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OPTIMAL INTERVAL LENGTHS FOR NONLOCAL BOUNDARY VALUE PROBLEMS FOR SECOND ORDER LIPSCHITZ EQUATIONS

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Dedicated to Professor J. R. L. Webb Upon His Retirement.

ABSTRACT. For the second order differential equation, $y'' = f(t, y, y')$, where $f(t, r_1, r_2)$ is Lipschitz continuous in terms of r_1 and r_2, we obtain optimal bounds on the length of intervals on which there exist unique solutions of certain nonlocal three point boundary value problems. These bounds are obtained through an application of the Pontryagin Maximum Principle from the theory of optimal control.

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1. INTRODUCTION

In this paper, we shall be concerned with the second order differential equation,

$$y'' = f(t, y, y'), \quad a < t < b, \tag{1.1}$$

for which the assumptions in the following hypothesis hold throughout.

Hypothesis. $f(t, r_1, r_2) : (a, b) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, and for nonnegative constants k_1 and k_2, satisfies the Lipschitz condition,

$$|f(t, r_1, r_2) - f(t, s_1, s_2)| \le k_1|r_1 - s_1| + k_2|r_2 - s_2|, \tag{1.2}$$

for each $(t, r_1, r_2), (t, s_1, s_2) \in (a, b) \times \mathbb{R}^2$.

In terms of the Lipschitz coefficients k_1 and k_2, we characterize the optimal length for subintervals of (a, b) on which there exist unique solutions of (1.1) satisfying the *nonlocal* three point boundary conditions,

$$y(t_1) = y_1,\ y(t_2) - y(t_3) = y_2, \tag{1.3}$$

where $a < t_1 < t_2 < t_3 < b$, and $y_1, y_2 \in \mathbb{R}$.

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More precisely, we characterize optimal length for subintervals of (a, b) on which solutions (1.1), (1.3) are unique. Existence of solutions follows as a consequence of "uniqueness implies existence" results for (1.1), (1.3) established by Henderson *et al.* [17, 18]. We state this "uniqueness implies existence" result as it appears in [17].

Theorem 1.1. *If solutions for the boundary value problem* (1.1), (1.3) *are unique, when they exist on* (a, b)*, then solutions for the boundary value problem* (1.1), (1.3) *exist on* (a, b).

Because of a close relationship between the boundary value problem (1.1), (1.3) and right focal boundary value problems for (1.1), we will ultimately establish that it suffices for us to characterize optimal length subintervals of (a, b) on which solutions are unique for (1.1) satisfying the right focal boundary conditions,

$$y(t_1) = y_1,\ y'(t_2) = y_2, \tag{1.4}$$

where $a < t_1 < t_2 < b$, and $y_1, y_2 \in \mathbb{R}$. The connection between this characterization and the characterization for our three point nonlocal problems is through a simple application of the Mean Value Theorem to Theorem 1.1.

Theorem 1.2. *If solutions for* (1.1), (1.4) *are unique, when they exist on* (a, b)*, then solutions for* (1.1), (1.3) *are unique, when they exist on* (a, b).

Thus, in view of Theorem 1.2, conditions sufficient to provide uniqueness of solutions, when they exist on (a, b), for two point right focal boundary value problems (1.1), (1.4), are sufficient to provide uniqueness of solutions, when they exist on (a, b) for three point nonlocal boundary value problems (1.1), (1.3).

Much of our process will involve developing a scenario in which the Pontryagin Maximum Principle can be applied. The manner in which we do this has an extensive history, with first motivation found in the papers by Melentsova [29] and Melentsova and Mil'shtein [30, 31]. Those papers were subsequently adapted to the context of several types of boundary value problems by Jackson [22, 23], Eloe and Henderson [6], Hankerson and Henderson [14] and Henderson *et al.* [5, 15, 16, 19].

Nonlocal boundary value problems have also been of tremendous interest both in application and theory, as can be seen in this list of papers and the references therein: [1]–[4], [7, 8], [10]–[13], [20, 21], [24, 25], [27, 28], [32]–[40].

2. OPTIMAL INTERVALS FOR UNIQUENESS OF SOLUTIONS

In this section, we apply the Pontryagin Maximum Principle to obtain a characterization, in terms of the Lipschitz constants k_1 and k_2, for the optimal length of subintervals of (a, b) on which solutions are unique, when they exist for the right focal boundary value problem (1.1), (1.4). This length, it will be argued, is optimal for uniqueness of solutions for the three point nonlocal boundary value problem (1.1), (1.3).

