and $\lambda \in (\lambda_k(p), \lambda_{k+1}(p))$ (the cases $1 < p < 2$ and $\lambda < \lambda_1(p)$ are similar). By Theorem 4.1 and Corollary 4.9, we can choose a continuous function $\rho : [2, p] \to \mathbb{R}$ such that $\rho(p) = \lambda$ and

$$\lambda_k(\widetilde{p}) < \rho(\widetilde{p}) < \lambda_{k+1}(\widetilde{p}), \quad \widetilde{p} \in [2, p].$$

It now follows from [18, Lemma 3.5] that the homotopy

$$H(\widetilde{p}, u) := K_{\rho(\widetilde{p}),\widetilde{p}}(u) : [2, p] \times Y \to Y$$

is completely continuous. Furthermore, for each $\widetilde{p} \in [2, p]$ the equation $u + H(\widetilde{p}, u) = 0$ has no non-trivial solution u, since $\rho(\widetilde{p})$ is not an eigenvalue of $-\Delta_{\widetilde{p}}$. Hence the result follows from the homotopy invariance of the Leray-Schauder degree. □

5.2. A global bifurcation result.

We consider the bifurcation problem

$$-\Delta_p(u) = \lambda f(u), \quad (\lambda, u) \in \mathbb{R} \times X, \tag{5.2}$$

where the function $f : \mathbb{R} \to \mathbb{R}$ is continuous, satisfies $sf(s) > 0$ for $s \in \mathbb{R} \setminus \{0\}$, and

$$f_0 := \lim_{\xi \to 0} \frac{f(\xi)}{\phi_p(\xi)} > 0, \tag{5.3}$$

where we suppose that the limit exists and is finite. Clearly, $u \equiv 0$ is a solution of (5.2) for any $\lambda \in \mathbb{R}$; such solutions will be called *trivial*. Let $\mathcal{S} \subset \mathbb{R} \times X$ denote the set of non-trivial solutions (λ, u) of (5.2), and let $\overline{\mathcal{S}}$ denote the closure of $\mathcal{S}$ in $\mathbb{R} \times X$.

We now state various results on the set of non-trivial solutions of (5.2). These results, and their proofs, are formally identical to those in [5, Section 4] (which dealt with the case $\alpha^- = 0$, $\alpha^+ > 0$), so we simply state the results here. The following lemma ensures that nodal properties are preserved along connected components of $\mathcal{S}$ (see property (b) in Theorem 5.4 below).

Lemma 5.3. *If $(\lambda, u) \in \mathcal{S}$ then $\lambda > 0$ and $u \in T_k$ for some $k \geq 1$.*

We now state the following Rabinowitz-type global bifurcation result for the solution set of (5.2). The proof uses the conclusions of Theorems 2.2 and 5.2, so we assume that the hypotheses of Theorem 2.2 hold.

Theorem 5.4. *Suppose that the hypotheses of Theorem 2.2 hold. Then, for each $k \geq 1$ there exists closed, connected sets $\mathcal{C}_k^\pm \subset (0,\infty) \times X$ of solutions of (5.2) with the properties*:

(a) $(\lambda_k/f_0, 0) \in \mathcal{C}_k^\pm$;
(b) $\mathcal{C}_k^\pm \backslash \{(\lambda_k/f_0, 0)\} \subset (0,\infty) \times T_k^\pm$;
(c) $\mathcal{C}_k^\pm$ *is unbounded in* $(0,\infty) \times Y$.

5.3. A non-resonance condition and nodal solutions.

In this subsection we obtain solutions of the problem

$$-\Delta_p(u) = f(u) + h, \quad u \in X, \tag{5.4}$$

for arbitrary $h \in Y$, and also nodal solutions of the problem

$$-\Delta_p(u) = f(u), \quad u \in X. \tag{5.5}$$

Again, the proofs are formally identical to those of Theorems 5.1, 5.3 and 5.5 in [5], so we only state the results. We suppose that the following limit exists,

$$f_\infty := \lim_{|\xi|\to\infty} \frac{f(\xi)}{\phi_p(\xi)} \geq 0.$$

Theorem 5.5.

