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HIGHER ORDER BOUNDARY VALUE PROBLEMS WITH TWO POINT SEPARATED NONHOMOGENEOUS BOUNDARY CONDITIONS

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Dedicated to Jeff Webb on the Occasion of his Retirement

ABSTRACT. The authors study the $2n$-th order nonlinear boundary value problems with two point separated nonhomogeneous boundary conditions

$$u^{(2n)} = g(t)f(t,u), \quad t \in (0,1),$$

$$\begin{cases} \alpha u^{(2i)}(0) - \beta u^{(2i+1)}(0) = (-1)^i \lambda_{2i}, \\ \gamma u^{(2i)}(1) + \delta u^{(2i+1)}(1) = (-1)^i \lambda_{2i+1}. \end{cases} \quad i = 0, \ldots, n-1,$$

Criteria are established for the existence of nontrivial solutions, positive solutions, and negative solutions of the above problem. Conditions are determined by the relationship between the behavior of the quotient $f(t,x)/x$ for x near 0 and ∞, and the smallest positive characteristic values of some associated linear integral operator. This work improves and extends a number of recent results in the literature on this topic. The results are illustrated with examples.

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1. INTRODUCTION

In this paper, we are concerned with the existence of nontrivial solutions of the $2n$-th order boundary value problem (BVP) consisting of the equation

$$u^{(2n)} = g(t)f(t,u), \quad t \in (0,1), \tag{1.1}$$

and the two point separated nonhomogeneous boundary condition (BC)

$$\begin{cases} \alpha u^{(2i)}(0) - \beta u^{(2i+1)}(0) = (-1)^i \lambda_{2i}, \\ \gamma u^{(2i)}(1) + \delta u^{(2i+1)}(1) = (-1)^i \lambda_{2i+1}, \end{cases} \quad i = 0, \dots, n-1, \tag{1.2}$$

where $n \geq 1$ is an integer, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : (0,1) \to \mathbb{R}_+ := [0,\infty)$ are continuous, $g \not\equiv 0$ on any subinterval of $(0,1)$, $\lambda_i \in \mathbb{R}_+$ for $i = 0, \dots, 2n-1$, and α, β, γ, $\delta \in \mathbb{R}_+$ with

$$\rho := \beta\gamma + \alpha\gamma + \alpha\delta > 0. \tag{1.3}$$

By a *nontrivial solution* of BVP (1.1), (1.2), we mean a function $u \in C^{2n-1}[0,1] \cap C^{2n}(0,1)$ such that $u(t) \not\equiv 0$ on $(0,1)$, and $u(t)$ satisfies Eq. (1.1) and BC (1.2). If $u(t) > 0$ on $(0,1)$, then $u(t)$ is a *positive solution*, and if $u(t) < 0$ on $(0,1)$, then $u(t)$ is a *negative solution*.

In case $\lambda_{2i} = \lambda_{2i+1} = 0$ for $i = 0, \dots, n-1$, BC (1.2) becomes the homogeneous BC

$$\begin{cases} \alpha u^{(2i)}(0) - \beta u^{(2i+1)}(0) = 0, \\ \gamma u^{(2i)}(1) + \delta u^{(2i+1)}(1) = 0, \end{cases} \quad i = 0, \dots, n-1. \tag{1.4}$$

When f is positone (i.e., $f \geq 0$), BVP (1.1), (1.4) and its special forms have been extensively studied in the literature by means of various tools and techniques such as the shooting method, fixed point theory in cones, fixed point index theory, the bifurcation approach, etc. For instance, papers [2, 4, 5, 13, 14] considered the problem with $n = 1$, papers [1, 9, 20, 32, 33] studied the problem with $n = 2$, and papers [3, 7, 8, 13, 19] studied the problem with a general positive integer n. All these papers investigated the existence of positive solutions of the problems. When f is sign-changing and bounded from below by a constant, the existence of nontrivial solutions of BVP (1.1), (1.4) with $n = 1$ has been studied in [22, 23] using Leray-Schauder degree theory. Results in [22] were subsequently improved in [12] to the case when f is sign-changing and not necessarily unbounded from below.

In recent years, the existence of positive solutions of various positone BVPs with nonhomogeneous BCs have also been studied in the literature; see, for example, [10, 15, 16, 18, 21, 35] and the references therein. Motivated partially by the recent papers [6, 12, 15, 16, 22, 23], here we will study the general nonhomogeneous BVP (1.1), (1.2), and derive several new criteria for the existence of nontrivial solutions, positive solutions, and negative solutions. Our analysis mainly relies on the Krein–Rutman theorem and topological degree theory. In our results, the nonlinear term f in (1.1) may be sign-changing and unbounded from below. Some of our existence conditions are optimal and determined by the relationship between the behavior of the quotient $f(t,x)/x$ for x near 0 and $\pm\infty$ and the smallest positive characteristic values (given by (3.1) below) of some related linear operator L (defined by (2.13) in Section). The results obtained in this paper extend many results in the literature, for example, those in [1, 3, 4, 5, 8, 12, 14, 20, 22, 23, 32, 33]. For papers on boundary value problems involving differential equations of the form (1.1) with nonlocal boundary conditions, we refer the reader to [24, 25, 26, 27, 28, 29, 30, 31].

We assume the following condition holds throughout without further mention:

(H) $\int_0^1 \mu(s)g(s)ds < \infty$, where

$$\mu(t) = (\alpha t + \beta)(\gamma + \delta - \gamma t) \quad \text{for } t \in [0,1]. \tag{1.5}$$

The rest of this paper is organized as follows. Section 2 contains some preliminary lemmas, Sections 3 contains the main results of this paper and several examples, and the proofs of the main results are presented in Section 4.

2. PRELIMINARY RESULTS

In this section, we present some preliminary results that will be used in the statements and the proofs of our main results. In the rest of this paper, the bold $\mathbf{0}$ stands for the zero element in any given Banach space. We refer the reader to [11, Lemma 2.5.1] for the proof of the following well known lemma.

Lemma 2.1. *Let Ω be a bounded open set in a real Banach space X with $\mathbf{0} \in \Omega$ and let $T : \overline{\Omega} \to X$ be compact. If*

$$Tu \neq \tau u \quad \textit{for all } u \in \partial\Omega \textit{ and } \tau \geq 1,$$

then the Leray-Schauder degree

$$\deg(I - T, \Omega, \mathbf{0}) = 1.$$

Now assume that X is a real Banach space with the norm $||\cdot||$, X^* is the dual space of X, P is a total cone in X, i.e., $X = \overline{P - P}$, and P^* is the dual cone of P, i.e.,

$$P^* = \{g \in X^* \ : \ g(u) \geq 0 \text{ for all } u \in P\}.$$

We recall that λ is an *eigenvalue* of an operator $L : X \to X$ with a corresponding *eigenfunction* φ if φ is nontrivial and $L\varphi = \lambda\varphi$. The reciprocals of eigenvalues are called the *characteristic values* of L. The operator L is said to be *positive* if $L(P) \subset P$.

The following Krein-Rutman theorem can be found in [34, Proposition 7.26].

Lemma 2.2. *Let $L : X \to X$ be a positive compact linear operator, L^* be the dual operator of L, and r_L be the spectral radius of L. If $r_L > 0$, then r_L is an eigenvalue of L and L^* with eigenfunctions in $P \setminus \{\mathbf{0}\}$ and $P^* \setminus \{\mathbf{0}\}$, respectively.*

Let L, L^*, and r_L be given as in Lemma 2.2 with $r_L > 0$. Then, from Lemma 2.2, there exist $\varphi_L \in P \setminus \{\mathbf{0}\}$ and $h \in P^* \setminus \{\mathbf{0}\}$ such that

$$L\varphi_L = r_L \varphi_L \quad \text{and } L^*h = r_L h. \tag{2.1}$$

Choose $\delta > 0$ and define

$$P(h, \delta) = \{u \in P \ : \ h(u) \geq \delta ||u||\}. \tag{2.2}$$

Then $P(h, \delta)$ is a cone in X.

