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VARIATIONAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR MONOTONE DISCRETE INCLUSIONS IN HILBERT SPACES

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ABSTRACT. We use variational methods to investigate the existence and uniqueness of solutions of a two-point boundary value problem concerning a system of second order difference inclusions in a Hilbert space, when the operators involved are maximal monotone.

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1. INTRODUCTION

In a Hilbert space H we study the existence of unique solutions of a system of monotone difference inclusions of the form

$$x_{i+1}^n - (1+\theta_i^n)x_i^n + \zeta_i^n x_{i-1}^n + \sum_{j\neq n} \varepsilon_i^j x_i^j \in c_i^n T^n x_i^n + u_i^n, \;\; i = 1, 2, \dots, N, \tag{1.1}$$

satisfying the two-point boundary condition

$$x_0^n =: a^n, \;\; x_{N+1}^n =: b^n. \tag{1.2}$$

Variational inclusion problems are among the most interesting classes of mathematical problems and have a variety of applications in control theory, optimization, economics, transportation equilibrium, engineering science. Many existence results for various variational inclusion (initial and boundary) problems have been studied in the literature. First we refer to the work due to Morosanu [30], which is our main motivation of this paper. Morosanu considered in a Hilbert space a two-point Dirichlet type boundary value problem associated with a second order difference equation of the form

$$\begin{gathered} u_{i+1} - 2u_i + u_{i-1} \in c_i T u_i, \;\; i = 1, 2, \dots, N, \\ u_0 = x, \;\; u_{N+1} = y, \end{gathered} \tag{1.3}$$

In that work it is assumed that T is a set-valued monotone operator and the main result states as follows:

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Theorem 1.1 ([30],Theorem 1.1)**.** *Let x, y be some given elements of the Hilbert space H and consider a positive integer N. Then there exists a unique N-vector $(u_i)_{i=1,2,\dots,N}$ in H^N such that u_i belongs to the domain of the operator T and it satisfies system* (1.3).

Actually Morosanu applies a method consisted of the following steps:

Step 1. Find maximal monotone operators $\mathcal{F}$ and $\mathcal{T}$ defined on a suitable Hilbert space $\mathcal{H}$ such that problem (1.3) is equivalent to a contingent equation of the form

$$0 \in \mathcal{F}(w) + \mathcal{T}(w). \tag{1.4}$$

Step 2. Show that the sum $\mathcal{F} + \mathcal{T}$ is a coercive (or m-accretive) operator.

The conclusion is that the range of the operator $\mathcal{F} + \mathcal{T}$ is the whole space $\mathcal{H}$, thus a (unique) point w exists satisfying (1.4). This fact would imply the result. We should notice that inclusions of the form (1.4) are discussed in the literature by many authors. More general forms of this inclusion is investigated elsewhere, see, e.g., Chidume, Zegeye and Kazmi [15], M. Noor, K. Noor and Rassias [33], Peng and Zhu [34] and the references therein.

In case the operator T is m-accretive, an analogous result is given by Poffald and Reich [35] and [36].

Monotonicity of the operator T is sufficient for the existence of solutions of problems of this kind, provided that the coefficients satisfy some specific conditions. However, in some cases, these problems might not have solutions. Consider, for instance, the problem

$$x_{i+1} - x_{i-1} = sgn x_i + 1, \;\; i = 1, 2, \dots, 2m-1 \tag{1.5}$$

with

$$x_0 = a, \;\; x_{2m} = b, \tag{1.6}$$

where $a < -2m$ and $b > 0$. Equation (1.5) is a selection of $x_{i+1} - x_{i-1} \in \partial x_i + 1, \;\; i = 1, 2, \dots, 2m-1$, where the operator ∂ is the sub-differential of the convex function $\phi(x) := |x|$. (Obviously the operator $x \to \partial|x|$ is maximal monotone and takes values in the interval $[-1, 1]$.) We can easily see that the solution of equation (1.5) is given by

$$x_{2j} := \begin{cases} a + 2j, & \text{if } \; x_1 > 0 \\ a, & \text{if } \; x_1 < 0 \\ a + j, & \text{if } \; x_1 = 0 \end{cases}$$

and

$$x_{2j+1} := x_1,$$

for all indices j. Therefore no solution of (1.5) exists satisfying the conditions (1.6).

A more general situation of the boundary value problem (1.3) was investigated by Apreutesei (see [8], Theorem 6.1.2), where inclusion in (1.3) becomes

$$\begin{gathered} u_{i+1}-(1+\theta_i)u_i+\theta_i u_{i-1}\in c_iTu_i+f_i,\ \ i=1,2,\dots,N \\ u_0=x,\ \ u_{N+1}=y. \end{gathered} \tag{1.7}$$

Here due to the first part of (1.7), Apreutesei does not apply the Morosanu's method, but, instead, he applies a variational method by using the Yosida approximation of the operator T. The method was already used previously elsewhere, see, e.g., Morosanu and Petrovanu [31], Aftabizadeh, Aizicovici and Pavel [2], Moudafi [32], and it works as follows:

Step 1. Formulate suitable maximal monotone operators $\mathcal{F}$ and $\mathcal{T}$ defined on an appropriate Hilbert space $\mathcal{H}$ such that problem (1.7) is equivalent to (1.4).

Since the form of the first part of (1.7) does not help to apply Morosanu's Step 2, we may consider the Yosida approximation $\mathcal{T}_\lambda$ of $\mathcal{T}$ for which it is known that it defines a single valued operator satisfying the convergence

$$\lim_{\lambda\to 0^+}\mathcal{T}_\lambda(x)=\mathcal{T}^0(x),$$

where $\mathcal{T}^0(x)$ is the element of minimum norm of $\mathcal{T}(x)$, see, e.g., Apreutesei [8] and the references therein. Next we discuss the second step:

Step 2. For each $\lambda>0$ show that the operator $\mathcal{F}+\mathcal{T}_\lambda$ is maximal monotone, thus given any $f\in\mathcal{H}$ and $r>0$ there exists a point $x_{\lambda,r}$ in $\mathcal{H}$ such that

$$\mathcal{F}(x_{\lambda,r})+\mathcal{T}_\lambda(x_{\lambda,r})+rx_{\lambda,r}+f=0. \tag{1.8}$$

Step 3. Show that the strong limit

$$\lim_{\lambda\to 0^+}x_{\lambda,r}=x_r,$$

say, exists and, due to (1.8), it satisfies $-f\in\mathcal{F}(x_r)+\mathcal{T}^0(x_r)+rx_r$.

