Communications in Applied Analysis **15** (2011), no. 2, 3 and 4, 457–474

VORTEX FILAMENTS AND 1D CUBIC SCHRÖDINGER EQUATIONS: SINGULARITY FORMATION

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ABSTRACT. In this paper we will give an overview on some recent results and work in progress on self-similar solutions of the *Localized Induction Approximation* (LIA) leading to a phenomenon of singularity formation in finite time. A special emphasis will be drawn to the connection of this geometrical flow with certain nonlinear cubic Schrödinger equations in one space dimension through the so-called Hasimoto transformation.

AMS (MOS) Subject Classification. 35Q35, 35J10, and 35Q55.

1. INTRODUCTION

We begin by introducing the physical context where the *localized induction approximation* appears in relation with the dynamics of a vortex filament.

The localized induction approximation, often abbreviated LIA, is a geometric flow in $\mathbb{R}^3$ described by the following system of nonlinear equations:

$$\mathbf{X}_t = \mathbf{X}_s \times \mathbf{X}_{ss}, \tag{1.1}$$

where $\mathbf{X} = \mathbf{X}(s,t)$ represents a curve in $\mathbb{R}^3$ with t and s denoting time and arclength, respectively.

Equation (1.1) was first proposed by Da Rios in 1906, and rediscovered independently by Arms-Hamma and Betchov in the early 1960s (see [9], [2] and [3]), as an approximation model for the evolution of a vortex filament in a 3D-incompressible inviscid fluid. The term localized induction approximation is used to highlight the fact that this approximation only retains the local effects of the Biot-Savart integral. We refer the reader to [7], [19], [1] and [14] for a detailed analysis of the model and its limitations.

Notice that, if one considers a curve $\mathbf{X}$ in $\mathbb{R}^3$ parametrized by arclength, then the associated tangent $\mathbf{T} = \mathbf{X}_s$, normal $\mathbf{n}$ and binormal $\mathbf{b}$ vectors satisfy the Serret-Frenet

Received November 2, 2010

system of equations

$$\begin{cases} \mathbf{T}_s = c\mathbf{n} \\ \mathbf{n}_s = -c\mathbf{T} + \tau\mathbf{b} \\ \mathbf{b}_s = -\tau\mathbf{n}, \end{cases} \tag{1.2}$$

where c and τ are the curvature and torsion associated to the curve $\mathbf{X}$. Thus, equation (1.1) can be rewritten as

$$\mathbf{X}_t = c\mathbf{b}. \tag{1.3}$$

This is the reason why LIA equation is also referred to as *binormal flow*.

The localized induction approximation has also been used to model the motion of a quantum vortex in superfluid ^{4}He. The use of LIA in this setting started with the work by Schwarz in 1985 ([20]). The reader is referred to the two papers by T. Lipniacki in [15] and [16] for further background and references about the use of LIA in the setting of superfluid helium.

In both the classical (ideal fluid) and the superfluid setting, the main advantage of LIA is that it describes the vortex motion in a much simpler way than the Biot-Savart law. However, it is not clear how robust are the solutions if LIA is replaced by the exact Biot-Savart law. Leaving to one side the limitations of the model equation versus the exact Biot-Savart integral, in the present work we will be restricted to LIA.

In this expository article we shall describe some particular solutions of (1.1) which develop a singularity in finite time, and present some recent work and work in progress on the study of the stability of these solutions.

Notice that, since (1.1) is a time-reversible flow (that is, if $\mathbf{X}(s,t)$ is a solution of (1.1), then $\mathbf{X}(-s, t_0 - t)$ for any $t_0 \in \mathbb{R}$, is also a solution), we can look at (1.1) backwards in time, thus the problem of singularity formation is equivalent to considering the problem of existence of solutions of the initial value problem for (1.1) with a singular initial datum $\mathbf{X}(s, 0)$.

Our analytical study of self-similar solutions of LIA started in [12], and carried on in the subsequent papers [10], [4], [5], [6] and [11] (see also [8], [15], and [16]).

In [12], we looked at self-similar solutions of LIA with respect to the unique scaling that preserves the arclength. These solutions are of the form

$$\mathbf{X}_{c_0}(s,t) = \sqrt{t}\mathbf{G}(s/\sqrt{t}), \qquad t > 0, \qquad c_0 > 0, \tag{1.4}$$

with $\mathbf{G}$ the curve determined by $c(s) = c_0$ and $\tau(s) = s/2$, and are found to solve the IVP for (1.1) with an initial curve $\mathbf{X}(s, 0)$ in the shape of a corner.

There is already quite a rich understanding of the dynamical behaviour of these particular solutions. A summary of the results related to these solutions will be described in Section 2.

In [10], solutions of LIA of the form

$$\mathbf{X}(s,t) = e^{\frac{A}{2}\log(t)}\sqrt{t}\mathbf{G}(s/\sqrt{t}), \tag{1.5}$$

with $\mathcal{A}$ a real antisymmetric 3×3 matrix of the form (w.l.o.g)

$$\mathcal{A} = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad a \in \mathbb{R} \tag{1.6}$$

are found to converge to an initial data $\mathbf{X}(s,0)$ which include a wide variety of 3d-spirals, whose rotation axis is the OZ-axis under the condition that the matrix $\mathcal{A}$ is of the form (1.6). In the case when the parameter $a \neq 0$, the singularity of the initial curve $\mathbf{X}(s,0)$ comes from the non-existence of the limit as $s \to 0$ of its tangent vector $\mathbf{T}(s,0)$. In Section 3, we shall focus on describing the results related to these solutions.

As we will see in the next sections, the understanding of the dynamical behaviour of the solutions (1.4) and (1.5), and in particular of their stability, is based on the remarkable connection of LIA with certain nonlinear Schrödinger equations in one dimension established by H. Hasimoto [13]. This connection is made by the so-called Hasimoto transformation and is as follows: Assume that $\mathbf{X} = \mathbf{X}(s,t)$ is a regular solution of (1.1) with a strictly positive curvature at all points[1], and define the *filament function*

$$u(s,t) = c(s,t)\exp\left(i\int_0^s \tau(s',t)\,ds'\right)$$

with c and τ the curvature and torsion of the curve $\mathbf{X}(s,t)$, respectively. Then u solves the nonlinear Schrödinger equation

$$iu_t + u_{ss} + \frac{u}{2}(|u|^2 - A(t)) = 0, \tag{1.7}$$

where $A(t)$ is a time-dependent function which depends on the values of $c(s,t)$ and $\tau(s,t)$ at $s=0$.[2] The filament function $u(s,t)$ is defined through the curvature and torsion, and thus reflects the geometric properties of the filament curve $\mathbf{X}$.