The following two lemmas are taken from [12, Theorem 2.1] and [6, Lemma 3.5], respectively. ¿From here on, for any $R > 0$, let $B(\mathbf{0}, R) = \{u \in X \ : \ ||u|| < R\}$ be the open ball of X centered at $\mathbf{0}$ with radius R.

Lemma 2.3. *Assume that the following conditions hold:*

(C1) *there exist $\varphi \in P\setminus\{\mathbf{0}\}$ and $h \in P^*\setminus\{\mathbf{0}\}$ such that (2.1) holds and $L(P) \subseteq P(h, \delta)$;*
(C2) *$A : X \to P$ is a continuous operator and there exist $0 < \nu < 1$ and $K > 0$ such that $||Au|| \leq K||u||^\nu$ for all $u \in X$;*
(C3) *$F : X \to X$ is a bounded continuous operator and there exists $u_0 \in X$ such that $Fu + Au + u_0 \in P$ for all $u \in X$;*
(C4) *there exists $\epsilon > 0$ and $v_0 \in X$ such that $LFu \geq r_L^{-1}(1 + \epsilon)Lu - LAu - v_0$ for all $u \in X$.*

Let $T = LF$. Then there exists $R > 0$ such that the Leray-Schauder degree

$$\deg(I - T, B(\mathbf{0}, R), \mathbf{0}) = 0.$$

Lemma 2.4. *Assume that (C1) and the following conditions hold:*

(C2)* *$A : X \to P$ is a continuous operator and there exist $\nu > 1$ and $K > 0$ such that $||Au|| \leq K||u||^\nu$ for all $u \in X$;*
(C3)* *$F : X \to X$ is a bounded continuous operator and there exists $r_1 > 0$ such that*

$$Fu + Au \in P \quad \textit{for all } u \in X \textit{ with } ||u|| < r_1;$$

(C4)* *there exist $\epsilon > 0$ and $r_2 > 0$ such that*

$$LFu \geq r_L^{-1}(1 + \epsilon)Lu \quad \textit{for all } u \in X \textit{ with } ||u|| < r_2.$$

Let $T = LF$. Then there exists $0 < R < \min\{r_1, r_2\}$ such that the Leray-Schauder degree

$$\deg(I - T, B(\mathbf{0}, R), \mathbf{0}) = 0.$$

Let

$$H(t,s)=\frac{1}{\rho}\begin{cases}(\alpha t+\beta)(\gamma s-\gamma-\delta), & 0\le t\le s\le 1,\\ (\alpha s+\beta)(\gamma t-\gamma-\delta), & 0\le s\le t\le 1,\end{cases} \tag{2.3}$$

where ρ is defined by (1.3). Then, $H(t,s)$ is the Green's function for the BVP

$$u''=0,\quad t\in(0,1),$$

$$\alpha u(0)-\beta u'(0)=0,\ \gamma u(1)+\delta u'(1)=0.$$

Let $H_1(t,s)=H(t,s)$, and for $j=2,\dots,n$, recursively define

$$H_j(t,s)=\int_0^1 H(t,\tau)H_{j-1}(\tau,s)d\tau, \tag{2.4}$$

which is the Green's function for the BVP

$$u^{(2j)}=0,\quad t\in(0,1)$$

$$\begin{cases}\alpha u^{(2i)}(0)-\beta u^{(2i+1)}(0)=0,\\ \gamma u^{(2i)}(1)+\delta u^{(2i+1)}(1)=0,\end{cases}\quad i=0,\dots,j-1.$$

For $j=1,\dots,n$, let

$$c_j=\left(\frac{1}{\rho(\alpha+\beta)(\gamma+\delta)}\right)^j\left(\int_0^1\mu^2(\tau)d\tau\right)^{j-1}, \tag{2.5}$$

$$d_j=\frac{1}{\rho^j}\left(\int_0^1\mu(\tau)d\tau\right)^{j-1}. \tag{2.6}$$

Lemma 2.5 below obtains some useful estimates for $H_j(t,s)$.

Lemma 2.5. *For $j=1,\dots,n$, the function $H_j(t,s)$ satisfies*

$$c_j\mu(t)\mu(s)\le(-1)^jH_j(t,s)\le d_j\mu(s)\quad \textit{for } t,s\in[0,1], \tag{2.7}$$

where $\mu(t)$ is defined by (1.5).

Note from (2.3) that

$$\frac{1}{\rho(\alpha+\beta)(\gamma+\delta)}\mu(t)\mu(s)\le -H(t,s)\le\frac{1}{\rho}\mu(s)\quad\text{for } t,s\in[0,1].$$

Then, using (2.4) and induction, (2.7) can be proved. We omit the details here.

Let $\phi(t)$ be the unique solution of the BVP

$$u''=0,\quad t\in(0,1),$$

$$\alpha u(0)-\beta u'(0)=1,\ \gamma u(1)+\delta u'(1)=0,$$

and let $\psi(t)$ be the unique solution of the BVP

$$u''=0,\quad t\in(0,1),$$

$$\alpha u(0)-\beta u'(0)=0,\ \gamma u(1)+\delta u'(1)=1.$$

Then,

$$\phi(t) = \frac{1}{\rho}(-\gamma t + \gamma + \delta) \quad \text{and} \quad \psi(t) = \frac{1}{\rho}(\alpha t + \beta). \tag{2.8}$$

Clearly,

$$\phi(t) > 0 \quad \text{and} \quad \psi(t) > 0 \quad \text{for } t \in (0, 1). \tag{2.9}$$

The next lemma yields information about solutions of some special BVPs.

Lemma 2.6. *For $l = 0, \dots, n-1$, let $y_{2l}(t)$ be the unique solution of the BVP*

$$u^{(2n)} = 0, \quad t \in (0,1),$$

$$\begin{cases} \alpha u^{(2l)}(0) - \beta u^{(2l+1)}(0) = 1, \\ \alpha u^{(2i)}(0) - \beta u^{(2i+1)}(0) = 0, \ i = 0, \dots, n-1, \ i \neq l, \\ \gamma u^{(2i)}(1) + \delta u^{(2i+1)}(1) = 0, \ i = 0, \dots, n-1, \end{cases}$$

and let $y_{2l+1}(t)$ be the unique solution of the BVP

$$u^{(2n)} = 0, \quad t \in (0,1),$$

$$\begin{cases} \alpha u^{(2i)}(0) - \beta u^{(2i+1)}(0) = 0, \ i = 0, \dots, n-1, \\ \gamma u^{(2i)}(1) + \delta u^{(2i+1)}(1) = 0, \ i = 0, \dots, n-1, \ i \neq l, \\ \gamma u^{(2l)}(1) + \delta u^{(2l+1)}(1) = 1. \end{cases}$$

Then, we have

(a) $y_{2l}^{(2i)}(t) = y_{2l+1}^{(2i)}(t) = 0$ *for* $t \in [0,1]$ *and* $i = l+1, \dots, n$;
(b) $(-1)^{l-i} y_{2l}^{(2i)}(t) > 0$ *and* $(-1)^{l-i} y_{2l+1}^{(2i)}(t) > 0$ *for* $t \in (0,1)$ *and* $i = 0, \dots, l$.

Proof. We first show part (a). If $l = n-1$, then part (a) trivially holds. In the following, we assume $l < n-1$. For $i = 0, \dots, n$, let $w_i(t) = y_{2l}^{(2i)}(t)$. Note that

$$w''_{n-1}(t) = w_n(t) = y_{2l}^{(2n)}(t) = 0, \ t \in (0,1), \tag{2.10}$$

$$\begin{cases} \alpha w_i(0) - \beta w'_i(0) = 0, \\ \gamma w_i(1) + \delta w'_i(1) = 0, \end{cases} \quad i = l+1, \dots, n-1. \tag{2.11}$$

Then, from (2.10) and (2.11) with $i = n-1$, we have $w_{n-1}(t) = 0$ on $[0,1]$. Thus,

$$w''_{n-2}(t) = w_{n-1}(t) = 0, \ t \in [0,1].$$

This, together with (2.11) with $i = n-2$, yields $w_{n-2}(t) = 0$ on $[0,1]$. By induction, we obtain that

$$w_i(t) = y_{2l}^{(2i)}(t) = 0 \quad \text{for } t \in [0,1] \text{ and } i = l+1, \dots, n.$$

Similarly, we can show that

$$y_{2l+1}^{(2i)}(t) = 0 \quad \text{for } t \in [0,1] \text{ and } i = l+1, \dots, n.$$

Hence, part (a) holds.