Step 4. Show that the strong limit

$$\lim_{r\to 0^+}x_r=:x^*$$

exists, which leads to the desired (unique) solution of the original problem.

In this work we apply the previous method to a system of difference inclusions, which might be generated from a system of differential inclusions of the form

$$y_n''(t)+p_n(t)y_n'(t)+\sum_{j=1}^{k}q_j(t)y_j(t)\in c_n(t)T^n(y_n(t))+f_n(t),\ \ t\in(0,1) \tag{1.9}$$

$n=1,2,\dots,k$, associated with the boundary conditions

$$y_n(0)=a^n,\ \ y_n(1)=b^n, \tag{1.10}$$

where, for each index n, T^n is a point-to-set valued maximal monotone operator with domain $\mathcal{D}(T^n)\subseteq H\to H$ and such that $0\in\cap_n\mathcal{D}\left(T^n\right)$.

To get a discrete version of problem (1.9) -(1.10) one can follow many methods exhibited in the literature. For instance, one can use equal steps of the time, though we should mention an interesting method where a linear approximation of the second derivative by four consecutive values of the state, together with a nonuniform mesh, which is suggested by Herceg [24]. See, also, Amodio and Sgura [5]. In our situation we prefer to follow the classical way, namely, use equal mesh: Indeed, we consider system (1.9) and, for a fixed $h \in (0,1)$, we let N be the integer part of $\frac{1}{h}$. Take h small enough so that $N > 1$. Define the quantities[1] $x_i^n := y_n(ih),\ \ i = 0, 1, \dots, N$ and $x_{N+1}^n := b^n$ and approximate the first and second derivatives of y_n at ih with the usual quotients $(y_n(ih) - y_n((i-1)h))(h)^{-1}$and $(y_n((i+1)h) - 2y_n(ih) + y_n((i-1)h))(h)^{-2}$, respectively. By this way we (can say that) approximate system (1.9)-(1.10) by the discrete inclusion (1.1) associated with the boundary conditions (1.2). The coefficients $\theta_i^n, \zeta_i^n, \varepsilon_i^n, c_i^n$ are real numbers defined by $\theta_i^n := 1 - hp_n(ih) - (h)^2 q_n(ih)$, $\zeta_i^n := 1 - hp_n(ih)$, $\varepsilon_i^j := (h)^2 q_j(ih)$, $c_i^n := (h)^2 c_n(ih)$, $\ u_i^n := (h)^2 f_n(ih)$.

And although in this case the coefficients satisfy the identity

$$\theta_i^n - \zeta_i^n = -\varepsilon_i^n, \tag{1.11}$$

in the sequel we shall not assume such a condition. Instead, we shall impose such conditions which do not require the truth of (1.11). For instance, in an example discussed in the last section we will use a restriction like $\theta_i^n - \zeta_i^n \geq |\varepsilon_i^n|$.

Obviously systems (1.3) and (1.7) are special cases of (1.1).

A great number of existence results related to Dirichlet boundary value problems concerning difference equations can be found in the literature. For instance, in a Banach space E a two-point discrete boundary value problem of the form

$$\begin{gathered} \Delta^2 y(i-1) + f(i, y(i)) = 0,\ \ i = 1, 2, \dots, N \\ y(0) = 0,\ \ y(N+1) = 0, \end{gathered} \tag{1.12}$$

is discussed in [18], where the function $f : \{0, 1, \dots, N\} \times E \to E$ is weakly-weakly sequentially continuous in the usual sense, i.e. given any sequence (x_n) in E with $w - \lim x_n = x$ it follows that $w - \lim f(i, x_n) = f(i, x),\ \ i = 0, 1, \dots, N$.

The existence results obtained in [18] are proved by using the well known Darbo fixed point theorem (see, e.g., Kubiaczyk [26]) concerning the measure of noncompactness for weakly sequentially continuous mappings. The same problem (but in the real line case) was, also, discussed by Zhang and Liu [45], via the critical point theory (Mountain Pass Theorem). For another similar situation, where the discrete time is replaced by an abstract time - scale, there is an interesting discussion by Agarwal, Otero-Espinar, Perera and Vivero [4], where variational methods are used. The

[1]WARNING: To avoid confusion of exponents, powers and indices we would like to make clear the following:

The exponent n in symbols like x^n will denote the n index of the item x. For the n-th power of x we shall use a parenthesis, like $(x)^n$. Also, symbols like $(a_i)_i$ will mean the k-dimensional vector $(a_1, a_2, \dots, a_k)$.

method of lower and upper solutions in the real number space was used by Drábek, Thompson and Tisdell [19] to investigate existence of solutions of the difference equation

$$y(i+1) - 2y(i) + y(i-1) = f(i, y(i)), \ \ i = 1, 2, \ldots, n$$

with boundary conditions involving two, three and four points. Henderson and Thompson [23] are interested in the existence of triple solutions of problem (1.12). Other forms of Dirichlet, or mixed type two-point or multi-point boundary value problems for difference equations of second or higher order can be found in the literature, see, e.g., [1, 3, 6, 7, 9, 12, 13, 14, 16, 20, 21, 22, 25, 27, 28, 29, 37, 38, 39, 40, 43, 44] and the references therein. For systems of difference equations, the problem of the existence of solutions was investigated elsewhere, see, e.g., [41, 42].

In this work we obtain existence and uniqueness results for the problem (1.1)–(1.2), by using the variational method based on the main idea of Apreutesei [8] consisted of the four previous steps. Notice that we use a fairly different approach, due to the system form of the problem, and also because of the fact that the coefficients θ_i and ζ_i might not be equal.

The paper is organized as follows:

Section 2 is devoted to some auxiliary facts from linear algebra and some topological properties of monotone operators. The basic setting of the problem, as well as the main results are presented in Section 3. In Section 4 we present in the form of lemmas some useful statements corresponding to the various steps of the method, which we apply and in Section 5 we give the proof of the main results. Finally, in Section 6 we present an illustrative example and show that some conditions imposed are sufficient for the existence of solutions. It is, also, shown that in some cases these conditions can not be omitted.

2. SOME AUXILIARY FACTS

Since in the sequel we shall use some facts from elementary algebra, we shall present some of them here, just for completeness of this work.

Let $\langle \cdot, \cdot \rangle > 0$ be the inner product and $\|\cdot\|$ the usual norm in the real k dimensional space.