Note that equation (1.7) can be reduced to the cubic nonlinear Schrödinger equation by using an integrating factor. Precisely, the substitution

$$\tilde{u}(s,t) = u(s,t)\exp\left(\frac{i}{2}\int_1^t A(t')\,dt'\right)$$

[1]The restriction that the curvature associated to $\mathbf{X}$ should not vanish can be avoided by using a different parallel frame. Precisely, one can consider the parallel frame of vectors $\{\mathbf{T}, \mathbf{e_1}, \mathbf{e_2}\}$ given by the system of equations:

$$\mathbf{T}_s = \alpha\mathbf{e_1} + \beta\mathbf{e_2}, \qquad \mathbf{e_1}_s = -\alpha\mathbf{T} \qquad \mathbf{e_2}_s = -\beta\mathbf{T},$$

in terms of the quantities α and β. Using the above frame, it can be proved that if $\mathbf{X}(s,t)$ is a regular solution of LIA, and define the function ψ by $\psi = \alpha + i\beta$, then ψ solves the nonlinear equation (1.7), with $A(t) = -|\psi|^2(0,t)/2 - \langle \mathbf{e_1}_t, \mathbf{e_2}\rangle(0,t)$.

[2]Precisely,

$$A(t) = \left(2\frac{c_{ss} - c\tau^2}{c} + c^2\right)(0,t).$$

transforms equation (1.7) into the cubic NLS

$$i\tilde{u}_t + \tilde{u}_{ss} + \frac{1}{2}|\tilde{u}|^2\tilde{u} = 0.$$

This equivalence to NLS implies that there exists a countable number of conserved quantities. It is precisely this connection which endows LIA with a strong structure.

The common ground of solutions of LIA of the form (1.4) and (1.5) is that their associated curvature and torsion functions are of the following self-similar form[3]

$$c(s,t) = \frac{1}{\sqrt{t}}\, c\left(\frac{s}{\sqrt{t}}\right), \qquad \tau(s,t) = \frac{1}{\sqrt{t}}\, \tau\left(\frac{s}{\sqrt{t}}\right) \tag{1.8}$$

(this is why we refer to these solutions as *self-similar*). Thus, their associated filament function $u(s,t)$ is a self-similar solution of the form

$$u(t,s) = \frac{1}{\sqrt{t}}\, u\left(\frac{s}{\sqrt{t}}\right) \qquad \left(\text{with} \qquad u(s) = c(s)e^{i\int_0^s \tau(s')\,ds'}\right) \tag{1.9}$$

of the nonlinear Schrödinger equation

$$iu_t + u_{ss} + \frac{u}{2}(|u|^2 - \frac{A}{t}) = 0 \tag{1.10}$$

for some constant A.

Unlike the case of the self-similar solutions $\mathbf{X}(s,t)$ in (1.4), where we have explicit expressions for the curvature and torsion (see Section 2 below)

$$c(s,t) = \frac{c_0}{\sqrt{t}}, \qquad \tau(s,t) = \frac{s}{2t},$$

in the study of the self-similar solutions (1.5) with $a \neq 0$ the only information we have is that their curvature and torsion are of the self-similiar form (1.8). This is one of the reasons why the study of the solutions (1.5) is more delicate and less understood.

In Section 3, we will see how the study of the properties and dynamical behaviour of solutions of LIA of the form (1.5) relies on a deep understanding of the asymptotic properties of the self-similar solutions of equation (1.10).

The outline of this paper is the following. In Section 2, we shall focus on giving an outline of the known results related to solutions of LIA of the form (1.4) which develop a singularity in the shape of a corner. In Section 3 we shall turn our attention to solutions of LIA of the form (1.5), which develop a spiral singularity. We conclude this paper by describing some recent results and work in progress related to the stability properties of the solutions of LIA described in Sections 2 and 3.

[3]With some abuse of notation, in what follows we denote a function of the two variables s and t evaluated at time $t = 1$ simply by $f(s)$, that is $f(s,1) = f(s)$. With this notation, $c(s)$ and $\tau(s)$ will denote hereafter the curvature and torsion associated to the curve $\mathbf{X}(s,1) = \mathbf{G}(s)$.

2. SINGULARITIES IN THE SHAPE OF A CORNER

Assume that $\mathbf{X}(s,t)$ is a (regular) self-similar solution of

$$\mathbf{X}_t = \mathbf{X}_s \times \mathbf{X}_{ss} \tag{2.1}$$

of the form (1.4); that is

$$\mathbf{X}(s,t) = \sqrt{t}\mathbf{G}\left(\frac{s}{\sqrt{t}}\right), \qquad t > 0 \tag{2.2}$$

for some curve $\mathbf{G}(s)$.

Then, after differentiation, we get that $\mathbf{G}(s) = \mathbf{X}(s,1)$ has to be a solution of

$$\frac{1}{2}\mathbf{G} - \frac{s}{2}\mathbf{G}' = \mathbf{G}' \times \mathbf{G}''. \tag{2.3}$$

A further differentiation yields the equation

$$-\frac{s}{2}\mathbf{G}'' = \mathbf{G}' \times \mathbf{G}''' \tag{2.4}$$

which, by using the Serret-Frenet equations (1.2), rewrites as

$$-\frac{s}{2}c\mathbf{n} = c'\mathbf{b} - c\tau\mathbf{n}.$$

As a consequence, we obtain that the curvature and torsion associated to the curve $\mathbf{G}$ are given by

$$c(s) = c_0 \qquad \text{and} \qquad \tau(s) = \frac{s}{2}, \qquad \text{with} \qquad c_0 > 0$$

(see also [8]).

For fixed $c_0 > 0$, define

$$\mathbf{X}_{c_0}(s,t) = \sqrt{t}\mathbf{G}\left(\frac{s}{\sqrt{t}}\right), \tag{2.5}$$

where $\mathbf{G}$ is the solution of the Serret-Frenet system of equations (1.2) with

$$c(s) = c_0 \qquad \text{and} \qquad \tau(s) = \frac{s}{2}, \tag{2.6}$$

and the initial conditions

$$\mathbf{G}(0) = 2c_0(0,0,1), \quad \mathbf{T}(0) = (1,0,0), \quad \mathbf{n}(0) = (0,1,0), \quad \text{and} \qquad \mathbf{b}(0) = (0,0,1). \tag{2.7}$$

A detailed analysis of the curves defined by (2.5)-(2.7) can be found in [12]. There, among other results, it is proved the following