We now show part (b). Note that $w_l(t) = y_{2l}^{(2l)}$ satisfies

$$w_l'' = y_{2l}^{(2l+2)}(t) = 0,$$
$$\begin{cases} \alpha w_l(0) - \beta w_l'(0) = 1, \\ \gamma w_l(1) + \delta w_l'(1) = 0. \end{cases}$$

Then, in view of (2.9), we have $w_l(t) = \phi(t) > 0$, where $\phi(t)$ is defined in (2.8), i.e., $y_{2l}^{(2l)}(t) > 0$ on $(0,1)$. For $i = 0, \ldots, l-1$, since $w_i(t)$ satisfies

$$w_i^{(2l-2i)}(t) = y_{2l}^{(2l)}(t) = \phi(t) > 0,\ t \in (0,1),$$
$$\begin{cases} \alpha w_i^{(2k)}(0) - \beta w_i^{(2k+1)}(0) = 0, \\ \gamma w_i^{(2k)}(1) + \delta w_i^{(2k+1)}(1) = 0, \end{cases} \quad k = 0, \ldots, l-i-1,$$

it follows that

$$w_i(t) = \int_0^1 H_{l-i}(t,s)\phi(s)ds \quad \text{on } [0,1],$$

where $H_{l-i}(t,s)$ is defined by (2.4) with $j = l-i$. Hence, by Lemma 2.5,

$$(-1)^{l-i} w_i(t) = (-1)^{l-i} \int_0^1 H_{l-i}(t,s)\phi(s)ds \geq c_{l-i}\mu(t) \int_0^1 \mu(s)\phi(s) > 0$$

for $t \in (0,1)$, i.e., $(-1)^{l-i} y_{2l}^{(2i)}(t) > 0$ on $(0,1)$. Using the function $\psi(t)$ defined in (2.8) and by a similar argument, we can show that

$$(-1)^{l-i} y_{2l+1}^{(2i)}(t) > 0 \quad \text{for } t \in (0,1) \text{ and } i = 0, \ldots, l.$$

Thus, part (b) holds and this completes the proof of the lemma. □

Remark 2.7. For the case when $n = 1$, $y_0(t) = \phi(t)$ and $y_1(t) = \psi(t)$.

Let $C[0,1]$ be the Banach space of continuous functions on $[0,1]$ equipped with the norm $||u|| = \max_{t\in[0,1]} |u(t)|$. Define a cone P in $C[0,1]$ by

$$P = \{u \in C[0,1] \ : \ u(t) \geq 0 \text{ for } t \in [0,1]\}. \tag{2.12}$$

Let the linear operator $L : C[0,1] \to C[0,1]$ be defined by

$$Lu(t) = (-1)^n \int_0^1 H_n(t,s)g(s)u(s)ds, \tag{2.13}$$

where $H_n(t,s)$ is given by (2.4) with $j = n$.

¿From here on, let r_L be the spectral radius of L, L^* be the dual operator of L, and P^* be the dual cone of P. The following lemma provides some information about the operators L and L^*.

Lemma 2.8. *The operator L is compact and maps P into P. Moreover, $r_L > 0$ and r_L is an eigenvalue of L and L^* with eigenfunctions $\varphi_L \in P \setminus \{\mathbf{0}\}$ and $h \in P^* \setminus \{\mathbf{0}\}$, respectively.*

Proof. The proof that L is compact and maps P into P is standard and will be omitted. By Lemma 2.5, we see that there exist $t_1, t_2 \in (0,1)$ such that $(-1)^n H_n(t,s) > 0$ for $t, s \in [t_1, t_2]$. Choose $u \in C[0,1]$ such that $u(t) \geq 0$ on $[0,1]$, $u(t^*) > 0$ for some $t^* \in [t_1, t_2]$, and $u(t) = 0$ for $t \in [0,1] \setminus [t_1, t_2]$. Then, for $t \in [t_1, t_2]$, we have

$$Lu(t) \geq (-1)^n \int_{t_1}^{t_2} H_n(t,s) g(s) u(s) ds > 0.$$

Thus, there exists $c > 0$ such that $cLu(t) \geq u(t)$ for $t \in [0,1]$. Now, from [17, Chapter 5, Theorem 2.1], it follows that $r_L > 0$. Finally, in view of $r_L > 0$ and the fact that the cone P defined by (2.13) is a total cone, the remaining part of the lemma readily follows from Lemma 2.2 and the first statement of this lemma. This completes the proof. □

3. MAIN RESULTS

For convenience, we introduce the following notations:

$$f_0 = \liminf_{x \to 0^+} \min_{t \in [0,1]} \frac{(-1)^n f(t,x)}{x}, \quad f_0^* = \liminf_{x \to 0} \min_{t \in [0,1]} \frac{(-1)^n f(t,x)}{x},$$

$$f_\infty = \liminf_{x \to \infty} \min_{t \in [0,1]} \frac{(-1)^n f(t,x)}{x}, \quad f_\infty^* = \liminf_{|x| \to \infty} \min_{t \in [0,1]} \frac{(-1)^n f(t,x)}{x},$$

$$F_0 = \limsup_{x \to 0} \max_{t \in [0,1]} \left| \frac{f(t,x)}{x} \right|, \quad F_\infty = \limsup_{|x| \to \infty} \max_{t \in [0,1]} \left| \frac{f(t,x)}{x} \right|.$$

Let r_L and φ_L be given as in Lemma 2.8 and define

$$\mu_L = \frac{1}{r_L}. \tag{3.1}$$

Clearly, μ_L is the smallest positive characteristic value of L and satisfies $\varphi_L = \mu_L L \varphi_L$.

We need the following assumptions.

(H1) There exist $a, b > 0$ and $0 < \xi < 1$ such that

$$(-1)^n f(t,x) \geq -a|x|^\xi - b \quad \text{for all } (t,x) \in [0,1] \times \mathbb{R}. \tag{3.2}$$

(H2) There exist $c > 0$, $\eta > 1$, and $0 < r < 1$ such that

$$(-1)^n f(t,x) \geq -c|x|^\eta \quad \text{for } (t,x) \in [0,1] \times [-r, 0]. \tag{3.3}$$

(H3) $(-1)^n x f(t,x) \geq 0$ for $(t,x) \in [0,1] \times \mathbb{R}$.

Remark 3.1. We wish to emphasize that, in (H1), we assume that $(-1)^n f(t,x)$ is bounded from below by $-a|x|^\xi - b$ for all $(t,x) \in [0,1] \times \mathbb{R}$; however in (H2), we only require that $(-1)^n f(t,x)$ is bounded from below by $-c|x|^\eta$ for $t \in [0,1]$ and x in a small left-neighborhood of 0.

Next, we state our existence results.