(i) *Suppose that $f_\infty < \infty$ and f_∞ is not an eigenvalue of $-\Delta_p$. Then*:

(a) *equation* (5.4) *has a solution $u \in X$ for any $h \in Y$*;
(b) *if $(\lambda_k - f_0)(\lambda_k - f_\infty) < 0$, for some $k \geq 1$, then* (5.5) *has solutions $u_k^\pm \in T_k^\pm$.*

(ii) *Suppose that $f_\infty = \infty$. If $\lambda_{k_0}/f_0 > 1$, for some $k_0 \geq 1$, then* (5.5) *has solutions $u_k^\pm \in T_k^\pm$, for all $k \geq k_0$.*

6. EXTENSION OF THE RESULTS TO THE CONDITIONS (1.6)

In this section we briefly describe how to extend the proofs of the above results to deal with the integral boundary conditions (1.6). A fundamental requirement (to continue solutions by applying the implicit function theorem as before) is that, for fixed $A^\pm$, the functions

$$(s,\theta) \to \widetilde{\Gamma}^\pm(s,\theta,A^\pm) := \sin_p(\pm s + \theta) - \int_{-1}^{1} \sin_p(sy+\theta)\, d\mu_{A^\pm}(y)$$

be of class $C^1((0,\infty)\times\mathbb{R})$ (the functions $\widetilde{\Gamma}^{\pm}$ are the obvious analogues of the functions $\Gamma^{\pm}$ used above, and serve the same role here). However, this follows readily from the mean-value theorem and dominated convergence. This observation is sufficient to follow the above proof of Theorem 4.1 in the current setting — most of the required arguments are obvious modifications of the previous arguments, using condition (1.7).

A slightly more involved matter is the continuous dependence of various objects on the boundary conditions in some of the other proofs. In particular, the proof of the degree-theoretic result in Theorem 5.2 relies on the fact that both the inverse operator Δ_p^{-1} and the eigenvalues λ_k depend continuously on the parameters $\alpha^{\pm}$ (see [18, Corollary 3.3] and Corollary 4.9 above, respectively). In the context of (1.6), the 'parameters' $A^{\pm} \in BV[-1,1]$, the space of functions of bounded variation on $[-1,1]$. This is a Banach space when endowed with the norm

$$\|A\|_{BV} := |A(-1)| + V_{-1}^1(A),$$

where $V_{-1}^1(A)$ is the total variation of A on $[-1,1]$. Assuming (1.7), the existence and the basic properties of the inverse operator $\Delta_p^{-1} : Y \to X_D$ can be obtained under analogous assumptions to those of Theorem 2.2. The continuous dependence of Δ_p^{-1} and of the eigenvalues λ_k upon $A^{\pm}$ is then proved similarly to [18, Corollary 3.3] and Corollary 4.9 above, by observing that, for any fixed $h \in C^0[-1,1]$, the linear mapping

$$BV[-1,1] \ni A \to \varphi(A) := \int_{-1}^{1} h \, d\mu_A$$

is continuous. Indeed, for any $A \in BV[-1,1]$, we have

$$|\varphi(A)| \le |h|_0 \left| \int_{-1}^{1} d\mu_A \right| \le |h|_0 V_{-1}^1(A) \le |h|_0 \|A\|_{BV},$$

where the second inequality is a consequence of the Riesz representation theorem as stated in [12, Section 36.6].

We also observe that, by using the above methods, the single multi-point boundary condition (3.2) can be replaced with an integral boundary condition, over some interval, and the obvious analogue of Theorem 3.1 can readily be proved.

7. SOME COUNTEREXAMPLES AND OPEN PROBLEMS

In this section we briefly describe some examples and some open problems concerning the spectral properties of the above multi-point problems. At the present stage of development of the theory there are many such open problems, so we only mention a small number directly related to the above results.

One obvious open question has already been mentioned in Remark 2.4 — that is, whether the additional conditions in Theorem 2.2 are actually necessary for the invertibility of the general Dirichlet-type operator Δ_p.

7.1. The condition (1.5) and the sets T_k, S_k — some counterexamples.

The spectral properties described in Theorem 4.1 are similar to those of the standard Sturm-Liouville problem with separated boundary conditions, see [3], except that for any separated boundary conditions the kth eigenfunction u_k has exactly $k-1$ zeros in $(0,1)$ — that is, the numbering of the eigenvalues can be characterized in terms of the number of nodal zeros of the corresponding eigenfunctions. On the other hand, the following simple 3-point example shows that in the case of Dirichlet-type multi-point boundary conditions this characterization in terms of nodal zeros need not be valid. Hence, this shows the necessity of counting bumps rather than zeros of the eigenfunctions in Theorem 4.1, in the Dirichlet-type case. In other words, it is necessary to use the sets T_k (which count bumps) rather than sets akin to S_k (which count zeros).