Theorem 2.1. *For any fixed $c_0 > 0$, $\mathbf{X}_{c_0}(s,t)$ defined by (2.5), (2.6) and (2.7) is a solution of LIA which is C^∞ for $t > 0$. Moreover, there exist vectors unitary vectors $\mathbf{A}^{\pm}_{c_0}$, and vectors $\mathbf{B}^{\pm}_{c_0}$ such that*

i) $|\mathbf{X}_{c_0}(s,t) - \mathbf{A}^{+}_{c_0} s\chi_{[0,+\infty)}(s) - \mathbf{A}^{-}_{c_0} s\chi_{(-\infty,0]}(s)| \le c_0\sqrt{t}$*;*

ii) Let $|s| > \max(2c_0, 4)$*, then the following asymptotics hold as* $s \to \pm\infty$*:*

$$\mathbf{G}(s) = \mathbf{A}^{\pm}_{c_0}(s + 2\frac{c_0^2}{s}) - 4c_0\frac{\mathbf{n}}{s^2} + O(1/s^3)$$

$$\mathbf{T}(s) = \mathbf{A}^{\pm}_{c_0} - 2c_0\frac{\mathbf{b}}{s} + O(1/s^2)$$

$$(\mathbf{n} - i\mathbf{b})(s) = \mathbf{B}^{\pm}_{c_0}e^{is^2/4}e^{ic_0^2\log s} + O(1/s)$$

iii) $\mathbf{A}^{\pm}_{c_0} = (A^{\pm}_{1,c_0}, A^{\pm}_{2,c_0}, A^{\pm}_{3,c_0})$ *are unitary vectors and*

$$A^{+}_{1,c_0} = A^{-}_{1,c_0} = e^{-\frac{c_0^2}{2}\pi}, \qquad A^{+}_{2,c_0} = -A^{-}_{2,c_0} \quad \textit{and} \quad A^{+}_{3,c_0} = -A^{-}_{3,c_0}$$

iv)

$$\sin(\theta/2) = A^{\pm}_{1,c_0} = e^{-\frac{c_0^2}{2}\pi} \tag{2.8}$$

where θ *is the angle between the vectors* $\mathbf{A}^{+}_{c_0}$ *and* $-\mathbf{A}^{-}_{c_0}$*.*

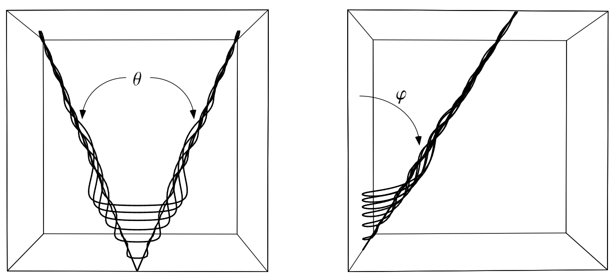

FIGURE 1. Vortex evolution. $\mathbf{X}(s,t)$ wraps asymptotically around two half-lines with angle θ such that $\sin(\theta/2) = e^{-\frac{c_0^2}{2}\pi}$.

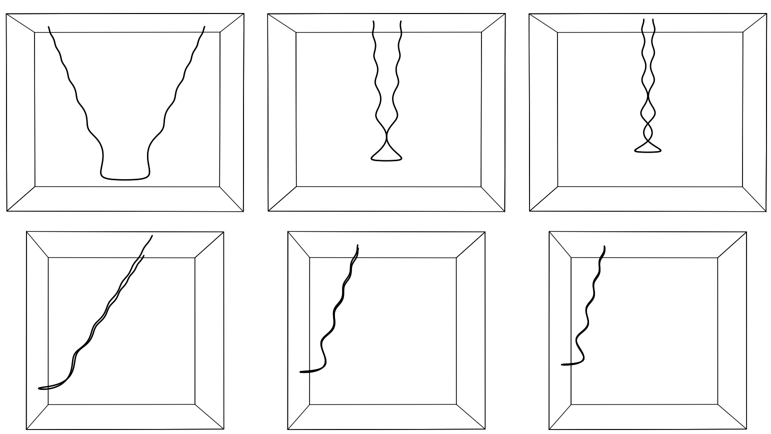

FIGURE 2. The curve $\mathbf{G}_{c_0}(s) = \mathbf{X}_{c_0}(s,1)$ for different values of the parameter $c_0 > 0$.

In Figure 1, the time evolution of $\mathbf{X}_{c_0}(s,t)$, for some $c_0 > 0$ is plotted. Figure 2 illustrates the curve $\mathbf{X}_{c_0}(s,1) = \mathbf{G}(s)$ for different values of the parameter c_0.

Some remarks are in order. First, part *i)* in Theorem 2.1 asserts the convergence of $\mathbf{X}_{c_0}(s,t)$ to a curve which is the union of the two half-lines determined by the vectors $\mathbf{A}_{c_0}^+$ and $\mathbf{A}_{c_0}^-$. Second, from the formulae for $\mathbf{G}$ and $\mathbf{n} - i\mathbf{b}$ in part *ii)* in Theorem 2.1, we observe that the wave-like behaviour of the solutions (see Figure 2) are due to the oscillating behaviour of the normal vector through the term $(4c_0\mathbf{n})/s^2$ in the asymptotic formula for $\mathbf{G}$; therefore the oscillations are of very small amplitude as $s \to \pm\infty$. Finally, formula (2.8) has two important consequences. On the one hand, notice that formula (2.8) implies that $\mathbf{A}^+(c_0) + \mathbf{A}^-(c_0) \neq \mathbf{0}$ for all $c_0 > 0$, and thus the two lines defining the initial data are different (in other words, for any $c_0 > 0$ the initial curve is a curve in the shape of a corner). As a by-product, Theorem 2.1 asserts the existence of regular solutions of LIA which develop a corner singularity in finite time. On the other hand, formula (2.8) allows to prove the following converse: For any given pair of unit vectors $\mathbf{A}^+$ and $\mathbf{A}^-$ different and non-opposite, there exists a regular solution $\mathbf{X}(s,t)$ of LIA for $t > 0$ such that

$$\mathbf{X}(s,0) = \mathbf{A}^+ s\chi_{[0,+\infty)}(s) + \mathbf{A}^- s\chi_{(-\infty,0]}(s).$$

The quantification of the wave-like behaviour of the solutions given in *ii)* in Theorem 2.1, and formula (2.8) rely on precise formulae for the components of the Frenet frame of the regular curve $\mathbf{G}(s)$ in terms of the solutions of the linear ODE

$$\theta'' + i\frac{s}{2}\theta' + \frac{c_0^2}{4}\theta = 0,$$

which can be integrated using Fourier transform methods (see details in [12]).

Since we have a priori knowledge of explicit expressions for the curvature and torsion associated to the curve $\mathbf{X}_{c_0}(s,1) = \mathbf{G}(s)$, there is no need to use the Hasimoto transform to study their properties, and, in this sense, one could say that Theorem 2.1 is not about nonlinear Schrödinger equations. As we will see in Section 4, the Hasimoto transform, and hence the study of the nonlinear Schrödinger equation (1.10), plays an essential role in studying the stability of this family of solutions.

We conclude this section by motivating the solutions that we will continue to consider in Secton 3.

Observe that the filament function associated to the solutions of LIA $\mathbf{X}_{c_0}(s,t)$ is given by

$$u_{c_0}(s,t) = \frac{c_0}{\sqrt{t}} e^{i\frac{s^2}{4t}},$$

which solves

$$iu_t + u_{ss} + \frac{u}{2}(|u|^2 - \frac{A}{t}) = 0, \qquad \text{with} \quad A = c_0 \tag{2.9}$$

and

$$u_{c_0}(s,t) \longrightarrow c\delta_0, \qquad \text{as} \quad t \to 0^+,$$

for some constant $c \in \mathbb{C}$. Notice that the filament function u_{c_0} is of self-similar form

$$u_{c_0}(s,t) = \frac{1}{\sqrt{t}} u(s/\sqrt{t}) \qquad \text{with} \qquad u(s) = c_0 e^{i\frac{s^2}{4}}$$

and, as a consequence, the filament function $u_{c_0}(s,t)$ satisfies the following scaling property

$$u(s,t) = \lambda u(\lambda s, \lambda^2 t), \qquad \forall\, \lambda > 0. \tag{2.10}$$

Due to the invariance of (2.9) under the above scaling, if we look for initial data that can develop into self-similar solutions with respect to this scaling, from (2.10) we see that the initial data has to be homogeneous of degree -1.