Theorem 3.2. *Assume that (H1) holds and $F_0 < \mu_L < f_\infty$. Then, for $(\lambda_0, \ldots, \lambda_{2n-1}) \in \mathbb{R}_+^{2n}$ with $\sum_{i=0}^{2n-1} \lambda_i$ sufficiently small, BVP* (1.1), (1.2) *has at least one nontrivial solution.*

Theorem 3.3. *Assume that (H2) holds and $F_\infty < \mu_L < f_0$. Then, for $(\lambda_0, \ldots, \lambda_{2n-1}) \in \mathbb{R}_+^{2n}$ with $\sum_{i=0}^{2n-1} \lambda_i$ sufficiently small, BVP* (1.1), (1.2) *has at least one nontrivial solution.*

Theorem 3.4. *Assume that (H3) holds, $F_0 < \mu_L < f_\infty^*$, and $\lambda_i = 0$ for $i = 0, \ldots, 2n-1$. Then BVP* (1.1), (1.2) *has at least one positive solution and one negative solution.*

Theorem 3.5. *Assume that (H3) holds, $F_\infty < \mu_L < f_0^*$, and $\lambda_i = 0$ for $i = 0, \ldots, 2n-1$. Then BVP* (1.1), (1.2) *has at least one positive solution and one negative solution.*

Remark 3.6. If the nonlinear term $f(t,x)$ is separable, say $f(t,x) = f_1(t)f_2(x)$, then conditions like $\mu_L < f_\infty$ and $\mu_L < f_0$ imply that $f_1(t) > 0$ on $[0,1]$. However, the function $g(t)$ in Eq. (1.1) may have zeros on $(0,1)$.

We now present some applications of the above theorems. To this end, let

$$A = \frac{1}{d_n \int_0^1 \mu(s)g(s)ds} \quad \text{and} \quad B = \frac{d_n}{c_n^2 \underline{\mu} \int_0^1 \mu^2(s)g(s)ds}, \tag{3.4}$$

where $\mu(t)$ is defined by (1.5), c_n and d_n are defined by (2.5) and (2.6) with $j = n$, respectively, and $\underline{\mu} = \min_{t\in[\theta_1,\theta_2]} \mu(t)$ with $0 < \theta_1 < \theta_2 < 1$ being fixed constants.

The following lemma will allow us to formulate versions of Theorems 3.2–3.5 that are fairly easy to apply.

Lemma 3.7. *Let μ_L be defined in* (3.1), *and A and B be given in* (3.4). *Then, we have $A \le \mu_L \le B$.*

In view of Lemma 3.7, the following corollaries become immediate consequences of Theorems 3.2–3.5.

Corollary 3.8. *Assume that (H1) holds and $F_0/A < 1 < f_\infty/B$. Then the conclusion of Theorem 3.2 holds.*

Corollary 3.9. *Assume that (H2) holds and $F_\infty/A < 1 < f_0/B$. Then the conclusion of Theorem 3.3 holds.*

Corollary 3.10. *Assume that (H3) holds, $F_0/A < 1 < f_\infty^*/B$, and $\lambda_i = 0$ for $i = 0, \ldots, 2n-1$. Then the conclusion of Theorem 3.4 holds.*

Corollary 3.11. *Assume that (H3) holds, $F_\infty/A < 1 < f_0^*/B$, and $\lambda_i = 0$ for $i = 0, \ldots, 2n-1$. Then the conclusion of Theorem 3.5 holds.*

In the remainder of this section, we present several examples to illustrate our results. For each of these examples we assume that for BVP (1.1), (1.2), $n \geq 1$ is an integer, α, β, γ, $\delta \in \mathbb{R}^+$ are such that (1.3) holds, $g : (0,1) \to \mathbb{R}_+$ is a continuous function, $g \not\equiv 0$ on any subinterval of $(0,1)$, and $\int_0^1 \mu(s)g(s)ds < \infty$, where $\mu(t)$ is defined by (1.5).

Example 3.12. Let

$$(-1)^n f(t,x) = \begin{cases} \sum_{i=1}^m \bar{a}_i(t)x^i, & x \in [-1,\infty), \\ \sum_{i=1}^m (-1)^i \bar{a}_i(t) - \bar{b}(t)|x|^\kappa + \bar{b}(t), & x \in (-\infty,-1), \end{cases} \tag{3.5}$$

where $m > 1$ is an integer, $0 \leq \kappa < 1$, $\bar{a}_i$, $\bar{b} \in C[0,1]$ with $0 \leq ||\bar{a}_1|| < A$ and $\bar{a}_m(t) > 0$ on $[0,1]$, where A is defined in (3.4). Then, for $(\lambda_0,\ldots,\lambda_{2n-1}) \in \mathbb{R}_+^{2n}$ with $\sum_{i=0}^{2n-1} \lambda_i$ sufficiently small, BVP (1.1), (1.2) has at least one nontrivial solution.

To see this, we first note that $f \in C([0,1]\times\mathbb{R})$ and assumption (H) is satisfied. Let

$$a = ||\bar{b}||, \quad b = \sum_{i=1}^n ||\bar{a}_i|| + ||\bar{b}||, \quad \text{and} \quad \xi = \kappa.$$

Then, f satisfies (3.2), i.e., (H1) holds. For A and B defined in (3.4), from (3.5), we have

$$F_0 = \limsup_{x\to 0} \max_{t\in[0,1]} \left|\frac{f(t,x)}{x}\right| = ||\bar{a}_1|| < A \text{ and } f_\infty = \liminf_{x\to\infty} \min_{t\in[0,1]} \frac{f(t,x)}{x} = \infty > B.$$

Hence, $F_0/A < 1 < f_\infty/B$. The conclusion then follows from Corollary 3.8.

In particular, let us take $n = 2$, $g(t) = t$, and in the boundary conditions set $\alpha = \beta = \gamma = \delta = 1$; then $\rho = 3$ and $\mu(t) = 2 + t - t^2$. With $\theta_1 = 1/4$ and $\theta_2 = 3/4$, we see that $\underline{\mu} = 35/16$, $d_n = 13/54$, $c_n = 47/1440$, $A = 648/169$, and $B = [13/54]/(47/1440)^2(35/16)(47/20) = 862617600/19622547$. Thus, we need $0 \leq ||\bar{a}_1|| < 648/169 \approx 3.834$.

Example 3.13. Let

$$(-1)^n f(t,x) = \begin{cases} -t^2 + 3 + (|x|^{1/3} - 1)|x|^{1/2}, & x < -1, \\ -t^2x^2 + 2|x| + 1, & -1 \leq x \leq 0, \\ 1 - tx^{1/3}, & x > 0. \end{cases} \tag{3.6}$$

Then, for $(\lambda_0,\ldots,\lambda_{2n-1}) \in \mathbb{R}_+^{2n}$ with $\sum_{i=0}^{n-1} \lambda_i$ sufficiently small, BVP (1.1), (1.2) has at least one nontrivial solution.

To see this, we first note that $f \in C([0,1]\times\mathbb{R})$ and assumption (H) is satisfied. Now with $c = 1$ and $\eta = 2$, we see that f satisfies (3.3) for any $r \in (0,1)$, i.e., (H2) holds. Moreover, from (3.6), we have $f_0 = \infty$ and $F_\infty = 0$. Thus, $F_\infty < \mu_L < f_0$, where μ_L is defined by (3.1). The conclusion then follows from Theorems 3.3.

Example 3.14. Let $\lambda_i = 0$ for $i = 0, \dots, 2n-1$, $\alpha = \gamma = 1$, $\beta = \delta = 0$, and

$$(-1)^n f(t,x) = \begin{cases} -k_1|x|^{l_1} + k_1 - k_2, & x \in (-\infty, -1), \\ -k_2|x|^{l_2}, & x \in [-1, 0), \\ k_3 x^{l_3}, & x \in (0, 1], \\ k_4 x^{l_4} - k_4 + k_3, & x \in (1, \infty), \end{cases} \tag{3.7}$$

where $k_i, l_i \in \mathbb{R}_+$ be such that either

$$\begin{cases} l_1 \geq 1,\ l_4 \geq 1, \quad \text{and} \quad k_1 > \pi^{2n},\ k_4 > \pi^{2n} \text{ if } l_1 = l_4 = 1, \\ l_2 \leq 1,\ l_3 \leq 1, \quad \text{and} \quad k_2 < \pi^{2n},\ k_3 < \pi^{2n} \text{ if } l_2 = l_3 = 1, \end{cases} \tag{3.8}$$

or

$$\begin{cases} l_1 \leq 1,\ l_4 \leq 1, \quad \text{and} \quad k_1 < \pi^{2n},\ k_4 < \pi^{2n} \text{ if } l_1 = l_4 = 1, \\ l_2 \geq 1,\ l_3 \geq 1, \quad \text{and} \quad k_2 > \pi^{2n},\ k_3 > \pi^{2n} \text{ if } l_2 = l_3 = 1. \end{cases} \tag{3.9}$$

Then, BVP (1.1), (1.2) has at least one positive solution and one negative solution.