We begin with a set of vector-valued *control functions*

$$\mathcal{U} := \{\mathbf{v}(t) = (v_1(t), v_2(t))^T \in \mathbb{R}^2 \mid v_1(t) \text{ and } v_2(t) \text{ are Lebesgue measurable and } |v_i(t)| \le k_i \text{ on } (a,b), i = 1,2\}.$$

We will be concerned with boundary value problems associated with linear differential equations of the form

$$x'' = u_1(t)x + u_2(t)x', \tag{2.1}$$

where $\mathbf{u}(t) = (u_1(t), u_2(t))^T \in \mathcal{U}$.

If $y(t)$ and $z(t)$ are distinct solutions of (1.1), (1.4), then their difference $x(t) := y(t) - z(t)$ satisfies

$$x(t_1) = x'(t_2) = 0, \tag{2.2}$$

for some $a < t_1 < t_2 < b$, and if

$$u_1(t) := \begin{cases} \frac{f(t,y(t),y'(t)) - f(t,z(t),y'(t))}{y(t)-z(t)}, & y(t) \ne z(t), \\ 0, & y(t) = z(t), \end{cases}$$

and

$$u_2(t) := \begin{cases} \frac{f(t,z(t),y'(t)) - f(t,z(t),z'(t))}{y'(t)-z'(t)}, & y'(t) \ne z'(t), \\ 0, & y'(t) = z'(t), \end{cases}$$

then $u_1(t)$ and $u_2(t)$ are Lebesgue measurable, and $|u_i(t)| \le k_i$, $i = 1, 2$, so that $\mathbf{u}(t) = (u_1(t), u_2(t))^T \in \mathcal{U}$, and $x(t)$ is a nontrivial solution of the boundary value problem (2.1), (2.2). It follows from optimal control theory (cf. Gamkrelidze [9, p. 147] and Lee and Markus [26, p. 259]), there is a boundary value problem in the class (2.1), (2.2), which has a nontrivial time optimal solution; that is, there exists at least one nontrivial $\mathbf{u}^* \in \mathcal{U}$ and points $t_1 \le c < d \le t_2$ such that

$$x'' = u_1^*(t)x + u_2^*(t)x', \tag{2.3}$$

$$x(c) = x'(d) = 0, \tag{2.4}$$

has a nontrivial solution, $x_0(t)$, and $d - c$ is a minimum over all such solutions. For this time optimal solution, $x_0(t)$, we set $\mathbf{x_0}(t) = (x_0(t), x_0'(t))^T$. Then $\mathbf{x_0}(t) \in \mathcal{U}$ is a solution of a first order system,

$$\mathbf{r}' = A[\mathbf{u}^*(t)]\mathbf{r}.$$

By the Pontryagin Maximum Principle, the adjoint system, whose form is given by

$$\mathbf{x}' = -A^T[\mathbf{u}^*(t)]\mathbf{x}, \quad a < t < b, \tag{2.5}$$

has a nontrivial solution, $\mathbf{x}^*(t) = (x_1^*(t), x_2^*(t))^T$ such that, for *a. e.* $t \in [c, d]$,

(i) $x_0'(t)x_1^*(t) + x_0''(t)x_2^*(t) = \langle \mathbf{x_0}'(t), \mathbf{x}^*(t)\rangle = \max_{\mathbf{u}\in\mathcal{U}}\{\langle A[\mathbf{u}(t)]\mathbf{x_0}(t), \mathbf{x}^*(t)\rangle\}$,

(ii) $\langle \mathbf{x_0}'(t), \mathbf{x}^*(t)\rangle$ is a nonnegative constant,

(iii) $x_2^*(c) = x_1^*(d) = 0$.

The maximum condition in (i) can be rewritten as

$$x_2^*(t)[u_1^*(t)x_0(t) + u_2^*(t)x_0'(t)] = \max_{\mathbf{u}\in\mathcal{U}} \{x_2^*(t)[u_1(t)x_0(t) + u_2(t)x_0'(t)]\}, \tag{2.6}$$

for $a.\ e.\ t \in [c,d]$.

By its time optimality and Rolle's Theorem, $x_0(t) \neq 0$, $t \in (c,d]$. We may assume without loss of generality that $x_0(t) > 0$ on $(c,d]$. If $x_2^*(t)$ has no zeros on (c,d), then we can use (2.6) to determine an optimal control $\mathbf{u}^*(t)$, for $a.\ e.\ t \in [c,d]$. We now consider the single signature of $x_2^*(t)$ on (c,d).