The family of solutions $\{\mathbf{X}_{c_0}(s,t)\}_{c_0>0}$ in Theorem 2.1 are associated to a filament function which converges to the delta distribution as $t \to 0^+$, which is homogeneous of degree -1. In order to look for solutions of LIA whose associated filament function may converge as $t \to 0^+$ to a principal value distribution (homogeneous of degree -1), we need to modify the ansantz considered in (2.2).

3. SINGULARITIES IN THE SHAPE OF A 3d-SPIRAL

Given $\mathcal{A} \in \mathcal{M}_{3\times 3}$ real and antisymmetric, we look for solutions of LIA of the form

$$\mathbf{X}(s,t) = e^{\frac{\mathcal{A}}{2}\log t}\sqrt{t}\mathbf{G}(s/\sqrt{t}). \tag{3.1}$$

Notice that, due to the invariance of LIA under rotations, we may assume without loss of generality that the matrix $\mathcal{A}$ is of the form

$$\mathcal{A} = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad a \in \mathbb{R}.$$

It is important to mention that, solutions of the form (3.1) have been also considered by T. Lipniacki (see [15] and [16]) in the setting of the flow defined by

$$\mathbf{X}_t = \beta \mathbf{X}_s \times \mathbf{X}_{ss} + \alpha \mathbf{X}_{ss},$$

modeling the motion of quantum vortices in superfluid ^4He.

Assuming that $\mathbf{X}(s,t)$ defined as (3.1) is a solution of LIA (1.1), we obtain that $\mathbf{G}(s)$ has to be a solution of the following vector ODE:

$$(\mathcal{I}+\mathcal{A})\mathbf{G} - s\mathbf{G}' = 2\mathbf{G}' \times \mathbf{G}'', \qquad |\mathbf{G}'(s)|^2 = 1. \tag{3.2}$$

Straightforward calculations show that equation (3.2) can be written equivalently as

$$\mathbf{G}'' = \frac{1}{2}(\mathcal{I}+\mathcal{A})\mathbf{G} \times \mathbf{G}', \tag{3.3}$$

whenever the initial data $(\mathbf{G}(0), \mathbf{G}'(0))$ satisfies

$$|\mathbf{G}'(0)| = 1 \qquad \text{and} \qquad (\mathcal{I}+\mathcal{A})\mathbf{G}(0)\cdot\mathbf{G}'(0) = 0. \tag{3.4}$$

As a consequence, the problem of finding solutions of LIA of the form (3.1) reduces to the problem of proving the existence of solutions of the IVP (3.3)-(3.4).

The global existence of $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^3)$-solution of the above IVP follows from the classical theory of systems of ODEs and the fact that $|\mathbf{G}'(s)| = 1$ for all $s \in \mathbb{R}$.

As we will see, the evolution of each of the curves $\mathbf{G}(s)$ under the relation (3.1) leads to a solution of LIA which converges as $t \to 0^+$ to different curves in the shape of 3d-spirals. Notice that, since the solutions $\mathbf{X}(s,t)$ are of the form (3.1), the shape of the curve $\mathbf{X}(s,0)$ is related to the asymptotic behaviour of the curve $\mathbf{G}(s)$ as

$s \to \pm\infty$. Hence, it suffices to study the behaviour of $\mathbf{G}(s)$ for large values of $|s|$. To this end, we consider the quantity

$$e^{-\mathcal{A}\log|s|}\frac{\mathbf{G}(s)}{s}, \qquad s \neq 0.$$

Then, by using the equation (3.2), we obtain that

$$\left(e^{-\mathcal{A}\log|s|}\frac{\mathbf{G}}{s}\right)_s = -e^{-\mathcal{A}\log|s|}\frac{(\mathcal{I}+\mathcal{A})\mathbf{G}}{s^2} + e^{-\mathcal{A}\log|s|}\frac{\mathbf{G}'}{s} = -2\, e^{-\mathcal{A}\log|s|}\frac{c\mathbf{b}}{s^2}, \qquad (3.5)$$

where, as before, $\mathbf{b}$ and c denote respectively the binormal vector and curvature associated to $\mathbf{G}$.

Assuming, momentarily that the curvature $c(s)$ is bounded, since $|\mathbf{b}| = 1$, from the above formula it is straightforward to prove the existence of

$$\lim_{s\to\pm\infty} e^{-\mathcal{A}\log|s|}(\mathbf{G}(s)/s) =: \mathbf{A}^{\pm}.$$

Thus integrating (3.5) in the interval $[s, +\infty)$ with $s > 0$ (or $(-\infty, s]$ if $s < 0$) we obtain the following expressions for $\mathbf{G}(s)$

$$\mathbf{G}(s) = se^{\mathcal{A}\log|s|}\mathbf{A}^+ + 2se^{\mathcal{A}\log|s|}\int_s^{+\infty} e^{-\mathcal{A}\log|s'|}\frac{c\mathbf{b}}{(s')^2}\,ds', \qquad s > 0,$$

and

$$\mathbf{G}(s) = se^{\mathcal{A}\log|s|}\mathbf{A}^- - 2se^{\mathcal{A}\log|s|}\int_{-\infty}^{s} e^{-\mathcal{A}\log|s'|}\frac{c\mathbf{b}}{(s')^2}\,ds', \qquad s < 0.$$

The convergence of $\mathbf{X}(s,t) = e^{\frac{\mathcal{A}}{2}\log t}\sqrt{t}\mathbf{G}(s/\sqrt{t})$ as $t \to 0^+$ to a curve of a shape of a 3d-spiral is a direct consequence of these formulae. In [10] the following result is proved:

Proposition 3.1. *For any given $a \in \mathbb{R}$ and $\mathbf{G}(s)$ solution of* (3.3) *associated to a given initial data* $(\mathbf{G}(0), \mathbf{G}'(0))$ *satisfying* (3.4), *define*

$$\mathbf{X}_a(s,t) = e^{\frac{\mathcal{A}}{2}\log t}\sqrt{t}\mathbf{G}\left(\frac{s}{\sqrt{t}}\right), \quad \text{with} \quad \mathcal{A} = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, $\mathbf{X}_a(s,t)$ is an analytic solution of LIA for all $t > 0$, and there exists vectors $\mathbf{A}^+$ and $\mathbf{A}^- \in \mathbb{R}^3$ such that

$$\lim_{t\to0^+}\mathbf{X}_a(s,t) = se^{\mathcal{A}\log|s|}\left(\mathbf{A}^+\chi_{[0,+\infty)}(s) + \mathbf{A}^-\chi_{(-\infty,0]}(s)\right)$$

with

$$|\mathbf{X}_a(s,t) - se^{\mathcal{A}\log|s|}\mathbf{A}^{\pm}| \leq 2\sqrt{t}\left(\sup_{s\in\mathbb{R}}|c(s)|\right).$$

Here, $c(s)$ is the curvature of the curve $\mathbf{G}(s)$.