To see this, we first note that assumptions (H) and (H3) are satisfied. Moreover, from (3.7), we see that

$$F_0 < \pi^{2n} < f_\infty^* \text{ if (3.8) holds} \quad \text{and} \quad F_\infty < \pi^{2n} < f_0^* \text{ if (3.9) holds.}$$

It is known that $\mu_L = \pi^{2n}$ when $g(t) \equiv 1$, $\alpha = \gamma = 1$ and $\beta = \delta = 0$. The conclusion then follows from Theorems 3.4 and 3.5.

4. PROOFS OF THE MAIN RESULTS

Let $y_i(t)$, $i = 0, \dots, 2n-1$, be given as in Lemma 2.6. For any $\lambda = (\lambda_0, \dots, \lambda_{2n-1}) \in \mathbb{R}_+^{2n}$ and $u \in C^{2n-1}[0,1] \cap C^{2n}(0,1)$, let

$$v(t) = u(t) - \sum_{j=0}^{n-1} (-1)^j (\lambda_{2j} y_{2j}(t) + \lambda_{2j+1} y_{2j+1}(t)),\ t \in [0,1]. \tag{4.1}$$

Then BVP (1.1), (1.2) is equivalent to the BVP consisting of the equation

$$v^{(2n)} = g(t) f\left(t, v + \sum_{j=0}^{n-1} (-1)^j (\lambda_{2j} y_{2j}(t) + \lambda_{2j+1} y_{2j+1}(t))\right), \tag{4.2}$$

and the homogeneous BC

$$\begin{cases} \alpha v^{(2i)}(0) - \beta v^{(2i+1)}(0) = 0, \\ \gamma v^{(2i)}(1) + \delta v^{(2i+1)}(1) = 0, \end{cases} \quad i = 0, \dots, n-1. \tag{4.3}$$

Moreover, if $v(t)$ is a solution of BVP (4.2), (4.3), then $u(t)$ given by (4.1) is a solution of BVP (1.1), (1.2).

Let P and L be defined by (2.12) and (2.13), respectively. By Lemma 2.8, L maps P into P and is compact. Define operators F_λ, $T : C[0,1] \to C[0,1]$ by

$$F_\lambda v(t) = (-1)^n f\left(t, v + \sum_{j=0}^{n-1} (-1)^j (\lambda_{2j} y_{2j}(t) + \lambda_{2j+1} y_{2j+1}(t))\right) \tag{4.4}$$

and

$$Tv(t) = LF_\lambda v(t) = (-1)^n \int_0^1 H_n(t,s) g(s) F_\lambda v(s) ds, \tag{4.5}$$

where H_n is defined by (2.4) with $j = n$. Then, F_λ is bounded, and a standard argument shows that T is compact. Moreover, a solution of BVP (4.2), (4.3) is equivalent to a fixed point of T in $C[0,1]$.

Proof of Theorem 3.2. We first verify that conditions (C1)–(C4) of Lemma 2.3 are satisfied. By Lemma 2.8, there exist $\varphi_L \in P \setminus \{\mathbf{0}\}$ and $h \in P^* \setminus \{\mathbf{0}\}$ such that (2.1) holds. We now show that h can be explicitly given by

$$h(v) = \int_0^1 \varphi_L(t) g(t) v(t) dt, \; v \in C[0,1]. \tag{4.6}$$

Clearly, $h \in P^* \setminus \{\mathbf{0}\}$. Note from (2.3) that $H(t,s) = H(s,t)$ for t, $s \in [0,1]$. Then, from (2.4) and by induction, it is easy to see that

$$H_n(t,s) = H_n(s,t) \quad \text{for } t, s \in [0,1].$$

Thus, from the first equation in (2.1), (2.13), and (4.6), we have

$$\begin{aligned}
(L^*h)(v) = h(Lv) &= \int_0^1 \varphi_L(t) g(t) Lv(t) dt \\
&= \int_0^1 \varphi_L(t) g(t) \left((-1)^n \int_0^1 H_n(t,s) g(s) v(s) ds\right) dt \\
&= \int_0^1 g(s) v(s) \left((-1)^n \int_0^1 H_n(t,s) g(t) \varphi_L(t) dt\right) ds \\
&= \int_0^1 g(s) v(s) \left((-1)^n \int_0^1 H_n(s,t) g(t) \varphi_L(t) dt\right) ds \\
&= \int_0^1 g(s) v(s) L\varphi_L(s) ds \\
&= r_L \int_0^1 g(s) v(s) \varphi_L(s) ds = r_L h(v),
\end{aligned}$$

i.e., h satisfies the second equation in (2.2). Hence, h can be explicitly given by (4.6). Note that $\varphi_L = \mu_L L\varphi_L$, so from (2.7), (2.13), and (3.1), we have

$$\begin{aligned} r_L\varphi_L(s) &= (-1)^n \int_0^1 H_n(s,t)g(t)\varphi_L(t)dt \\ &\geq c_n\mu(s)\int_0^1 \mu(t)g(t)\varphi_L(t)dt \\ &\geq \frac{c_n}{d_n}(-1)^n H_n(t,s)\int_0^1 \mu(t)g(t)\varphi_L(t)dt \\ &= \delta(-1)^n H_n(t,s) \quad \text{for } t,s\in[0,1], \end{aligned}$$

where

$$\delta = \frac{c_n}{d_n}\int_0^1 \mu(t)g(t)\varphi_L(t)dt.$$

Thus,

$$\begin{aligned} h(Lv) = r_L h(v) &= r_L\int_0^1 \varphi_L(s)g(s)v(s)ds \\ &\geq \delta(-1)^n\int_0^1 H_n(t,s)g(s)v(s)ds \\ &= \delta Lv(t) \quad \text{for } t\in[0,1]. \end{aligned}$$

Hence, $h(Lv)\geq \delta||Lv||$, i.e., $L(P)\subseteq P(h,\delta)$. Therefore, condition (C1) of Lemma 2.3 holds.

Let $Av(t) = a|v(t)|^\xi$ for $v\in C[0,1]$, where a and ξ are given in (H1). Then, with $K = a$ and $\nu = \xi$, condition (C2) of Lemma 2.3 holds.

Let $\lambda = (\lambda_0,\ldots,\lambda_{2n-1})\in\mathbb{R}_+^{2n}$, F_λ be defined by (4.4), and $u_0(t)\equiv b$, where b is given in (H1). Then (3.2) implies $F_\lambda u + Au + u_0\in P$ for all $u\in C[0,1]$, i.e., condition (C3) of Lemma 2.3 holds.

Since $f_\infty > \mu_L$, there exist $\epsilon_1 > 0$ and $N_1 > 0$ such that

$$\begin{aligned} (-1)^n f(t,x) &\geq \mu_L(1+\epsilon_1)x \\ &= r_L^{-1}(1+\epsilon_1)x \quad \text{for } (t,x)\in[0,1]\times(N_1,\infty). \end{aligned} \tag{4.7}$$

For a and ξ given in (H1), and noting that $0<\xi<1$, we have

$$\lim_{x\to-\infty}\frac{-a|x|^\xi}{r_L^{-1}(1+\epsilon_1)x} = 0.$$

Thus, there exists $N_2 > 0$ such that

$$0 < \frac{-a|x|^\xi}{r_L^{-1}(1+\epsilon_1)x}\leq 1 \quad \text{for } x < -N_2,$$

or equivalently,

$$-a|x|^\xi \geq r_L^{-1}(1+\epsilon_1)x \quad \text{for } x < -N_2.$$

Then, from (3.2),

$$\begin{aligned}(-1)^n f(t,x) &\geq -a|x|^\xi - b \\ &\geq r_L^{-1}(1+\epsilon_1)x - b \quad \text{for } (t,x) \in [0,1] \times (-\infty, -N_2).\end{aligned} \tag{4.8}$$