We denote by M' the transpose of any square matrix M. As it is well known, a $k \times k$- real symmetric (Hermitian) square matrix M is positive definite, if it satisfies $\langle z, Mz \rangle > 0$, for all non-zero vectors z with real entries. Therefore, given a $k \times k$-real square matrix M with its symmetric part $\frac{1}{2}(M + M')$ being positive definite, it holds $\langle x, Mx \rangle = \langle x, \frac{1}{2}(M + M')x \rangle > 0$, for all non-zero real vectors x. Thus we can say that M is positive definite as well.

Notice that a Hermitian matrix is positive definite if and only if all of its eigenvalues are positive.

Lemma 2.1. *Assume that X_i , $i = 1, 2, \dots, N$ are real positive definite $k\times k$-matrices and let ρ_0, ρ_1 be positive real numbers. Also, let $\mathcal{E}$ be the set of all vectors*

$$(v_1, v_2, \dots, v_N)$$

in the cartesian product $\mathbb{R}^k \times \mathbb{R}^k \times \cdots \times \mathbb{R}^k$ (N factors) satisfying inequality

$$\sum_{i=1}^{N}\langle v_i, X_i v_i\rangle \le \rho_0\Big(\sum_{i=1}^{N}(\|v_i\|)^2\Big)^{\frac{1}{2}} + \rho_1. \tag{2.1}$$

Then it holds

$$\|v_i\| \le \frac{1}{2\mu}(\rho_0 + \sqrt{(\rho_0)^2 + 4\mu\rho_1}), \tag{2.2}$$

for all components v_i of all vectors $(v_1, v_2, \dots, v_N)$ in $\mathcal{E}$, where $\mu > 0$ is the smallest eigenvalue of all matrices X_i.

Proof. Indeed, for each $i = 1, 2, \dots, N$ choose an orthonormal basis made up of eigenvectors $v_{1,i}, v_{2,i}, \dots v_{k,i}$ of $\frac{1}{2}(X_i + X_i')$ corresponding to the (positive) eigenvalues $\mu_{1,i}, \mu_{2,i}, \dots, \mu_{k,i}$. Then any $v_i \in \mathbb{R}^k$ can be written in the form

$$v_i = \xi_i^1 v_{1,i} + \xi_i^2 v_{2,i} + \cdots + \xi_i^k v_{k,i}.$$

Therefore, if a point $(v_1, v_2, \dots, v_N) \in \mathcal{E}$ satisfies

$$\sum_{i=1}^{N}\langle v_i, \frac{1}{2}(X_i + X_i')v_i\rangle = \sum_{i=1}^{N}\langle v_i, X_i v_i\rangle \le \rho_0\Big(\sum_{i=1}^{N}(\|v_i\|)^2\Big)^{\frac{1}{2}} + \rho_1,$$

then it holds

$$\mu\sum_{i=1}^{N}(\|v_i\|)^2 \le \sum_{i=1}^{N}\sum_{j=1}^{k}\mu_{j,i}(|\xi_i^j|)^2 = \sum_{i=1}^{N}\langle v_i, \frac{1}{2}(X_i + X_i')v_i\rangle \le \rho_0\Big(\sum_{i=1}^{N}(\|v_i\|)^2\Big)^{\frac{1}{2}} + \rho_1.$$

Thus we get

$$\Big(\sum_{i=1}^{N}(\|v_i\|)^2\Big)^{\frac{1}{2}} \le \frac{1}{2\mu}(\rho_0 + \sqrt{(\rho_0)^2 + 4\mu\rho_1}),$$

from which (2.2) follows. □

Before we present next lemma we recall that a directed set is a nonempty set I, associated with a reflexive and transitive binary relation $\le$, with the additional property that every pair of elements has an upper bound. Directed sets are, for instance, the set of natural numbers with the usual order, as well as the set $\mathbb{N}\times\mathbb{N}$ of pairs of natural numbers with order defined by $(n_0, n_1) \le (m_0, m_1)$, if $n_0 \le m_0$ and $n_1 \le m_1$. Also, a net in a set X is a function $s : I \to X$, where I is a directed set.

Lemma 2.2. *Let X be a real positive definite matrix and consider a directed set I. Assume that s_i, $i \in I$ is a net of positive real numbers satisfying the relation $\lim s_i = 0$. Then every net (x_i) in $\mathbb{R}^k$ with $\langle x_i, X x_i\rangle \le s_i$, for all $i \in I$ satisfies $\lim x_i = 0$.*

Proof. As above, choose an orthonormal basis made up of eigenvectors $v_1, v_2, \ldots v_k$ of $\frac{1}{2}(X+X')$ corresponding to the (positive) eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$. Then we have $x_i = \xi_i^1 v_1 + \xi_i^2 v_2 + \cdots + \xi_i^k v_k$ and therefore

$$s_i \geq \langle x_i, Xx_i\rangle = \langle x_i, \frac{1}{2}(X+X')x_i\rangle = (\xi_i^1)^2\mu_1 + (\xi_i^2)^2\mu_2 + \cdots + (\xi_i^k)^2\mu_k \geq 0.$$

Thus, for each j, it holds $\lim_i \xi_i^j = 0$, which implies that $\lim x_i = 0$. □

Next we recall some facts from the theory of monotone operators. Many sources of such arguments can be found in the literature, but we prefer to use the book due to Cioranescu [17].

Let $S : \mathcal{D}(S) \subset H \to H$ be an operator. Then S^{-1} will denote the inverse relation of S, namely, $S^{-1}(y) := \{x \in H : \ y \in S(x)\}$. Obviously, S^{-1} is a maximal monotone operator if and only if S is maximal monotone. The resolvent with parameter $\lambda > 0$ of S is the single valued operator given by $J_\lambda^S := (I + \lambda S)^{-1}$. This is a nonnexpansive mapping defined everywhere. The Yosida approximate with parameter $\lambda > 0$ of S is a monotone operator given by the type

$$S_\lambda := \frac{1}{\lambda}(I - J_\lambda^S),$$

it is defined everywhere and it is Lipschitz continuous (with Lipschitz parameter $1/\lambda$). Obviously, we have

$$J_\lambda^S x + \lambda S_\lambda x = x, \tag{2.3}$$

for all $x \in H$ and $\lambda > 0$.