In the following examples we consider the linear ($p = 2$) problem

$$-u'' = \lambda u, \quad \text{on } (0,1),$$
$$u(0) = 0, \quad u(1) = \alpha u(\eta)$$

(to simplify the notation slightly we use the interval $(0,1)$, instead of $(-1,1)$; also we have $m^- = 0$ $m^+ = 1$, so we have omitted the subscript 1 and superscript + on the parameters α and η). Due to the Dirichlet condition at $x = 0$, any eigenfunction u must have the form $u(x) = C\sin sx$, with $s := \lambda^{1/2}$. Hence, $\lambda = s^2$ is an eigenvalue if and only if

$$\Gamma(s) := \sin s - \alpha \sin(\eta s) = 0, \quad s \in (0,\infty).$$

For an eigenfunction u, we let $Z(u)$ denote the number of zeros of u in $(0,1)$.

Example 7.1. Let $\alpha = \frac{1}{2}$. If $\eta = \frac{1}{2}$ then $u_2(1) = 0$, and simple estimates on the sign of the eigenfunctions also show that:

$$\begin{aligned} \eta \in (0,\tfrac{1}{2}) &\implies Z(u_2) = 2,\\ \eta \in (\tfrac{1}{2},1) &\implies Z(u_2) = 1,\\ \eta \in (0,\tfrac{1}{3}) \cup (\tfrac{2}{3},1) &\implies Z(u_3) = 2,\\ \eta \in (\tfrac{1}{3},\tfrac{2}{3}) &\implies Z(u_3) = 3.\end{aligned}$$

In particular, if $\eta \in (0,\frac{1}{3})$ then $Z(u_2) = Z(u_3) = 2$. Hence, counting the number of zeros does not distinguish between eigenfunctions corresponding to different eigenvalues. This also shows that varying η continuously causes the number of zeros in $(0,1)$ of the eigenfunctions to change.

Next, we examine the necessity of the hypothesis (1.5). The following examples show that Theorem 4.1 need not be true if $\alpha \notin \mathcal{A}^+ = (-1,1)$. In fact, the first one shows that Corollary 4.5 fails, even when $\alpha \in \partial\mathcal{A}^+ = \{\pm 1\}$.

Example 7.2. Let $\alpha = 1$ and $\eta = 1/5$. Then it can be verified that $\Gamma(15\pi/2) = 0$, and the corresponding eigenfunction $u(x) = \sin(\frac{15\pi}{2}x)$ has $u'(1) = 0$, that is, u is not in any set T_k, $k \geq 0$. In fact, the coefficient $\alpha = 1$ lies in $\partial\mathcal{A}^+$, while $u \in \partial T_7 \cap \partial T_8$.

The next example shows that if $\alpha > 1$ then η can be chosen in such a way that arbitrarily many nodal eigenfunctions may be 'missing', that is, Theorem 4.1 can fail 'very badly'.

Example 7.3. Let $\alpha = 1 + \epsilon$, for some arbitrarily small $\epsilon > 0$, and choose an arbitrarily large $\ell > 0$. Then we can choose $\delta \in (0, \pi/2)$ such that

$$\frac{\pi}{2} - \delta < t < \frac{\pi}{2} + \delta \implies \alpha \sin t > 1,$$

and set $\eta = \delta/\ell$. With these choices it can be seen that

$$s \in W := \left(\frac{\ell\pi}{2\delta} - \ell, \frac{\ell\pi}{2\delta} + \ell\right) \implies \Gamma(s) \leq 1 - \alpha \sin(\delta s/\ell) < 0,$$

that is, Γ has no zero on the interval $W \subset (0, \infty)$, and W has length 2ℓ. Thus, for any integer k with $((k-1)\pi, (k+1)\pi) \subset W$, there are no eigenfunctions lying in T_k.

7.2. Complex eigenvalues.

Heuristically, Examples 7.2 and 7.3 show that as α crosses the boundary of the set $\mathcal{A}^+$ eigenfunctions $u_k(\alpha)$ can move out of the nodal sets T_k, and then 'disappear'. The obvious question is: what has happened to these eigenfunctions, and the corresponding eigenvalues? One possible answer is that they have become complex. We have not considered complex eigenvalues and eigenfunctions here, for various reasons:
(a) in the p-Laplacian context, with $p \neq 2$, it is not trivial to even define the operator on complex valued functions;
(b) our main motivation has been the type of nonlinear problems considered in Section 5, rather than spectral theory in its own right. In this context, we are only interested in real solutions.

However, in the linear case it is natural to consider complex eigenvalues, and indeed these have been discussed in [6]. It is shown there that even if the assumption (1.5) does not hold then a sequence of eigenvalues still exists, but these eigenvalues may now be complex. Various properties of the spectrum in this case are also obtained in [6], but nodal properties are not considered so it is not clear if the above 'missing' eigenvalues have indeed become complex. There are many other natural, spectral theoretic, questions about the properties of this linear multi-point problem which remain open. Since our main interest lies in the nonlinear applications we will not consider this further.