To that end, if $\overline{\mathbf{u}} \in \mathcal{U}$ is such that the boundary value problem given by (2.1) and (2.2), for some $a < t_1 < t_2 < b$, has a nontrivial solution, then the adjoint system

$$\alpha' = -A^T[\overline{\mathbf{u}}(t)]\alpha, \quad t \in (a,b), \tag{2.7}$$

$$\alpha_2(t_1) = \alpha_1(t_2) = 0, \tag{2.8}$$

also has a nontrivial solution, and conversely. Hence, the Pontryagin Maximum Principle associates with a time optimal solution of boundary value problem (2.1), (2.2) a time optimal solution of boundary value problem (2.7), (2.8), and conversely. Hence, it follows by its own time optimality that $x_2^*(t)$ does not vanish on (c,d).

Now, $x_0(t) > 0$ on $(c,d]$, and so we have from (2.6) that, if $x_2^*(t) < 0$ on (c,d), then the time optimal solution $x_0(t)$ is a solution of

$$x'' = -k_1 x - k_2|x'| \tag{2.9}$$

on $[c,d]$, while if $x_2^*(t) > 0$ on (c,d), then the time optimal solution $x_0(t)$ is a solution of

$$x'' = k_1 x + k_2|x'| \tag{2.10}$$

on $[c,d]$.

Now, we may assume without loss of generality that $x_0'(c) > 0$ so that, indeed, $x_0(t) > 0$ on $(c,d]$. If $\mathbf{x}^*(t) = (x_1^*(t), x_2^*(t))^T$ is a nontrivial solution of the adjoint system (2.5) associated with $x_0(t)$, then by the Pontryagin Maximum Principle, $x_2^*(c) = x_1^*(d) = 0$, and by its own time optimality, $x_2^*(t) \neq 0$ on (c,d). From the nature of equations (2.9) or (2.10), $x_0''(t)$ is of one sign on (c,d), and so $x_0'(t)$ is strictly monotone on $[c,d]$. From the assumption that $x_0'(c) > 0$ and the boundary conditions $x_0(c) = x_0'(d) = 0$, it follows that $x_0''(t) < 0$ on (c,d), that $x_0'(t) > 0$ on $[c,d)$, and that $x_0(t)$ is a solution of (2.9) on $[c,d]$, and moreover, (2.9) now takes the form

$$x'' = -k_1 x - k_2 x'. \tag{2.11}$$

Our discussion thus far is based on the premise that (1.1) has distinct solutions whose difference satisfies (2.2). Moreover, if the appropriate sign conditions are satisfied by the optimal solution $x_0(t)$ of the boundary value problem (2.1), (2.2) and by the component $x_2^*(t)$ of the solution of the associated adjoint system (2.5), then optimal intervals can be determined on which only trivial solutions exist for boundary value problems (2.9), (2.2) or (2.10), (2.2). Further detailed sign analysis led to determination of optimal intervals on which only trivial solutions exist for only the boundary value problem (2.11), (2.2). As a consequence, solutions of the boundary value problem (1.1), (1.4) will be unique on such subintervals.

Theorem 2.1. *If there is a vector-valued* $\mathbf{u}(t) \in \mathcal{U}$ *for all* $a < t < b$*, for which the boundary value problem* (2.1)*,* (2.2) *has a nontrivial solution for some* $a < t_1 < t_2 < b$*, and if* $x_0(t)$ *is a time optimal solution satisfing* (2.4)*, where* $d - c$ *is a minimum, then* $x_0(t)$ *is a solution of* (2.11) *on* $[c, d]$*.*

Theorem 2.2. *Let* $\ell = \ell(k_1, k_2) > 0$ *be the smallest positive number such that there exists a solution* $x(t)$ *of the boundary value problem for* (2.11) *satisfying*

$$x(0) = x'(\ell) = 0, \tag{2.12}$$

with $x(t) > 0$ *on* $(0, \ell]$*, or* $\ell = \infty$ *if no such solution exists. If* $y(t)$ *and* $z(t)$ *are solutions of the boundary value problem* (1.1)*,* (1.4)*, for some* $a < t_1 < t_2 < b$*, and if* $t_2 - t_1 < \ell$*, it follows that* $y(t) \equiv z(t)$ *on* $[t_1, t_2]$*, and this is best possible for the class of all differential equations satisfying the Lipschitz condition* (1.2)*.*

Proof. Since equation (2.11) is autonomous, translations of solutions are again solutions of (2.11). Hence, it suffices to apply Theorem 2.1 with respect to the boundary conditions at 0 and ℓ.