Observe that, given $x \in H$ and setting $y := S_\lambda x$, it holds $\lambda y = x - J_\lambda^S x$, or $J_\lambda^S x = x - \lambda y$. Therefore we have that $x \in (I + \lambda S)(x - \lambda y)$, which implies that

$$S_\lambda x = y \in S(x - \lambda y) = S(J_\lambda^S x). \tag{2.4}$$

More properties of these operators can be found in, for example, [10], Proposition 1.1, page 42.

The following fact is significant in our approach and although it can be found elsewhere, we shall present the proof here just for completeness of the present work.

Proposition 2.3. *Every maximal monotone operator is $w-s$-demiclosed and $s-w$-demiclosed.*

Proof. Assume that A is a maximal monotone operator and consider sequences (x_n) and (y_n) such that $y_n \in Ax_n$, $x_n \xrightarrow{w} x_0$ and $y_n \to y_0$. First we claim that

$$\langle x_n, y_n\rangle \to \langle x_0, y_0\rangle .$$

Indeed, we observe that

$$|\langle x_n, y_n\rangle - \langle x_0, y_0\rangle| \leq \|x_n - x_0\|\|y_n - y_0\| + |\langle x_n - x_0, y_0\rangle| + \|x_0\|\|y_n - y_0\|. \tag{2.5}$$

Since $x_n \xrightarrow{w} x_0$ and H is a Hilbert space, it follows, that $\lim_{n\to\infty} \langle x_n - x_0, y_0\rangle = 0$. On the other hand the classical Uniform Boundedness Principle, guarantees that the sequence (x_n) is bounded. Hence, from (2.5), we get $\lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x_0, y_0\rangle$.

We take some $y \in Ax$. Since A is a monotone operator, it holds $\langle x_n - x, y_n - y\rangle \geq 0$. But, clearly, $\langle x_n - x, y_n - y\rangle = \langle x_n, y_n\rangle - \langle x, y\rangle - \langle x_n - x, y\rangle - \langle x, y_n - y\rangle$. Therefore we get

$$\lim_{n\to\infty} \langle x_n - x, y_n - y\rangle = \langle x_0, y_0\rangle - \langle x, y\rangle - \langle x_0 - x, y\rangle - \langle x, y_0 - y\rangle =$$
$$= \langle x_0, y_0 - y\rangle - \langle x, y_0 - y\rangle = \langle x_0 - x, y_0 - y\rangle$$

and so $\langle x_0 - x, y_0 - y\rangle \geq 0$. Due to maximality of A the latter gives $y_0 \in Ax_0$, which shows that A is a $w - s$-demiclosed operator.

The proof that A is $s - w$-demiclosed is similar. □

Finally we state a useful result concerning the sum of two maximal monotone operators:

Lemma 2.4 ([11, Theorem 9, case (i)]). *If $S_i : \mathcal{D}(S_i) \subseteq H \longrightarrow H$, $i = 1, 2$, are maximal monotone operators such that the domain of at least one of them is the entire space H, then their (point-wise) sum*

$$S_1 + S_2 : \mathcal{D}(S_1) \cap \mathcal{D}(S_2) \to H$$

is a maximal monotone operator.

3. SETTING OF THE PROBLEM

Everywhere in this work we shall assume that the coefficients c_i^n and ζ_i^n are positive real numbers. Also, we shall assume that the operators T^n, $n = 1, 2, \ldots, k$ are maximal monotone having the property that $0 \in \cap_n \mathcal{D}(T^n)$ and moreover, without loss of generality, $0 \in T^n(0)$ for all n. If the last condition is not true, then we can set

$$S^n u := T^n u - w_0^n,$$

where w_0^n is the minimum-norm point of $T^n(0)$. (The latter exists because the set $T^n(0)$ is closed and convex.) Then we can take a system of the form (1.1) having in the right side the quantity $c_i^n S^n u_i^n + v_i^n$, where the perturbation v_i^n is the vector $u_i^n + c_i^n w_0^n$.

First, for each $n = 1, 2, \ldots, k$ and $i \in \{1, 2, \ldots, N+1\}$ define the quantities

$$\alpha_0^n := 1, \;\; \alpha_i^n \zeta_i^n = \alpha_{i-1}^n, \tag{3.1}$$

and we formulate the following $k \times k$ matrices:

$$\Theta_i := \begin{pmatrix} \theta_i^1 & 0 & \cdots & 0 \\ 0 & \theta_i^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \theta_i^k \end{pmatrix}, \;\; Z_i := \begin{pmatrix} \zeta_i^1 & 0 & \cdots & 0 \\ 0 & \zeta_i^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \zeta_i^k \end{pmatrix}, \tag{3.2}$$

as well as

$$E_i := \begin{pmatrix} 0 & -\varepsilon_i^2 & \cdots & -\varepsilon_i^k \\ -\varepsilon_i^1 & 0 & \cdots & -\varepsilon_i^k \\ \vdots & \vdots & \cdots & \vdots \\ -\varepsilon_i^1 & -\varepsilon_i^2 & \cdots & 0 \end{pmatrix}.$$

Moreover, for simplicity, we consider the $k \times k$ - matrix $\bar{E}_i := 2I_{k\times k} + \Theta_i + E_i$, as well as the $2k \times 2k$-matrix

$$M := \prod_{i=0}^{N-2} \begin{pmatrix} \bar{E}_{N-i} & -Z_{N-i} \\ I_{k\times k} & 0_{k\times k} \end{pmatrix} =: \begin{pmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{pmatrix},$$

where the blocks M^{ij} $(i, j \in \{1, 2\})$ are $k \times k$ -matrices.

We create the following condition:

$(C1)$ The $k \times k$-matrix

$$M^{11}\bar{E}_1 + M^{12}$$

is non-singular.

Lemma 3.1. *Under the condition* (C1)*, the N-term of the $k \times k$ matrix solution Λ_n of the recursive equation*

$$\Lambda_{i+1} = \bar{E}_{i+1}\Lambda_i - Z_{i+1}\Lambda_{i-1}, \quad i \geq 1 \tag{3.3}$$

with initial values $\Lambda_0 = I_{k\times k}$ and $\Lambda_1 = \bar{E}_1$, is a nonsingular matrix.