7.3. Variable coefficients.

Consider the linear problem

$$-u'' + qu = \lambda u, \quad \text{on } (-1, 1), \tag{7.1}$$

with a variable coefficient function $q \in C^0[-1,1]$, together with Dirichlet-type boundary conditions. We will call the lowest eigenvalue the *principal eigenvalue* and this eigenvalue has a corresponding *principal eigenfunction*. Due to the nodal count used in the Dirichlet-type case, these are denoted by λ_1 and $u_1 \in T_1$ in Theorem 4.1.

In the variable coefficient case we can easily construct examples of 'bad' nodal behaviour of the principal eigenfunction as follows. Choose a function $u_1 \in C^2[-1,1]$, with $u_1 > 0$ on $[-1,1]$, and define a coefficient function $q := u_1''/u_1 \in C^0[-1,1]$. With this choice of q it is clear that u_1 is a non-trivial solution of (7.1) with $\lambda = 0$. Next, if we define the coefficients $\alpha^{\pm} := u_1(\pm 1)/u_1(0)$, then u_1 satisfies the Dirichlet-type boundary conditions

$$u_1(\pm 1) = \alpha^{\pm} u_1(0). \tag{7.2}$$

Of course, we may need to do some more work on the choice of u_1 to ensure that the coefficients $\alpha^{\pm}$ satisfy (1.5), but this is not difficult. This procedure enables us to construct eigenvalues $\lambda = 0$ of the problem (7.1), (7.2), whose principal eigenfunction u_1 can have:

(a) $u' = 0$ at an end point ± 1;
(b) arbitrarily many zeros of u' in $(-1, 1)$.

This shows that for the variable coefficient problem the principal eigenfunction u_1 need not belong to the set T_1, and so the basic nodal counting method used in Theorem 4.1 may fail in the variable coefficient case. Hence, it is not clear what to count to obtain nodal properties that are preserved in this case.

The variable coefficient case is also considered in [6], and the existence of complex eigenvalues is proved for this problem, but, as remarked above, nodal properties are not considered there. So, it is not clear if a spectral result such as Theorem 4.1 is true for the variable coefficient case.

7.4. Positivity of the principal eigenfunctions.

In the previous subsection we used the principal eigenfunctions to construct 'bad' nodal behaviour, simply because their positivity made the construction easy. In fact, the positivity of the principal eigenfunction is also an important property in its own right, which is often used to obtain positive solutions of nonlinear problems, see [21] or [22] for example. However, this is not a trivial property and it is not clear under what conditions it remains true for general multi-point problems.

The hypothesis (1.5) alone is not sufficient to yield positivity, even for constant coefficient problems — the simple Dirichlet-type problem

$$-u'' = \lambda u, \quad \text{on } (-1,1),$$
$$u(\pm 1) = \pm\frac{1}{2}u(0),$$

cannot have a positive principal eigenfunction. Of course, imposing additional sign conditions on $\alpha^{\pm}$, for example $\alpha^{\pm} > 0$, would seem natural for Dirichlet-type boundary conditions. Such sign conditions are sufficient to yield positivity of the principal eigenfunction for the constant coefficient problems considered here, see [18, Corollary 5.11] for the Dirichlet-type case (with both $\alpha^{\pm} > 0$) and [19, Theorem 9.10] for the mixed case (with $\alpha^{\nu} > 0$ at the Dirichlet-type end-point $x = \nu$); the Neumann-type case is trivial since the principal eigenfunction is constant.

For the variable coefficient problem the situation is not clear in general. In the Dirichlet-type case, with both $\alpha^{\pm} > 0$, the existence of a positive principal eigenfunction is proved in [20], for a coefficient function $q \in L^1(-1,1)$. However, the arguments in [20] only show the positivity of the principal eigenfunction — they give no indication of what nodal properties might hold for this, or any other, eigenfunction. For Neumann-type conditions, in general there seems little reason to expect that sign conditions on the coefficients $\alpha^{\pm}$ can ensure the positivity of the principal eigenfunction u (since, in this case, such conditions only affect the values of the derivative u', rather than the values of u, and by the method of Section 7.3 we can easily construct examples of variable coefficient problems for which the derivative u' of the principal eigenfunction is highly oscillatory). The following is an example of a Neumann-type problem which does not have a positive principal eigenfunction.