Figures 3-7 represent different solutions $\mathbf{G}(s)$ of (3.3)-(3.4). The vortex filament wraps asymptotically around different cones, depending on the initial conditions $\mathbf{G}(0)$ and $\mathbf{G}'(0)$, and on the parameter $a \in \mathbb{R}$.

A more exhaustive analysis of the asymptotic behaviour of both $\mathbf{G}(s)$ and $\mathbf{G}'(s)$, where their wave-like behaviour is quantified, was made in [10]. In particular, in [10] the following result is proved:

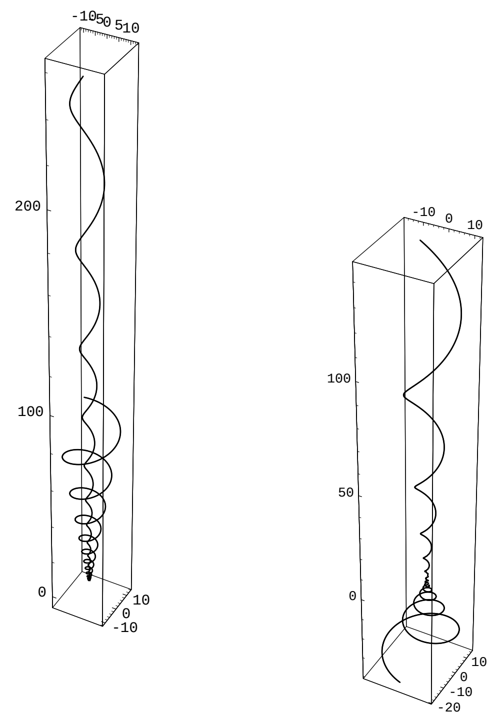

FIGURE 3. Non-Symmetric solutions. $\mathbf{G}(0) = (0,0,\beta)$, $\mathbf{G}'(0) = (1,0,0)$, l.h.s $a = 10$ and $\beta = 1$ and r.h.s $a = 20$, $\beta = 6$.

Theorem 3.2 (see [10, Theorem 1])**.** *Let $G(s)$ the solution of* (3.3) *associated to a given initial data* $(\mathbf{G}(0), \mathbf{G}'(0))$ *satisfying* (3.4). *Then, there exist unique vectors $\mathbf{B}^{\pm}$, with $|\mathbf{B}^{\pm}| = 1$ and $\mathbf{A}^{\pm} = (\mathcal{I} + \mathcal{A})^{-1}\mathbf{B}^{\pm}$ such that the following asymptotics hold as $s \to \pm\infty$*

i) $$\mathbf{G}(s) = s e^{\mathcal{A}\log|s|}\mathbf{A}^{\pm} + \frac{e^{\mathcal{A}\log|s|}}{s}\{2c_{\pm\infty}^2\mathbf{B}^{\pm} - \mathcal{A}\mathbf{B}^{\pm} \times \mathbf{B}^{\pm}\} - 4\frac{c_{\pm\infty}\mathbf{n}}{s^2} + \mathcal{O}\left(\frac{1}{s^3}\right),$$

ii) $$\mathbf{T}(s) = e^{\mathcal{A}\log|s|}\mathbf{B}^{\pm} - 2\frac{c_{\pm\infty}\mathbf{b}}{s} + \mathcal{O}\left(\frac{1}{s^2}\right).$$

iii) Moreover, if $a \neq 0$, $B_3^{\pm} \neq 1$, and $c_{\pm\infty} \neq 0$, then

$$c_{\pm\infty}(\mathbf{n} - i\mathbf{b})(s) = \frac{b_{\pm}e^{ia_{\pm}}}{|\mathcal{A}\mathbf{B}^{\pm}|^2} e^{i(\frac{s^2}{4} - \gamma_{\pm}\log|s|)} e^{\mathcal{A}\log|s|}\{\mathcal{A}\mathbf{B}^{\pm} \times \mathbf{B}^{\pm} - i\mathcal{A}\mathbf{B}^{\pm}\} + \mathcal{O}\left(\frac{1}{|s|}\right).$$

Here,[4] *$a_{\pm} \in [0, 2\pi)$, and $c_{\pm\infty}$, $\gamma_{\pm}$ and $b_{\pm}$ are constants which depend on $B_3^{\pm}$ and A, with $A = aT_3(0) + \frac{1}{4}|(\mathcal{I} + \mathcal{A})\mathbf{G}(0)|^2$.*

[4] $\{\mathbf{T}, \mathbf{n}, \mathbf{b}\}$ is the Frenet frame associated to $\mathbf{G}$, $c(s)$ the curvature function, and $c_{\pm\infty} = \lim_{s\to\infty} c(s)$.

The proof of Proposition 3.1, that is the analysis of $\mathbf{X}(s,t)$ as $t \to 0^+$, depends on the apriori knowledge that the curvature associated to the curve $\mathbf{G}(s)$ is bounded. In the same way, the proof of Theorem 3.2 relies on a very careful analysis of the curvature and torsion related to the curve $\mathbf{G}(s)$. The analysis of the properties of these two quantities is based, through the Hasimoto transformation, on an analysis of self-similar solutions of certain cubic Schrödinger equations. An outline is the following.

Assuming that $\mathbf{X}(s,t)$ is of the form (3.1), then it is straightforward to verify that the curvature and torsion associated to $\mathbf{X}(s,t)$ are of the self-similar form

$$c(s,t) = \frac{1}{\sqrt{t}} c(s/\sqrt{t}), \qquad \tau(s,t) = \frac{1}{\sqrt{t}} \tau(s/\sqrt{t}).$$

As a consequence the associated filament function is given by

$$u(s,t) = \frac{1}{\sqrt{t}} u(s/\sqrt{t}), \qquad \text{with} \qquad {}^5 u(s) = c(s) e^{i \int_0^s \tau(s')\, ds'} \tag{3.6}$$

and

$$A(t) = \frac{A}{t} \qquad \text{with} \qquad A = A(1).$$

Since $\mathbf{X}(s,t)$ is a solution of LIA, through the Hasimoto transformation, we know that its filament function $u(s,t) = \frac{1}{\sqrt{t}} u(s/\sqrt{t})$ solves

$$i u_t + u_{ss} + \frac{u}{2}\left(|u|^2 - \frac{A}{t}\right) = 0. \tag{3.7}$$

Then, the function $u(s)$ is a solution of the complex complex ODE

$$u'' - \frac{i}{2}(u + s u') + \frac{u}{2}(|u|^2 - A) = 0. \tag{3.8}$$

In order to study the solutions of this equation, it turns out that it is more convenient to introduce a new variable f through the definition

$$u(s) = f(s) e^{is^2/4} \qquad \left(\text{i.e} \qquad f(s) = e^{-i\frac{s^2}{4}} u(s)\right). \tag{3.9}$$

Then, u is a solution of (3.8) if and only if f is a solution of

$$f'' + i\frac{s}{2} f' + \frac{f}{2}(|f|^2 - A) = 0 \tag{3.10}$$

Notice that from (3.10) easily follows that

$$\frac{d}{ds}\left(|f|^2 + \frac{1}{4}(|f|^2 - A)^2\right) = 0$$

so, in particular we conclude that $|f|^2 = c^2$ is bounded. A detailed study of the solutions of (3.9) (on which the results in Theorem 3.2 rely) can be found in [10, Theorem 3].