Now, if $y(t)$ and $z(t)$ are distinct solutions of (1.1) whose difference $w(t) := y(t) - z(t)$ satisfies (2.2), where $t_2 - t_1 < \ell$, then $w(t)$ is a nontrivial solution of the boundary value problem (2.1), (2.2), for appropriately defined $\mathbf{u} \in \mathcal{U}$. Then, from the discussion and Theorem 2.1, equation (2.11) has a nontrivial solution on a subinterval of length less than ℓ. But, by the minimality of ℓ, such a boundary value problem can have only the trivial solution; this is a contradiction. Therefore, solutions of the boundary value problem (1.1), (1.4) are unique, whenever $t_2 - t_1 < \ell$.

This is best possible from the fact that (2.11) satisfies the Lipschitz condition (1.2), and if $\ell \neq \infty$, then $x(t)$ is a nontrivial solution of (2.11) and (2.2) on $[0, \ell]$. The boundary value problem also has the trivial solution. □

Because of the uniqueness relations stated in Theorem 1.2, we can apply Theorem 2.2 to obtain optimal intervals for uniqueness of solutions of the boundary value problem (1.1), (1.3).

Theorem 2.3. *Let* ℓ *be as in Theorem* 2.2*. If* $y(t)$ *and* $z(t)$ *are solutions of the boundary value problem* (1.1)*,* (1.3)*, for some* $a < t_1 < t_2 < t_3 < b$*, and if* $t_3 - t_1 \leq \ell$*,* *it follows that* $y(t) \equiv z(t)$ *on* $[t_1, t_3]$*, and this is best possible for the class of all differential equations satisfying the Lipschitz condition* (1.2)*.*

Proof. In view of Theorem 1.2 and Theorem 2.2, solutions of the boundary value problem (1.1), (1.3) are unique, when $t_3 - t_1 \leq \ell$. To see again that this is best possible, consider the solution $x(t)$ in Theorem 2.2. This is a nontrivial solution of (2.11) and (2.12).

Let $\epsilon > 0$ be sufficiently small that $x(t)$ is a solution of (2.11) on $[0, \ell + \epsilon]$. Now, $x''(t) < 0$ on $[0, \ell + \epsilon]$. From (2.12), $x'(\ell) = 0$, and since $x''(\ell) < 0$, we have that $x(t)$ has a positive maximum at ℓ. So, there exist $0 < \tau_1 < \ell < \tau_2 < \ell + \epsilon$ such that $x(t)$ is a nontrivial solution of (2.11) satisfying $x(0) = x(\tau_1) - x(\tau_2) = 0$. This boundary value problem also has the trivial solution. Since $\epsilon > 0$ was arbitrary, the "best possible" statement follows for uniqueness of solutions of the boundary value problem (1.1), (1.3). □

3. OPTIMAL INTERVALS FOR EXISTENCE OF SOLUTIONS

In this section, we make an immediate application of Theorem 2.3 in conjunction with the uniqueness implies existence result for nonlocal boundary value problems stated in Theorem 1.1.

Theorem 3.1. *Let* ℓ *be as in Theorem* 2.3*. Then, the boundary value problem* (1.1)*,* (1.3) *has a unique solution, provided* $t_3 - t_1 < \ell$*. Moreover, this result is best possible for the class of second order ordinary differential equations* (1.1) *satisfying the Lipschitz condition* (1.2)*.*

Example. In this example, for a few values of k_1 and k_2, we compute the optimal interval length ℓ for which there exist unique solutions for the boundary value problem (1.1), (1.3) on subintervals whose length is no more than ℓ.

In particular, let $x(t)$ be the solution of (2.11) satisfying the initial conditions

$$x(0) = 0, \quad x'(0) = 1,$$

and let $\eta > 0$ be the first positive number such that $x'(\eta) = 0$. Then, $\eta = \ell$ of Theorem 2.3, and we find by elementary methods that,

(i) if $k_1 = 1$ and $k_2 = 2$, then $\eta = \ell = 1$,
(ii) if $k_1 = 1$ and $k_2 = 0$, then $\eta = \ell = \frac{\pi}{2}$,
(iii) if $k_1 = 0$ and $k_2 = 1$, then $\eta = \ell = \infty$.

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