Let $m > 0$ be large enough so that

$$-m \leq \min_{(t,x)\in[0,1]\times[-N_2,N_1]} \left\{(-1)^n f(t,x) - r_L^{-1}(1+\epsilon_1)x\right\}.$$

Then,

$$(-1)^n f(t,x) \geq r_L^{-1}(1+\epsilon_1)x - m \quad \text{for } (t,x) \in [0,1] \times [-N_2, N_1]. \tag{4.9}$$

Let $b_1 = \max\{b, m\}$. Then, from (4.7), (4.8), and (4.9), it follows that

$$(-1)^n f(t,x) \geq r_L^{-1}(1+\epsilon_1)x - b_1 \quad \text{for all } (t,x) \in [0,1] \times \mathbb{R}. \tag{4.10}$$

¿From Lemma 2.6 (b), (4.4), and (4.10), we see that

$$\begin{aligned}F_\lambda v(t) &\geq r_L^{-1}(1+\epsilon_1)\left(v(t) + \sum_{j=0}^{n-1}(-1)^j(\lambda_{2j}y_{2j}(t) + \lambda_{2j+1}y_{2j+1}(t))\right) - b_1 \\ &\geq r_L^{-1}(1+\epsilon_1)v(t) - b_1 \quad \text{for } v \in C[0,1].\end{aligned}$$

Note that $Av(t) \geq 0$ for any $v \in C[0,1]$. Then,

$$F_\lambda v(t) \geq r_L^{-1}(1+\epsilon_1)v(t) - Av(t) - b_1 \quad \text{for } v \in C[0,1].$$

Thus,

$$LF_\lambda v(t) \geq r_L^{-1}(1+\epsilon_1)Lv(t) - LAv(t) - Lb_1.$$

Hence, (C4) of Lemma 2.3 holds with $F = F_\lambda$ and $v_0 = L(b_1)$.

Since the hypotheses of Lemma 2.3 are satisfied, there exists $R_1 > 0$ such that

$$\deg(I - T, B(\mathbf{0}, R_1), \mathbf{0}) = 0. \tag{4.11}$$

Next, since $F_0 < \mu_L$, there exist $0 < \epsilon_2 < 1$ and $0 < R_2 < R_1$ such that

$$\begin{aligned}|f(t,x)| &\leq \mu_L(1-\epsilon_2)|x| \\ &= r_L^{-1}(1-\epsilon_2)|x| \quad \text{for } (t,x) \in [0,1] \times [-2R_2, 2R_2].\end{aligned} \tag{4.12}$$

In what follows, let $\lambda = (\lambda_0, \ldots, \lambda_{2n-1}) \in \mathbb{R}_+^{2n}$ with $\sum_{i=0}^{2n-1} \lambda_i$ small enough so that

$$\sum_{j=0}^{n-1}(\lambda_{2j}||y_{2j}|| + \lambda_{2j+1}||y_{2j+1}||) < R_2 \quad \text{and} \quad C_1 < \epsilon_2 R_2, \tag{4.13}$$

where

$$C_1 = r_L^{-1}(1-\epsilon_2)\sum_{j=0}^{n-1}(\lambda_{2j}||y_{2j}|| + \lambda_{2j+1}||y_{2j+1}||) \max_{t\in[0,1]} \int_0^1 |H_n(t,s)|g(s)ds. \tag{4.14}$$

We claim that

$$Tv \neq \tau v \quad \text{for all } v \in \partial B(\mathbf{0}, R_2) \text{ and } \tau \geq 1. \tag{4.15}$$

If this is not the case, then there exist $\bar{v} \in \partial B(\mathbf{0}, R_2)$ and $\bar{\tau} \geq 1$ such that $T\bar{v} = \bar{\tau}\bar{v}$. Thus, $\bar{v} = \bar{s}T\bar{v}$, where $\bar{s} = 1/\bar{\tau}$. Clearly, $\bar{s} \in (0, 1]$. ¿From (4.4) and (4.12), we have

$$\begin{aligned} |F_\lambda \bar{v}(t)| &\leq r_L^{-1}(1-\epsilon_2)\left|\bar{v}(t) + \sum_{j=0}^{n-1}(-1)^j(\lambda_{2j}y_{2j}(t) + \lambda_{2j+1}y_{2j+1}(t))\right| \\ &\leq r_L^{-1}(1-\epsilon_2)\left(|\bar{v}(t)| + \sum_{j=0}^{n-1}(\lambda_{2j}||y_{2j}|| + \lambda_{2j+1}||y_{2j+1}||)\right). \end{aligned} \tag{4.16}$$

Assume $R_2 = ||\bar{v}|| = |\bar{v}(\bar{t})|$ for some $\bar{t} \in [0, 1]$. Then, from (2.13), (4.5), and (4.16), we obtain

$$\begin{aligned} R_2 &= |\bar{v}(\bar{t})| = \bar{s}|T\bar{v}(\bar{t})| \leq (-1)^n \int_0^1 H_n(\bar{t}, s)g(s)|F_\lambda \bar{v}(s)|ds \\ &\leq r_L^{-1}(1-\epsilon_2)(-1)^n \int_0^1 H_n(\bar{t}, s)g(s)|\bar{v}(s)|ds \\ &\quad + r_L^{-1}(1-\epsilon_2)\sum_{j=0}^{n-1}(\lambda_{2j}||y_{2j}|| + \lambda_{2j+1}||y_{2j+1}||)\int_0^1 |H_n(\bar{t}, s)|g(s)ds \\ &\leq r_L^{-1}(1-\epsilon_2)L|\bar{v}(\bar{t})| + C_1 = r_L^{-1}(1-\epsilon_2)LR_2 + C_1. \end{aligned}$$

Consequently,

$$\begin{aligned} h(R_2) &\leq r_L^{-1}(1-\epsilon_2)h(LR_2) + h(C_1) \\ &= r_L^{-1}(1-\epsilon_2)(L^*h)(R_2) + h(C_1) \\ &= r_L^{-1}(1-\epsilon_2)r_L h(R_2) + h(C_1) \\ &= (1-\epsilon_2)h(R_2) + h(C_1). \end{aligned}$$

Thus,

$$(C_1 - \epsilon_2 R_2)h(1) \geq 0.$$

Since $h(1) > 0$, we have $C_1 \geq \epsilon_2 R_2$. But this contradicts the second inequality in (4.13). Thus, (4.15) holds. Now, Lemma 2.1 implies

$$\deg(I - T, B(\mathbf{0}, R_2), \mathbf{0}) = 1. \tag{4.17}$$

By the additivity property of the Leray-Schauder degree, (4.11), and (4.17), we have

$$\deg(I - T, B(\mathbf{0}, R_1) \setminus \overline{B(\mathbf{0}, R_2)}) = -1.$$

Then, from the solution property of the Leray-Schauder degree, T has at least one fixed point v in $B(\mathbf{0}, R_1) \setminus \overline{B(\mathbf{0}, R_2)}$, which is a solution of BVP (4.2), (4.3). Therefore, we have shown that, for $\lambda = (\lambda_0, \ldots, \lambda_{2n-1}) \in \mathbb{R}_+^{2n}$ satisfying (4.13), BVP (4.2), (4.3) has at least one solution $v(t)$ satisfying $||v|| \geq R_2$. Thus, for each $\lambda = (\lambda_0, \ldots, \lambda_{2n-1}) \in \mathbb{R}_+^{2n}$ with $\sum_{i=0}^{2n-1} \lambda_i$ sufficiently small, BVP (1.1), (1.2) has at least one solution $u(t)$, which is given by (4.1) with the above solution $v(t)$ and satisfies

$$\begin{aligned} ||u|| &\geq ||v|| - \sum_{j=0}^{n-1}(\lambda_{2j}||y_{2j}|| + \lambda_{2j+1}||y_{2j+1}||) \\ &\geq R_2 - \sum_{j=0}^{n-1}(\lambda_{2j}||y_{2j}|| + \lambda_{2j+1}||y_{2j+1}||) > 0. \end{aligned}$$

This complete the proof of the theorem. □

Proof of Theorem 3.3. We first verify that conditions (C1) and (C2)*–(C4)* of Lemma 2.4 are satisfied. As in the proof of Theorem 3.2, there exist $\varphi_L \in P \setminus \{\mathbf{0}\}$ and $h \in P^* \setminus \{\mathbf{0}\}$ defined by (4.6) such that (C1) holds.