As we have previously mentioned, unlike the solutions considered in Section 2 where we had an a-priori knowledge of an *explicit* formula for their associated filament function (recall that their filament function was $u_{c_0}(s,t) = \frac{c_0}{\sqrt{t}} e^{i\frac{s^2}{4t}}$), in the case of

[5] Notice that $u(s)$ is nothing but the filament function associated to the curve $\mathbf{G}(s) = \mathbf{X}(s,1)$.

solutions of the form (3.1) the only information we have is that their filament function of the self-similar form (3.6). Hence, here we are working in a more general setting. It is precisely due to this generality, and lack of more precise information of the filament function, that the results stated in Theorem 3.2 are weaker than the ones given in Theorem 2.1. In particular, in this general setting, we are not able to find closed formulae for the parameters which characterize these solutions, and the constants and vectors describing the asymptotic behaviour, in terms of the initial conditions of the problem (unlike *iii)* and *iv)* of Theorem 2.1 for the family of curves developing a corner singularity). In some particular situations though, where the solutions have a bit "more" structure, we are able to obtain more specific properties than the ones stated in Theorem 3.2.

More precisely, the following two distinct types of solutions of LIA of the form (3.1) come from the symmetries of the equation symmetries of the equation

$$\mathbf{G}'' = \frac{1}{2}(\mathcal{A} + \mathcal{I})\mathbf{G} \times \mathbf{G}', \tag{3.11}$$

and deserve a special mention. In what follows we will continue to define what we will refer to as *odd solutions* and *mixed-symmetry solutions* of the equation (3.11). We refer the reader to [10] for further properties of these odd and mixed-symmetry solutions.

Odd solutions. For fixed $a \in \mathbb{R}$, let $\mathbf{G}$ a solution of (3.11) with the initial conditions

$$\mathbf{G}(0) = (0,0,0) \qquad \text{and} \qquad \mathbf{G}'(0) = (0, \sqrt{1-\delta^2}, \delta), \qquad -1 \le \delta \le 1. \tag{3.12}$$

Then

$$\mathbf{G}(s) = -\mathbf{G}(-s).$$

In particular, it can be shown (see [11]) that if $\mathbf{G}(s)$ is an odd solution, then the associated function f, through the Hasimoto transformation and the change of variables in (3.9), is an odd solution of (3.10) with $A = a\delta$.

In Figure 4 and Figure 5, we display the graphics of different solutions of (3.11) associated to an initial data of the form (3.12). The right-handside pictures represent the solution near the point $s = 0$.

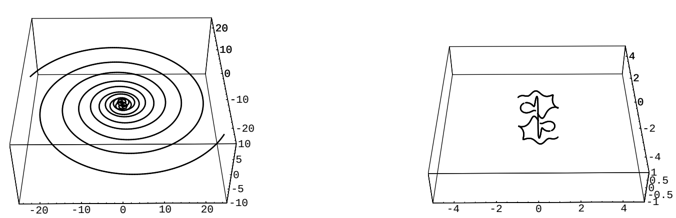

FIGURE 4. Odd solutions. $\mathbf{G}(0) = \mathbf{0}$, $\mathbf{G}'(0) = (0, \sqrt{1-\delta^2}, \delta)$, $a = 10$ and $\delta = 0.956$.

Mixed-symmetry solutions. Given any $a \in \mathbb{R}$, assume $\mathbf{G}(s)$ is a solution of (3.11) with the initial conditions

$$\mathbf{G}(0) = \left(\frac{2c_0}{\sqrt{1+a^2}}, 0, 0\right) \qquad \text{and} \qquad \mathbf{G}'(0) = (0,0,1) \quad (\text{ or } \ \mathbf{G}'(0) = (0,0,-1)). \tag{3.13}$$

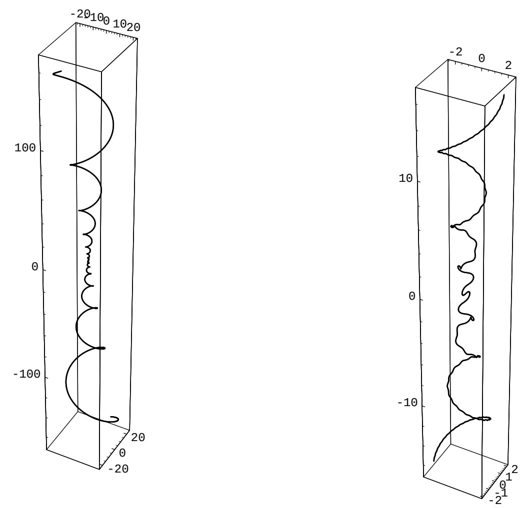

FIGURE 5. Odd solutions. $\mathbf{G}(0) = \mathbf{0}$, $\mathbf{G}'(0) = (0, \sqrt{1-\delta^2}, \delta)$, $a = 10$ and $\delta = -0.1$.

with $c_0 > 0$. Then

$$\begin{cases} G_1(s) = G_1(-s) \\ G_2(s) = G_2(-s) \\ G_3(s) = -G_3(-s). \end{cases}$$

In particular, it can be shown (see [11]) that if $\mathbf{G}(s)$ is a mixed-symmetry solution of LIA, then the associated function f, through the Hasimoto transformation and the change of variables in (3.9), is an even solution of (3.10) with $A = a + c_0^2$ (or $A = -a + c_0^2$).

Two examples of solutions of (3.11) with initial data of the form (3.13) are plotted in Figures 6 and Figure 7.

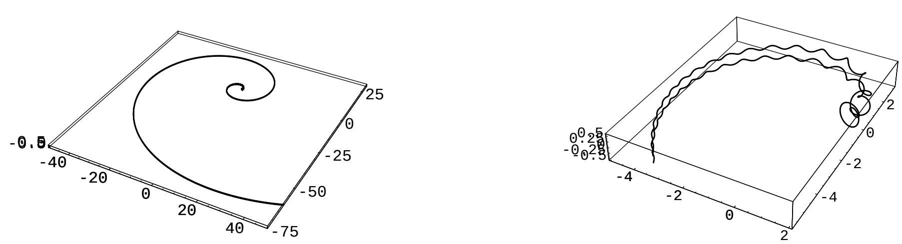

FIGURE 6. Mixed-symmetry solutions. $\mathbf{G}(0) = (2c_0/\sqrt{1+a^2}, 0, 0)$, $\mathbf{G}'(0) = (0, 0, 1)$, $a = 3$, $c_0 = 1.8$.