Proof. Define the $2k \times k$-matrix

$$\bar{\Lambda}_i := \begin{pmatrix} \Lambda_i \\ \Lambda_{i-1} \end{pmatrix}$$

and observe that

$$\bar{\Lambda}_1 = \begin{pmatrix} \Lambda_1 \\ \Lambda_0 \end{pmatrix} = \begin{pmatrix} \bar{E}_1 \\ I_{k\times k} \end{pmatrix}.$$

Also, from (3.3) it follows that it satisfies the linear recursive relation

$$\bar{\Lambda}_{i+1} = \begin{pmatrix} \bar{E}_{i+1} & -Z_{i+1} \\ I_{k\times k} & 0_{k\times k} \end{pmatrix} \bar{\Lambda}_i.$$

This gives

$$\bar{\Lambda}_N = \prod_{i=0}^{N-2} \begin{pmatrix} \bar{E}_{N-i} & -Z_{N-i} \\ I_{k\times k} & 0_{k\times k} \end{pmatrix} \bar{\Lambda}_1 = M\bar{\Lambda}_1,$$

from which the result follows, since we have $\Lambda_N = M^{11}\bar{E}_1 + M^{12}$. □

Next we define the matrices

$$C_i := \begin{pmatrix} \alpha_i^1 & 0 & \cdots & 0 \\ 0 & \alpha_i^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \alpha_i^k \end{pmatrix}, \quad G_i := \begin{pmatrix} 0 & -|\varepsilon_i^2| & \cdots & -|\varepsilon_i^k| \\ -|\varepsilon_i^1| & 0 & \cdots & -|\varepsilon_i^k| \\ \vdots & \vdots & \cdots & \vdots \\ -|\varepsilon_i^1| & -|\varepsilon_i^2| & \cdots & 0 \end{pmatrix}$$

and we assume the following condition:

$(C2)$: The $k \times k$-matrices

$$C_i(\Theta_i - Z_i + G_i), \;\; i = 1, 2, \dots, N$$

are positive definite.

Remark 3.2. Since the matrices C_i and Z_i are diagonal with positive elements, under the condition (C2), the $k \times k$ matrix $C_i(\Theta_i + G_i)$ is also positive definite.

To proceed, consider the matrix

$$\mathcal{K} := \begin{pmatrix} \kappa^1 & 0 & \cdots & 0 \\ 0 & \kappa^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \kappa^k \end{pmatrix},$$

where

$$\kappa^n := 4 \sum_{m=1}^{N} \alpha_m^n \sum_{i=1}^{m} (\alpha_{i-1}^n)^{-1}$$

and assume the following condition:

$(C3)$: For any $i = 1, 2, \dots, N$ the matrix

$$\tilde{C}_i := C_i \left(I_{k\times k} + \frac{1}{2}\mathcal{K}(\Theta_i - Z_i + G_i) \right), \;\; i = 1, 2, \dots, N$$

is positive definite.

Now we are ready to formulate our main result of this work:

Theorem 3.3. *Assume that $T^j : \mathcal{D}(T^j) \subset H \to H$, $j = 1, 2, \dots, k$ are maximal monotone operators. If the coefficients c_i^n and ζ_i^n are positive real numbers and the conditions* (C1)–(C3) *are satisfied, then the boundary value problem* (1.1)–(1.2) *admits a unique solution.*

We will give the proof of this theorem in Section 5.

4. USEFUL LEMMAS

We start with the following existence result:

Lemma 4.1. *Assume that condition* (C1) *is satisfied. Then given a^n, b^n in H and $(f_i^n)_i \in H^N$, $n = 1, \dots, k$, there exists a unique $(x_i^n)_i \in H^N$ satisfying the (finite) recursive scheme*

$$x_{i+1}^n - (2 + \theta_i^n)x_i^n + \zeta_i^n x_{i-1}^n + \sum_{j \neq n} \varepsilon_i^j x_i^j = f_i^n, \tag{4.1}$$

$i = 1, 2, \dots, N$, $n = 1, 2, \dots, k$, *where $x_0^n := a^n$ and $x_{N+1}^n := b^n$.*

Proof. Denote the column vectors $(x_i^n)_n$, $(a^n)_n$, $(b^n)_n$ and $(f_i^n)_n$ by the symbols X_i, A, B and F_i respectively. Then system (4.1) can be written in the equivalent form

$$X_{i+1} = \bar{E}_i X_i - Z_i X_{i-1} + F_i, \quad i = 1, 2, \ldots, N, \tag{4.2}$$

where

$$X_0 = A \text{ and } X_{N+1} = B.$$

In order to show that the scheme (4.1) is solvable, we will use a simple version of the so called *shooting method*. More precisely, we will show that there is a vector $U := (u^n) \in H^k$, such that, if $X_0 = A$ and $X_1 = U$ and X_i solves (4.2), then $X_{N+1} = B$.

To this end, for each $U = (u^n) \in H^k$ we take the unique solution $X_i(U)$ of the recursive relation (4.2) with $X_0(U) = A$ and $X_1(U) = U$. Then we get

$$X_2(U) = \bar{E}_1 X_1(U) - Z_1 X_0(U) + F_1 = \bar{E}_1 U - Z_1 A + F_1 =: \Lambda_1 U + \Phi_1.$$

Also we have

$$X_3(U) = \bar{E}_2 X_2(U) - Z_2 X_1(U) + F_2 = \bar{E}_2(\Lambda_1 U + \Phi_1) - Z_2 U + F_2 =: \Lambda_2 U + \Phi_2,$$

where the matrices

$$\Lambda_0 = I, \;\; \Lambda_1 := \bar{E}_1, \;\; \Lambda_2 := \bar{E}_2 \Lambda_1 - Z_2 I$$

and

$$\Phi_0 := 0, \;\; \Phi_1 := F_1 - Z_1 A, \;\; \Phi_2 := \bar{E}_2 \Phi_1 + F_2$$

do not depend on U. Similarly, by induction, we obtain

$$X_{i+1}(U) = \Lambda_i U + \Phi_i,$$

where

$$\Lambda_i := \bar{E}_i \Lambda_{i-1} - Z_i \Lambda_{i-2}, \;\; i \geq 2$$

and Φ_i do not depend on U.

Since the requirement

$$B = X_{N+1}(U) = \Lambda_N U + \Phi_N,$$

must be satisfied, it is sufficient the original vector U to be chosen in such way that

$$\Lambda_N U = B - \Phi_N. \tag{4.3}$$

By Lemma 3.1 we know that the matrix Λ_N is invertible, thus from (4.3) a unique vector U can be found. The lemma is proved. □

To formulate the problem under consideration, for each $j = 1, 2, \ldots, k$ denote by $\mathcal{L}_j$ the space $H^N := H \times H \times \cdots \times H$ (N factors) endowed with the inner product

$$\langle x, y \rangle_j := \sum_{i=1}^{N} \alpha_i^j \langle x_i, y_i \rangle_H .$$

We let $\mathcal{L}$ be the product $\prod_j \mathcal{L}^j$ of all these spaces furnished with inner product the sum of the inner products of all factors. In the sequel, for simplicity, we shall not use the index j in the symbol of the inner product $\langle x, y\rangle_j$.