Example 7.4. Consider the problem

$$-u'' = (\lambda - q)u, \quad \text{on } (-1,1), \tag{7.3}$$
$$u'(-1) = \alpha^{-}u'(0), \quad u'(1) = 0 \tag{7.4}$$

with q having the form

$$q(x) := \begin{cases} (10\pi)^2, & x \in [-1,0], \\ 0, & x \in (0,1]. \end{cases}$$

By some elementary calculations on the solutions of equation (7.3) it can be shown that there exists $\epsilon > 0$ such that if $1 - \epsilon < \alpha^{-} < 1$ then:

- if $\lambda - (10\pi)^2 < -1$ then the BC at -1 can only hold if u changes sign on $[-1,0]$;
- if $\lambda - (10\pi)^2 \geq -1$ then any nontrivial solution of (7.3) changes sign on $[0,1]$.

Hence, the boundary value problem (7.3)-(7.4) does not have a positive eigenfunction.

REFERENCES

[1] P. Binding, P. Drábek, Sturm-Liouville theory for the p-Laplacian, *Studia Sci. Math. Hungar.* **40** (2003), 375–396.

[2] C. Bai, J. Fang, Existence of multiple positive solutions for nonlinear m-point boundary value problems, *J. Math. Anal. Appl.* **281** (2003), 76–85.

[3] E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[4] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[5] N. Dodds, B. P. Rynne, Spectral properties and nodal solutions for second-order, m-point, p-Laplacian boundary value problems, *Topol. Methods Nonlinear Anal.* **32** (2008), 21–40.

[6] J. Gao, D. Sun, M. Zhang, Structure of Eigenvalues of Multi-point Boundary Value Problems, *preprint.*

[7] M. García-Huidobro, Ch. P. Gupta, R. Manásevich, Some multipoint boundary value problems of Neumann-Dirichlet type involving a multipoint p-Laplace like operator, *J. Math. Anal. Appl.* **333** (2007), 247–264.

[8] M. García-Huidobro, R. Manásevich, J. R. Ward, A homotopy along p for systems with a p-Laplace operator, *Adv. Differential Equations* **8** (2003), 337–356.

[9] C. P. Gupta, A non-resonant generalized multi-point boundary-value problem of Dirichelet type involving a p-Laplacian type operator, in: *Proceedings of the Sixth Mississippi State–UBA Conference on Differential Equations and Computational Simulations*, 127–139, Electron. J. Differ. Equ. Conf., 15, Southwest Texas State Univ., San Marcos, TX, 2007.

[10] E. Hewitt, K. Stromberg, *Real and Abstract Analysis*, Second Edition, Springer, 1969.

[11] Y. X. Huang, G. Metzen, The existence of solutions to a class of semilinear differential equations, *Differential Integral Equations* **8** (1995), 429–452.

[12] A. N. Kolmogorov, S. V. Fomin, *Introductory Real Analysis*, Dover, New York, 1975.

[13] P. Lindqvist, Some remarkable sine and cosine functions, *Ricerche di Matematica* **44** (1995), 269–290.

[14] Y. Liu, Non-homogeneous boundary-value problems of higher order differential equations with p-Laplacian, *Electron. J. Differential Equations* **2008**, No. 22.

[15] R. Ma, D. O'Regan, Nodal solutions for second-order m-point boundary value problems with nonlinearities across several eigenvalues, *Nonlinear Anal.* **64** (2006), 1562–1577.

[16] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Analysis* **7** (1971), 487-513

[17] B. P. Rynne, Spectral properties and nodal solutions for second-order, m-point, boundary value problems, *Nonlinear Analysis* **67** (2007), 3318–3327.

[18] B. P. Rynne, Spectral properties of second-order, multi-point, p-Laplacian boundary value problems, *Nonlinear Analysis* **72** (2010), 4244–4253.

[19] B. P. Rynne, Spectral properties of p-Laplacian problems with Neumann and mixed-type multi-point boundary conditions, *Nonlinear Analysis* **74** (2010), 1471–1484.

[20] B. P. Rynne, Eigenvalue criteria for existence of positive solutions of second-order, multi-point, p-Laplacian boundary value problems, *Topol. Methods Nonlinear Anal.* **36** (2010), 311–326.

[21] J. R. L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, *J. London Math. Soc.* **74** (2006), 673–693.

[22] J. R. L. Webb, K. Q. Lan, Eigenvalue criteria for existence of multiple positive solutions of nonlinear boundary value problems of local and nonlocal type, *Topol. Methods Nonlinear Anal.* **27** (2006), 91–115.

[23] X. Xu, Multiple sign-changing solutions for some m-point boundary-value problems, *Electron. J. Differential Equations* **89** (2004).

[24] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Vol. I - Fixed Point Theorems*, Springer-Verlag, New York, 1986.