The "extra" symmetry properties of the above two types of solutions allow us to obtain more specific properties for these solutions. We refer the reader to [10]. In addition, as we will see in the next section, one could expect to say something about the stability of these particular symmetry-solutions.

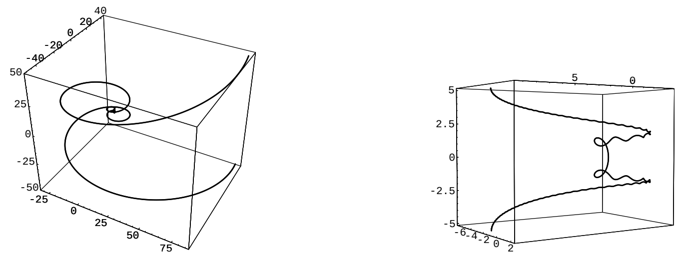

FIGURE 7. Mixed-symmetry solutions. $\mathbf{G}(0) = (2c_0/\sqrt{1+a^2}, 0, 0)$, $\mathbf{G}'(0) = (0,0,1)$, $a = 3$, $c_0 = 0.4$.

4. STABILITY OF THE SINGULAR VORTEX DYNAMICS

A both natural and interesting question about the solutions of LIA described in the previous sections is: are these solutions stable under small perturbations? The aim of this section is to provide a summary of what is known in this context.

The study of the stability properties of the solutions to LIA of the form

$$\mathbf{X}_{c_0}(s,t) = \sqrt{t}\mathbf{G}\left(\frac{s}{\sqrt{t}}\right), \qquad c_0 > 0$$

found in Theorem 2.1 started in [4], and continued in the subsequent papers [5] and [6]. In particular in [5], the authors proved that under a smallness assumption on c_0, there exist regular perturbations of $\mathbf{X}_{c_0}(s,t)$ that converge as $t \to 0^+$ to an initial data $\mathbf{X}_0(s)$ which is close to a curve in the shape of a corner. We refer the reader to [5, Theorem 1.5] for a precise statement of the result.

The construction of the appropriate perturbations of $\mathbf{X}_{c_0}(s,t)$ relies on the construction of appropriate perturbations for the associated filament function. In the case of the self-similar solutions $\mathbf{X}_{c_0}(s,t)$ the filament function is

$$u_{c_0}(s,t) = \frac{c_0}{\sqrt{t}} e^{i\frac{s^2}{4t}},$$

which solves the Schrödinger equation

$$iu_t + u_{ss} + \frac{u}{2}(|u|^2 - \frac{c_0^2}{t}) = 0, \tag{4.1}$$

and

$$u_{c_0}(s,t) \longrightarrow c\delta, \qquad \text{as} \qquad t \to 0^+,$$

for some constant $c \in \mathbb{C}$. Hence, we need to study the initial value problem for the above cubic Schrödinger equation with a "rough" initial data. In order to avoid this obstruction, in [5] the authors use the so-called pseudo-conformal transformation of (4.1). Briefly, given any solution u of (4.1), we define a new unknown v as follows

$$u(s,t) = \mathcal{T}v(s,t) = \frac{e^{i\frac{s^2}{4t}}}{\sqrt{t}} \bar{v}\left(\frac{1}{t}, \frac{x}{t}\right), \tag{4.2}$$

where hereafter the bar denotes complex conjugation. Then, v becomes a solution of

$$iv_t + v_{ss} + \frac{v}{2t}(|v|^2 - c_0^2) = 0, \tag{4.3}$$

and the filament function $u_{c_0}(s,t)$ corresponds to the solution of (4.3) given by

$$v_{c_0}(s,t) = c_0, \qquad t > 0. \tag{4.4}$$

The study of the stability of the solutions $\mathbf{X}_{c_0}(s,t)$ reduces, through the Hasimoto and pseudo-conformal transformations, to the study of the stability properties of the solution $v_{c_0}(s,t) = c_0$ of the equation (4.3). In this setting, we have the following

Theorem 4.1 (see [5, Theorem 1.2])**.** *Let $t_0 > 0$, and given any u_+ define*

$$v_1(s,t) = c_0 + e^{i\frac{c_0^2}{2}\log t} e^{it\partial_s^2} u_+(x).$$

Then, there exists a constant $C_0 > 0$ such that for all $c_0 < C_0$, and for all u_+ small in $L^1 \cap L^2$ with respect to C_0 and t_0, the equation

$$iv_t + v_{ss} + \frac{v}{2t}(|v|^2 - c_0^2) = 0$$

has a unique solution v such that

$$v - v_1 \in \mathcal{C}\left([t_0,\infty), L^2(\mathbb{R})\right) \cap L^4\left([t_0,\infty), L^\infty(\mathbb{R})\right)$$

satisfying, as $t \to +\infty$

$$\|v(t) - v_1(t)\|_{L^2} + \|v(t) - v_1(t)\|_{L^4((t,\infty),L^\infty)} = \mathcal{O}(t^{-\frac{1}{4}}). \tag{4.5}$$

Here, $e^{it\partial_s^2}u_+$ denotes the solution of the free Schrödinger equation with initial data u_+.

It turns out that the decay rate $t^{-\frac{1}{4}}$ in Theorem 4.1 needs to be improved in order to prove that the corner is preserved. This was achieved in [5, Theorem 1.4] imposing more restrictions of the asymptotic data u_+. Precisely, by assuming that u_+ belongs to the Sobolev spaces $H^{-2} \cap H^s \cap W^{s,1}$ with $s \in \mathbb{N}^*$, a decay rate of the order $\mathcal{O}(t^{-\frac{1}{2}})$ was obtained.

The construction of adequate perturbations of the solution $v_{c_0} = c_0$ in the interval $[t_0,\infty)$ with $t_0 \geq 1$ leads to the construction of desired perturbations of $\mathbf{X}_{c_0}$ developing a corner at the initial time $t = 0$ (see [5] for the precise statement and proof of the results).

We conclude this section by briefly mentioning some work in progress in relation with the study of the stability properties of the solutions of LIA of the form

$$\mathbf{X}_a(s,t) = e^{\frac{A}{2}\log t}\sqrt{t}\,\mathbf{G}\left(\frac{s}{\sqrt{t}}\right) \tag{4.6}$$

established in Proposition 3.1 (see notation in Section 3).

As we have learned from the results in [5], a first step in understanding the stability of the above solutions is to prove an analogue of Theorem 4.1 in the setting of these solutions.