The most important role in our approach will be played by an operator defined in the following way:

Given the vectors $A := (a^n)$ and $B := (b^n)$ appeared in the boundary condition (1.2), define the operator

$$\Omega(\cdot; A, B) : \mathcal{L} \to \mathcal{L}$$

such that for all $j, n = 1, 2, \ldots, k$

$$\Omega((x_i^j); A, B)_1^n = -x_2^n + (1+\theta_1^n)x_1^n - \zeta_1^n a^n - \sum_{j\neq n} \varepsilon_1^j x_1^j,$$

$$\Omega((x_i^j); A, B)_i^n = -x_{i+1}^n + (1+\theta_i^n)x_i^n - \zeta_i^n x_{i-1}^n - \sum_{j\neq n} \varepsilon_i^j x_i^j, \;\; i = 2, \ldots, N-1$$

$$\Omega((x_i^j); A, B)_N^n = -b^n + (1+\theta_N^n)x_N^n - \zeta_N^n x_{N-1}^n - \sum_{j\neq n} \varepsilon_N^j x_N^j.$$

Lemma 4.2. *Assume that the conditions* (C1) *and* (C2) *are satisfied. Then the operator* $\Omega(\cdot; A, B)$ *is maximal monotone.*

Proof. First we will show that $\Omega(\cdot; A, B)$ is a monotone operator acting on the Hilbert space $\mathcal{L}$. We take two points $x := (x_i^j)$ and $y := (y_i^j)$ in $\mathcal{L}$ and define their initial and the final values as follows:

$$x_0^n = y_0^n = a^n \;\text{ and }\; x_{N+1}^n = y_{N+1}^n = b^n.$$

For simplicity we put

$$\varphi_i^j := x_i^j - x_{i-1}^j, \;\; \psi_i^j := y_i^j - y_{i-1}^j, \;\; i = 1, \ldots, N+1$$

and

$$\delta_i^j := \theta_i^j - \zeta_i^j, \;\; i = 0, 1, \ldots, N+1.$$

Then observe that

$$\begin{aligned}
-\alpha_i^n(x_{i+1}^n - y_{i+1}^n) &+ \alpha_i^n(1+\theta_i^n)(x_i^n - y_i^n) \\
&\qquad - \alpha_i^n\zeta_i^n(x_{i-1}^n - y_{i-1}^n) - \sum_{j\neq n} \alpha_i^n \varepsilon_i^j (x_i^j - y_i^j) \\
&= \alpha_i^n\Big[-(\varphi_{i+1}^n - \psi_{i+1}^n) + \theta_i^n(\varphi_i^n - \psi_i^n) \\
&\qquad + \delta_i^n(x_{i-1}^n - y_{i-1}^n) - \sum_{j\neq n} \varepsilon_i^j (x_i^j - y_i^j)\Big] \\
&= \alpha_i^n\Big[-(\varphi_{i+1}^n - \psi_{i+1}^n) + \zeta_i^n(\varphi_i^n - \psi_i^n) \\
&\qquad + \delta_i^n(x_i^n - y_i^n) - \sum_{j\neq n} \varepsilon_i^j (x_i^j - y_i^j)\Big].
\end{aligned}$$

Therefore we have

$$\begin{aligned}&\langle \Omega(x;A,B)-\Omega(y;A,B),x-y\rangle\\&=\sum_{n=1}^{k}\sum_{i=1}^{N}\alpha_i^n\Big[-\left\langle \varphi_{i+1}^n-\psi_{i+1}^n,x_i^n-y_i^n\right\rangle+\zeta_i^n\left\langle \varphi_i^n-\psi_i^n,x_i^n-y_i^n\right\rangle\\&+\delta_i^n\left\langle x_i^n-y_i^n,x_i^n-y_i^n\right\rangle-\sum_{j\neq n}\varepsilon_i^j\left\langle x_i^j-y_i^j,x_i^n-y_i^n\right\rangle\Big]=:\sum_{n=1}^{k}\tau^n.\end{aligned}$$

But, for each $n=1,2,\dots,k$, it holds

$$\begin{aligned}\tau^n&=\sum_{i=1}^{N}\alpha_i^n\Big[-\left\langle \varphi_{i+1}^n-\psi_{i+1}^n,x_i^n-y_i^n\right\rangle+\zeta_i^n\left\langle \varphi_i^n-\psi_i^n,x_i^n-y_i^n\right\rangle\\&\quad+\delta_i^n\left\langle x_i^n-y_i^n,x_i^n-y_i^n\right\rangle-\sum_{j\neq n}\varepsilon_i^j\left\langle x_i^j-y_i^j,x_i^n-y_i^n\right\rangle\Big]\\&=\sum_{i=1}^{N}\alpha_i^n\Big[\left\langle \varphi_{i+1}^n-\psi_{i+1}^n,-(x_i^n-y_i^n)\right\rangle+\zeta_i^n\left\langle \varphi_i^n-\psi_i^n,x_i^n-y_i^n\right\rangle\\&\quad+\delta_i^n(\|x_i^n-y_i^n\|)^2-\sum_{j\neq n}\varepsilon_i^j\left\langle x_i^j-y_i^j,x_i^n-y_i^n\right\rangle\Big]\end{aligned}$$

and hence

$$\begin{aligned}\tau^n=&\sum_{i=1}^{N}\alpha_i^n\Big[\left\langle \varphi_{i+1}^n-\psi_{i+1}^n,\varphi_{i+1}^n-\psi_{i+1}^n\right\rangle-\left\langle \varphi_{i+1}^n-\psi_{i+1}^n,x_{i+1}^n-y_{i+1}^n\right\rangle\\&+\zeta_i^n\left\langle \varphi_i^n-\psi_i^n,x_i^n-y_i^n\right\rangle+\delta_i^n(\|x_i^n-y_i^n\|)^2-\sum_{j\neq n}\varepsilon_i^j\left\langle x_i^j-y_i^j,x_i^n-y_i^n\right\rangle\Big].\end{aligned}$$