Let $Av(t) = c|v(t)|^\eta$ for $v \in C[0,1]$, where c and η are given in (H2). Then, with $K = c$ and $\nu = \eta$, (C2)* of Lemma 2.4 holds.

Since $f_0 > \mu_L$, there exist $\epsilon_3 > 0$ and $0 < \zeta_1 < 1$ such that

$$\begin{aligned} (-1)^n f(t,x) &\geq \mu_L(1+\epsilon_3)x \\ &= r_L^{-1}(1+\epsilon_3)x \geq 0 \quad \text{for } (t,x) \in [0,1] \times [0, 2\zeta_1]. \end{aligned} \tag{4.18}$$

Let $\lambda = (\lambda_0, \ldots, \lambda_{2n-1}) \in \mathbb{R}_+^{2n}$ with $\sum_{i=0}^{2n-1} \lambda_i$ small enough so that

$$\sum_{j=0}^{n-1}(\lambda_{2j}||y_{2j}|| + \lambda_{2j+1}||y_{2j+1}||) \leq \zeta_1 \tag{4.19}$$

and F_λ be defined by (4.4). Then, (4.18) implies

$$\begin{aligned} F_\lambda v(t) &\geq r_L^{-1}(1+\epsilon_3)\left(v(t) + \sum_{j=0}^{n-1}(-1)^j(\lambda_{2j}y_{2j}(t) + \lambda_{2j+1}y_{2j+1}(t))\right) \\ &\geq r_L^{-1}(1+\epsilon_3)v(t) \quad \text{for all } v \in P \text{ with } ||v|| \leq \zeta_1. \end{aligned} \tag{4.20}$$

Let r be given in (H2). Now, in view of (3.3) and (4.20), we see that condition (C3)* of Lemma 2.4 holds with $F = F_\lambda$ and $r_1 = \min\{r, \zeta_1\}$.

Choose $0 < \zeta_2 < \min\{r, \zeta_1\}$ small enough so that $-c|x|^\eta \geq r_L^{-1}(1+\epsilon_3)x$ for $x \in [-\zeta_2, 0]$. Then, from (3.3),

$$(-1)^n f(t,x) \geq r_L^{-1}(1+\epsilon_3)x \quad \text{for } (t,x) \in [0,1] \times [-\zeta_2, 0]. \tag{4.21}$$

¿From (4.18) and (4.21), we have

$$\begin{aligned} F_\lambda v(t) &\geq r_L^{-1}(1+\epsilon_3)\left(v(t) + \sum_{j=0}^{n-1}(-1)^j(\lambda_{2j}y_{2j}(t) + \lambda_{2j+1}y_{2j+1}(t))\right) \\ &\geq r_L^{-1}(1+\epsilon_3)v(t) \quad \text{for all } v \in C[0,1] \text{ with } ||v|| \leq \zeta_2. \end{aligned}$$

which clearly implies that

$$LF_\lambda v \geq r_L^{-1}(1+\epsilon_3)Lv \quad \text{for all } v \in C[0,1] \text{ with } ||v|| < \zeta_2.$$

Hence, (C4)* of Lemma 2.4 holds with $F = F_\lambda$ and $r_2 = \zeta_2$.

We have verified that all the conditions of Lemma 2.4 hold, so there exists $R_3 > 0$ such that

$$\deg(I - T, B(\mathbf{0}, R_3), \mathbf{0}) = 0. \tag{4.22}$$

Next, since $F_\infty < \mu_L$, there exist $0 < \epsilon_4 < 1$ and $R_4 > R_3$ such that

$$|f(t,x)| \le \mu_L(1-\epsilon_4)|x| = r_L^{-1}(1-\epsilon_4)|x| \quad \text{for } (t,|x|) \in [0,1] \times (R_4, \infty). \tag{4.23}$$

Let

$$\begin{aligned} C_2 \;&=\; r_L^{-1}(1-\epsilon_4)\sum_{j=0}^{n-1}(\lambda_{2j}||y_{2j}|| + \lambda_{2j+1}||y_{2j+1}||)\max_{t\in[0,1]}\int_0^1 |H_n(t,s)|g(s)ds \\ &\quad + \max_{t\in[0,1],|x|\le R_4}|f(t,x)|\max_{t\in[0,1]}\int_0^1 |H_n(t,s)|g(s)ds. \end{aligned} \tag{4.24}$$

Then $0 < C_2 < \infty$. Choose R_5 large enough so that

$$R_5 > \max\{R_4,\ C_2/\epsilon_4\}. \tag{4.25}$$

We claim that

$$Tv \ne \tau v \quad \text{for all } v \in \partial B(\mathbf{0}, R_5) \text{ and } \tau \ge 1. \tag{4.26}$$

If this is not the case, then there exist $\bar v \in \partial B(\mathbf{0}, R_5)$ and $\bar\tau \ge 1$ such that $T\bar v = \bar\tau\bar v$. It follows that $\bar v = \bar s T\bar v$, where $\bar s = 1/\bar\tau$. Clearly, $\bar s \in (0,1]$. Assume $R_5 = ||\bar v|| = |\bar v(\bar t)|$ for some $\bar t \in [0,1]$. Let

$$J_1(\bar v) = \left\{ t \in [0,1]\ :\ \left|\bar v(t) + \sum_{j=0}^{n-1}(-1)^j(\lambda_{2j}y_{2j}(t) + \lambda_{2j+1}y_{2j+1}(t))\right| > R_4 \right\},$$

$$J_2(\bar v) = [0,1] \setminus J_1(\bar v),$$

and

$$p(\bar v(t)) = \min\left\{ \left|\bar v(t) + \sum_{j=0}^{n-1}(-1)^j(\lambda_{2j}y_{2j}(t) + \lambda_{2j+1}y_{2j+1}(t))\right|,\ R_4 \right\} \quad \text{for } t \in [0,1].$$

Then, from (2.13), (4.5), (4.23), and (4.24), it follows that

$$\begin{aligned}
&R_5 = |\bar{v}(\bar{t})| = \bar{s}|T\bar{v}(\bar{t})| \\
&\le (-1)^n \int_0^1 H_n(\bar{t},s)g(s)|F_\lambda \bar{v}(s)|ds \\
&= (-1)^n \int_{J_1(\bar{v})} H_n(\bar{t},s)g(s)|F_\lambda \bar{v}(s)|ds + (-1)^n \int_{J_2(\bar{v})} H_n(\bar{t},s)g(s)|F_\lambda \bar{v}(s)|ds \\
&\le r_L^{-1}(1-\epsilon_4)(-1)^n \int_{J_1(\bar{v})} H_n(\bar{t},s)g(s)\left|\bar{v}(s) + \sum_{j=0}^{n-1}(-1)^j(\lambda_{2j}y_{2j}(s) + \lambda_{2j+1}y_{2j+1}(s))\right|ds \\
&\quad + \int_{J_2(\bar{v})} H_{n-1}(\bar{t},s)g(s)|F_\lambda p(\bar{v}(s))|ds \\
&\le r_L^{-1}(1-\epsilon_4)(-1)^n \int_0^1 H_n(\bar{t},s)g(s)|\bar{v}(s)|ds \\
&\quad + r_L^{-1}(1-\epsilon_4)\sum_{j=0}^{n-1}(\lambda_{2j}||y_{2j}|| + \lambda_{2j+1}||y_{2j+1}||)\int_0^1 |H_n(\bar{t},s)|g(s)ds \\
&\quad + \int_0^1 |H_n(\bar{t},s)|g(s)|F_\lambda p(\bar{v}(s))|ds \\
&\le r_L^{-1}(1-\epsilon_4)L|v(\bar{t})| + C_2 = r_L^{-1}(1-\epsilon_4)LR_5 + C_2.
\end{aligned}$$