Following the arguments in [5], the stability of the solutions of LIA of the form (4.6) is related through the Hasimoto and pseudoconformal transformations to the stability properties of the solution of the equation

$$iv_t + v_{ss} + \frac{v}{2t}(|v|^2 - A) = 0$$

given by

$$v_f(s,t) = \bar{f}\left(\frac{s}{\sqrt{t}}\right), \tag{4.7}$$

with f a solution of the equation (3.10), that is

$$f'' + i\frac{s}{2}f' + \frac{f}{2}(|f|^2 - A) = 0. \tag{4.8}$$

Here, as before, the constant A is a constant which depends on the initial conditions $(\mathbf{G}(0), \mathbf{G}'(0))$. This follows from the fact that $\mathbf{X}_a(s,t)$ has an associated filament function $u(s,t)$ given by

$$u(s,t) = \frac{1}{\sqrt{t}}u(s/\sqrt{t}) = \frac{e^{i\frac{s^2}{4t}}}{\sqrt{t}}\, f\left(\frac{s}{\sqrt{t}}\right)$$

(recall the identities (3.6) and (3.9)), which leads to (4.7) by using the pseudoconformal transformation (4.2).

In order to give a precise statement of our result, we introduce some notation. We denote by $L^2(\langle x\rangle^\gamma)$ the L^2-spaces with Lebesgue measure replaced by $\langle x\rangle^\gamma\, dx = (1+|x|^2)^{\frac{\gamma}{2}}\, dx$, that is

$$L^2(\langle x\rangle^\gamma) = \{\phi : \mathbb{R} \longrightarrow \mathbb{C} \;:\; \|\phi\|_{L^2(\langle x\rangle^\gamma)} = \left(\int_{\mathbb{R}} |\phi(x)|^2 (1+|x|^2)^{\gamma/2}\, dx\right)^{1/2} < \infty\}.$$

Given u_+, and f solution of (4.8) such that $|f|_{+\infty} = |f|_{-\infty}$, we define $\tilde{v}_f$ by

$$\tilde{v}_f(s,t) = v_f(s,t) + e^{i\alpha\log t}\left(e^{it\partial_x^2}u_+\right)(x),$$

with

$$v_f(s,t) = \bar{f}\left(\frac{s}{\sqrt{t}}\right) \qquad \text{and} \qquad \alpha = \frac{1}{2}(2|f|_\infty^2 - A).^6$$

Our next result can be seen as an extension of Theorem 4.1.

Theorem 4.2. *Let $t_0 > 0$, and $0 < \gamma < 1$. There exist (small) positive constants A_0*[7] *and B_0, such that for all $|A| < A_0$, f a solution of*

$$f'' + i\frac{x}{2}f' + \frac{f}{2}(|f|^2 - A) = 0$$

such that $|f|_{-\infty} = |f|_{+\infty}$ with $\|f\|_{L^\infty} \le B_0$, and u_+ small in $L^1 \cap L^2(\langle x\rangle^\gamma)$ with respect to A_0, B_0, t_0, and f, the equation

$$iv_t + v_{xx} + \frac{v}{2t}(|v|^2 - A) = 0 \tag{4.9}$$

has a unique solution $v(t,x)$ in the time interval $[t_0,\infty)$ such that

$$v - \tilde{v}_f \in \mathcal{C}\left([t_0,\infty), L^2(\mathbb{R})\right) \cap L^4\left([t_0,\infty), L^\infty(\mathbb{R})\right).$$

Moreover, the solution v satisfies

$$\|v - \tilde{v}_f\|_{L^2(\mathbb{R})} + \|v - \tilde{v}_f\|_{L^4((t,\infty),L^\infty(\mathbb{R}))} = \mathcal{O}\left(\frac{1}{t^{\frac{\gamma}{4}}}\right), \tag{4.10}$$

[6] If $|f|_{+\infty} = |f|_{-\infty}$, then we will denote $|f|_{\pm\infty}$ simply by $|f|_\infty$.

[7] Since the submission of this paper, some work in progress seems to indicate that the smallness condition on the coefficient A in Theorem 4.2 can be removed (see [11]).

as t goes to infinity.

The previous theorem asserts the existence of the modified wave operator in the time interval $[t_0, \infty)$ with $t_0 > 0$, for any given final data u_+ in $L^1 \cap L^2(\langle x \rangle^\gamma)$ with $0 < \gamma < 1$, and any solution of (4.8) such that $|f|_{+\infty} = |f|_{-\infty}$, under the smallness conditions on the parameter $|A|$, $\|f\|_{L^\infty}$, and the data u_+.

To complete this program and obtain the required stability result in the setting of LIA, we will need, by "undoing" the pseudoconformal and Hasimoto transformation, to construct the associated perturbations of the solutions $\mathbf{X}_a$ of LIA and study the behaviour of the perturbations as t goes to 0. This will be done elsewhere.

One of the key ingredients in the proof of Theorem 4.1 and Theorem 4.2 is the study of the linearized equation (4.9) around the constant solution $v_{c_0}(s,t) = c_0$ $(A = c_0)$ in Theorem 4.1, and around the solution $v_f(s,t) = \bar{f}(s/\sqrt{t})$ in Theorem 4.2. Precisely, the associated linearized equations are given by

$$iz_t + z_{xx} + \frac{c_0^2}{2t}(z + \bar{z}) = 0, \tag{4.11}$$

and

$$iz_t + z_{xx} + \frac{1}{2t}[(2|v_f|^2 - A)z + v_f^2 \bar{z}] = 0, \tag{4.12}$$

respectively. Unlike the equation (4.11), where the coefficients only depend on t allowing the analysis of this equation using the Fourier transform in space, the coefficients in equation (4.12) are both time and space dependent making the study of the linear equation (4.12) more delicate. This is the reason why we have put ourselves in a more simple situation, and reduced our analysis to consider only those self-similar solutions $v_f(s,t) = \bar{f}(s/\sqrt{t})$ associated to a profile f satisfying the property that

$$|f|_{+\infty} = |f|_{-\infty},$$

and in particular those that f is an even or odd function. Even under the above assumption on the profile f, the equation (4.12) is not easy to analyse. In fact, as establised in Theorem 4.2, we need to consider the asymptotic data u_+ to be in $L^1 \cap L^2(\langle x \rangle^\gamma)$ with $0 < \gamma < 1$, rather than in $L^1 \cap L^2$ as in Theorem 4.1, and there is a loss in the decay rate (compare the inequalities (4.5) and (4.10)).

The main obstruction for proving the statement of Theorem 4.2 for an arbitrary profile f comes from the Duhamel term associated to the coefficient $v_f^2/2t$ in (4.12).

Finally, notice that here (see Theorem 4.2) we have only considered the problem of the existence of (modified) wave operators for the Schrödinger equation (4.9) for perturbations of the particular solutions v_f, leaving on a side the more difficult question of the asymptotic completeness of the scattering operators. The asymptotic completeness of the scattering operators for the equation (4.9) $(A = c_0^2)$ for perturbations of the solutions $v_{c_0}(s,t) = c_0$ was studied in [6].

ACKNOWLEDGMENTS

The author would like to thank L. Vega for the long-term and on going collaboration which underpins the contents of this article.

Part of this work was done while the author was visiting the Universidad del País Vasco under the PIC program. Financial support from the program "Euclidean Harmonic Analysis, Nilpotent Lie Groups and PDEs", held in the Centro di Ricerca Matematica Ennio De Giorgi in Pisa, is kindly acknowledged.

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