The expressions of the last quantities can be simplified if we set

$$w_i^j:=x_i^j-y_i^j\ \text{ and }\ \rho_i^j:=\left\langle \varphi_i^j-\psi_i^j,x_i^j-y_i^j\right\rangle.$$

Indeed, in this case we get

$$\begin{aligned}\tau^n&=\sum_{i=1}^{N}\alpha_i^n(\|\varphi_{i+1}^n-\psi_{i+1}^n\|)^2-\sum_{i=1}^{N}\alpha_i^n\rho_{i+1}^n+\sum_{i=1}^{N}\alpha_i^n\zeta_i^n\rho_i^n\\&\quad+\sum_{i=1}^{N}\alpha_i^n\delta_i^n(\|x_i^n-y_i^n\|)^2-\sum_{i=1}^{N}\sum_{j\neq n}\alpha_i^n\varepsilon_i^j\left\langle x_i^j-y_i^j,x_i^n-y_i^n\right\rangle\\&=\sum_{i=1}^{N}\alpha_i^n(\|\varphi_{i+1}^n-\psi_{i+1}^n\|)^2-\sum_{i=1}^{N}\alpha_i^n\rho_{i+1}^n+\sum_{i=1}^{N}\alpha_i^n\zeta_i^n\rho_i^n\\&\quad+\sum_{i=1}^{N}\alpha_i^n\delta_i^n(\|w_i^n\|)^2-\sum_{i=1}^{N}\sum_{j\neq n}\alpha_i^n\varepsilon_i^j\left\langle w_i^j,w_i^n\right\rangle,\end{aligned}$$

namely

$$\tau^n=\sum_{i=1}^{N}\alpha_i^n(\|\varphi_{i+1}^n-\psi_{i+1}^n\|)^2+\gamma^n,$$

where

$$\gamma^n := -\sum_{i=1}^{N}\alpha_i^n\rho_{i+1}^n + \sum_{i=1}^{N}\alpha_i^n\zeta_i^n\rho_i^n + \sum_{i=1}^{N}\alpha_i^n\delta_i^n(\|w_i^n\|)^2 - \sum_{i=1}^{N}\sum_{j\neq n}\alpha_i^n\varepsilon_i^j\left\langle w_i^j, w_i^n\right\rangle.$$

For the constant γ^n we have

$$\begin{aligned}\gamma^n &= \alpha_1^n\zeta_1^n\rho_1^n + [-\alpha_1^n + \alpha_2^n\zeta_2^n]\rho_2^n + [\alpha_3^n\zeta_3^n - \alpha_2^n]\rho_3^n + \cdots + [\alpha_N^n\zeta_N^n - \alpha_{N-1}^n]\rho_N^n \\ &\quad + \alpha_1^n\delta_1^n(\|w_1^n\|)^2 + \ldots + \alpha_N^n\delta_N^n(\|w_N^n\|)^2 - \sum_{i=1}^{N}\sum_{j\neq n}\alpha_i^n\varepsilon_i^j\left\langle w_i^j, w_i^n\right\rangle \\ &= \alpha_1^n\zeta_1^n\rho_1^n + \alpha_1^n\delta_1^n(\|w_1^n\|)^2 + \ldots + \alpha_N^n\delta_N^n(\|w_N^n\|)^2 - \sum_{i=1}^{N}\sum_{j\neq n}\alpha_i^n\varepsilon_i^j\left\langle w_i^j, w_i^n\right\rangle,\end{aligned}$$

because of the choice of the original constants α_i^n. But it holds

$$\begin{aligned}\rho_1^n &= \langle\varphi_1^n - \psi_1^n, x_1^n - y_1^n\rangle = \langle x_1^n - x_0^n - (y_1^n - y_0^n), x_1^n - y_1^n\rangle = \\ &= \langle x_1^n - a^n - y_1^n + a^n, x_1^n - y_1^n\rangle = (\|x_1^n - y_1^n\|)^2 = (\|w_1^n\|)^2\end{aligned}$$

and therefore

$$\begin{aligned}\sum_{n=1}^{k}\gamma^n &= \sum_{n=1}^{k}\Big[\alpha_1^n\theta_1^n(\|w_1^n\|)^2 + \alpha_2^n\delta_2^n(\|w_2^n\|)^2 + \ldots + \alpha_N^n\delta_N^n(\|w_N^n\|)^2 \\ &\quad - \sum_{i=1}^{N}\sum_{j\neq n}\alpha_i^n\varepsilon_i^j\left\langle w_i^j, w_i^n\right\rangle\Big] \\ &\geq \sum_{n=1}^{k}\Big[\alpha_1^n\theta_1^n(\|w_1^n\|)^2 + \alpha_2^n\delta_2^n(\|w_2^n\|)^2 + \ldots + \alpha_N^n\delta_N^n(\|w_N^n\|)^2 \\ &\quad - \sum_{i=1}^{N}\sum_{j\neq n}\alpha_i^n|\varepsilon_i^j|\|w_i^j\|\|w_i^n\|\Big].\end{aligned}$$

Finally, if for each $i = 1, 2, \ldots, N$ we denote by s_i the vector with entries $\|w_i^j\|$, then the right side of the previous relation takes the form

$$\langle s_1, C_1(\Theta_1 + G_1)s_1\rangle + \sum_{i=2}^{N}\langle s_i, C_i(\Theta_i - Z_i + G_i)s_i\rangle,$$

which is positive, because of assumption (C2) and Remark 3.2. This means that the quantity

$$U = \sum_{n=1}^{k}\sum_{i=1}^{N}\alpha_i^n(\|\varphi_{i+1}^n - \psi_{i+1}^n\|)^2 + \sum_{n=1}^{k}\gamma^n$$

is nonnegative and therefore the operator $\Omega(\cdot; A, B)$ is monotone.