Hence, for h defined by (4.6), we have

$$\begin{aligned}
h(R_5) &\le r_L^{-1}(1-\epsilon_4)h(R_5) + h(C_2) \\
&= r_L^{-1}(1-\epsilon_4)(L^*h)(R_5) + h(C_2) \\
&= r_L^{-1}(1-\epsilon_4)r_L h(R_5) + h(C_2) \\
&= (1-\epsilon_4)h(R_5) + h(C_2),
\end{aligned}$$

which implies

$$(\epsilon_4 R_5 - C_2)h(1) \le 0.$$

In view of the fact that $h(1) > 0$, it follows that $R_5 \le C_2/\epsilon_4$. This contradicts (4.25) and so (4.26) holds. Then, by Lemma 2.1, we have

$$\deg(I - T, B(\mathbf{0}, R_5), \mathbf{0}) = 1. \tag{4.27}$$

By the additivity property of the Leray-Schauder degree, (4.22), and (4.27), we obtain

$$\deg(I - T, B(\mathbf{0}, R_5) \setminus \overline{B(\mathbf{0}, R_3)}) = 1.$$

Thus, from the solution property of the Leray-Schauder degree, T has at least one fixed point v in $B(\mathbf{0}, R_5) \setminus \overline{B(\mathbf{0}, R_3)}$, which is a solution of BVP (4.2), (4.3). Therefore, we have shown that, for $\lambda = (\lambda_0, \dots, \lambda_{2n-1}) \in \mathbb{R}_+^{2n}$ satisfying (4.19), BVP (4.2), (4.3) has at least one solution $v(t)$ satisfying $||v|| \ge R_3$. Thus, for each $\lambda = (\lambda_0, \dots, \lambda_{2n-1}) \in \mathbb{R}_+^{2n}$ with $\sum_{i=0}^{2n-1} \lambda_i$ sufficiently small, BVP (1.1), (1.2) has at least one solution $u(t)$, which is given by (4.1) with the above solution $v(t)$ and satisfies

$$\begin{aligned} ||u|| &\geq ||v|| - \sum_{j=0}^{n-1}(\lambda_{2j}||y_{2j}|| + \lambda_{2j+1}||y_{2j+1}||) \\ &\geq R_3 - \sum_{j=0}^{n-1}(\lambda_{2j}||y_{2j}|| + \lambda_{2j+1}||y_{2j+1}||) > 0. \end{aligned}$$

This complete the proof of the theorem. □

Proof of Theorem 3.4. For $(t,x) \in [0,1] \times \mathbb{R}$, let

$$f_1(t,x) = \begin{cases} f(t,x), & x \geq 0, \\ -f(t,x), & x < 0. \end{cases} \tag{4.28}$$

By virtue of (H3), we see that $f_1 : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $(-1)^n f_1(t,x) \geq 0$ for $(t,x) \in [0,1] \times \mathbb{R}$. So (H1) with $f = f_1$ is trivially satisfied. Moreover, from $F_0 < \mu_L < f_\infty^*$, it follows that $F_{1,0} < \mu_L < f_{1,\infty}$, where

$$F_{1,0} = \limsup_{x\to 0} \max_{t\in[0,1]} \left|\frac{f_1(t,x)}{x}\right| \quad \text{and} \quad f_{1,\infty} = \liminf_{x\to\infty} \min_{t\in[0,1]} \frac{(-1)^n f_1(t,x)}{x}.$$

Thus, by Theorem 3.2, we know that the BVP consisting of the equation

$$u^{(2n)} + g(t) f_1(t,u) = 0, \ t \in (0,1),$$

and BC (1.2) has at least one nontrivial solution $u_1(t)$ satisfying

$$\begin{aligned} u_1(t) &= \int_0^1 H_n(t,s) g(s) f_1(s,u_1(s)) ds \\ &= \int_0^1 (-1)^n H_n(t,s) g(s) (-1)^n f_1(s,u_1(s)) ds. \end{aligned}$$

Then, by Lemma 2.5, $u_1(t) > 0$ on $(0,1)$. Therefore, from (4.28), $f_1(t,u(t)) = f(t,u(t))$, and so $u_1(t)$ is a positive solution of BVP (1.1), (1.2).

For $(t,x) \in [0,1] \times \mathbb{R}$, let

$$f_2(t,x) = \begin{cases} -f(t,-x), & x \geq 0, \\ f(t,-x), & x < 0. \end{cases} \tag{4.29}$$

In virtue of (H3), we see that $f_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $(-1)^n f_2(t,x) \geq 0$ for $(t,x) \in [0,1] \times \mathbb{R}$. Then, (H1) with $f = f_2$ is trivially satisfied. Moreover, from $F_0 < \mu_L < f_\infty^*$, it follows that $F_{2,0} < \mu_L < f_{2,\infty}$, where

$$F_{2,0} = \limsup_{x\to 0} \max_{t\in[0,1]} \left|\frac{f_2(t,x)}{x}\right| \quad \text{and} \quad f_{2,\infty} = \liminf_{x\to\infty} \min_{t\in[0,1]} \frac{(-1)^n f_2(t,x)}{x}.$$

Thus, as above, we know that the BVP consisting of the equation

$$u^{(2n)} + g(t) f_2(t,u) = 0, \ t \in (0,1),$$

and BC (1.2) has at least one solution $v(t)$ satisfying $v(t) > 0$ on $(0,1)$ and

$$v(t) = \int_0^1 H_n(t,s)g(s)f_2(s,v(s))ds.$$

Then, from (4.29),

$$-v(t) = \int_0^1 H_n(t,s)g(s)f(s,-v(s))ds.$$

Therefore, $u_2(t) := -v(t)$ is a negative solution of BVP (1.1), (1.2), and the theorem is proved. □

Using Theorem 3.3, Theorem 3.5 can be proved by similar ideas as those given in the proof of Theorem 3.4. We omit the details here.

Proof of Lemma 3.7. Let φ_L be given as in Lemma 2.8. Then,

$$\varphi_L(t) = \mu_L(-1)^n \int_0^1 H_n(t,s)g(s)\varphi_L(s)ds \quad \text{for } t \in [0,1].$$

By Lemma 2.5, we have

$$\varphi_L(t) \le \mu_L d_n \int_0^1 \mu(s)g(s)\varphi_L(s)ds \quad \text{on } [0,1] \tag{4.30}$$

and

$$\varphi_L(t) \ge \mu_L c_n \mu(t) \int_0^1 \mu(s)g(s)\varphi_L(s)ds \quad \text{on } [0,1]. \tag{4.31}$$

Thus,

$$\varphi_L(t) \ge \frac{c_n}{d_n}\mu(t)||\varphi_L|| \quad \text{on } [0,1]. \tag{4.32}$$

¿From (4.30), we have

$$\varphi_L(t) \le \mu_L d_n ||\varphi_L|| \int_0^1 \mu(s)g(s)ds \quad \text{on } [0,1].$$

Hence,

$$\mu_L \ge \frac{1}{d_n \int_0^1 \mu(s)g(s)ds} = A.$$

¿From (4.31) and (4.32), we see that

$$\begin{aligned} \varphi_L(t) &\ge \frac{c_n^2}{d_n}\mu_L\mu(t)||\varphi_L|| \int_0^1 \mu^2(s)g(s)ds \\ &\ge \frac{c_n^2}{d_n}\mu_L\underline{\mu}\ ||\varphi_L|| \int_0^1 \mu^2(s)g(s)ds \quad \text{for } t \in [\theta_1,\theta_2]. \end{aligned}$$

Hence,

$$\mu_L \le \frac{d_n}{c_n^2\underline{\mu} \int_0^1 \mu^2(s)g(s)ds} = B.$$

This proves the lemma. □

Finally, in view of Lemma 3.7, Corollaries 3.8–3.11 follow immediately from Theorems 3.2–3.5, respectively.

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