To complete the proof of the lemma it remains to show that the range of the operator $I + \Omega(\cdot; A, B)$ is the entire space, namely

$$\mathcal{R}(I + \Omega(\cdot; A, B)) = \mathcal{L}.$$

To do that it is enough to prove that given $(f_i^j) \in \mathcal{L}$, there exists a point (x_i^j) in $\mathcal{L}$ such that by setting $x_0^j := a^j$ and $x_{N+1}^j := b^j$, system (4.1) is satisfied. But this fact is guaranteed from Lemma 4.1. This completes the proof of the lemma. □

Lemma 4.3. *Assume that condition* (C3) *is satisfied and let* T_λ^j, $(\lambda > 0)$ *be the Yosida approximation of each maximal monotone operator* T^j. *Then, if for given* $r \in (0,1]$ *and* $(u_i^j) \in \mathcal{L}$ *a (finite) sequence* $(x_i^{n,\lambda r})$ *exists satisfying*

$$x_{i+1}^{n,\lambda r} - (1+\theta_i^n)x_i^{n,\lambda r} + \zeta_i^n x_{i-1}^{n,\lambda r} + \sum_{j\neq n} \varepsilon_i^j x_i^{j,\lambda r} = c_i^n T_\lambda^n x_i^{n,\lambda r} + r x_i^{n,\lambda r} + u_i^n, \tag{4.4}$$

for $i = 1, 2, \ldots, N$, *where*

$$x_0^{n,\lambda r} = a^n \ \text{ and } \ x_{N+1}^{n,\lambda r} = b^n,$$

then there exists $K > 0$ *not depending on the indices* i, n *and the parameters* λ, r *such that*

$$\|x_i^{n,\lambda r}\| \leq K. \tag{4.5}$$

Moreover the quantity $(T_\lambda^n x_i^{n,\lambda r})$ *is also bounded with respect to* λ *and* r.

Proof. We use relation (4.4) and have

$$\begin{aligned}
&\sum_{i=1}^N \alpha_i^n \left\langle x_{i+1}^{n,\lambda r} - x_i^{n,\lambda r}, x_i^{n,\lambda r} \right\rangle - \sum_{i=1}^N \alpha_i^n \delta_i^n (\|x_i^{n,\lambda r}\|)^2 \\
&- \sum_{i=1}^N \alpha_i^n \zeta_i^n \left\langle x_i^{n,\lambda r} - x_{i-1}^{n,\lambda r}, x_i^{n,\lambda r} \right\rangle + \sum_{i=1}^N \alpha_i^n \sum_{j\neq n} \varepsilon_i^j \left\langle x_i^{j,\lambda r}, x_i^{n,\lambda r} \right\rangle \\
&= \sum_{i=1}^N \alpha_i^n c_i^n \left\langle T_\lambda^n x_i^{n,\lambda r}, x_i^{n,\lambda r} \right\rangle + r \sum_{i=1}^N \alpha_i^n (\|x_i^{n,\lambda r}\|)^2 + \sum_{i=1}^N \alpha_i^n \left\langle u_i^n, x_i^{n,\lambda r} \right\rangle,
\end{aligned} \tag{4.6}$$

where

$$\delta_i^n := \theta_i^n - \zeta_i^n.$$

Due to the fact that $0 \in \cap_n T^n(0)$, from the monotonicity of T^n, (2.3) and (3.1) it follows that

$$\begin{aligned}
&\sum_{i=1}^N \alpha_i^n \left\langle x_{i+1}^{n,\lambda r} - x_i^{n,\lambda r}, x_i^{n,\lambda r} \right\rangle - \sum_{i=1}^N \alpha_i^n \delta_i^n (\|x_i^{n,\lambda r}\|)^2 \\
&- \sum_{i=1}^N \alpha_{i-1}^n \left\langle x_i^{n,\lambda r} - x_{i-1}^{n,\lambda r}, x_i^{n,\lambda r} \right\rangle + \sum_{i=1}^N \alpha_i^n \sum_{j\neq n} \varepsilon_i^j \left\langle x_i^{j,\lambda r}, x_i^{n,\lambda r} \right\rangle \\
&\geq r \sum_{i=1}^N \alpha_i^n (\|x_i^{n,\lambda r}\|)^2 + \sum_{i=1}^N \alpha_i^n \left\langle u_i^n, x_i^{n,\lambda r} \right\rangle \geq \sum_{i=1}^N \alpha_i^n \left\langle u_i^n, x_i^{n,\lambda r} \right\rangle.
\end{aligned} \tag{4.7}$$

Therefore we have

$$\sum_{i=1}^{N}\alpha_i^n\delta_i^n(\|x_i^{n,\lambda r}\|)^2 \le \sum_{i=1}^{N}\Big[\alpha_i^n\left\langle x_{i+1}^{n,\lambda r}-x_i^{n,\lambda r},x_i^{n,\lambda r}\right\rangle - \alpha_{i-1}^n\left\langle x_i^{n,\lambda r}-x_{i-1}^{n,\lambda r},x_{i-1}^{n,\lambda r}\right\rangle\Big] - \sum_{i=1}^{N}\alpha_{i-1}^n(\|x_i^{n,\lambda r}-x_{i-1}^{n,\lambda r}\|)^2 + \sum_{i=1}^{N}\alpha_i^n\sum_{j\neq n}\varepsilon_i^j\left\langle x_i^{j,\lambda r},x_i^{n,\lambda r}\right\rangle - \sum_{i=1}^{N}\alpha_i^n\left\langle u_i^n,x_i^{n,\lambda r}\right\rangle,$$

from which we get

$$\sum_{i=1}^{N}\alpha_i^n\delta_i^n(\|x_i^{n,\lambda r}\|)^2 \le \alpha_N^n\left\langle b^n-x_N^{n,\lambda r},x_N^{n,\lambda r}\right\rangle - \left\langle x_1^{n,\lambda r}-a^n,a^n\right\rangle - \sum_{i=1}^{N}\alpha_{i-1}^n(\|x_i^{n,\lambda r}-x_{i-1}^{n,\lambda r}\|)^2 + \sum_{i=1}^{N}\alpha_i^n\sum_{j\neq n}\varepsilon_i^j\left\langle x_i^{j,\lambda r},x_i^{n,\lambda r}\right\rangle - \sum_{i=1}^{N}\alpha_i^n\left\langle u_i^n,x_i^{n,\lambda r}\right\rangle. \tag{4.8}$$

From this relation it follows that

$$\sum_{i=1}^{N}\alpha_i^n\delta_i^n(\|x_i^{n,\lambda r}\|)^2 + \sum_{i=1}^{N}\alpha_{i-1}^n(\|x_i^{n,\lambda r}-x_{i-1}^{n,\lambda r}\|)^2 \le \alpha_N^n\|b^n\|\|x_N^{n,\lambda r}\| + \|x_1^{n,\lambda r}\|\|a^n\| + (\|a^n\|)^2 + \sum_{i=1}^{N}\alpha_i^n\sum_{j\neq n}|\varepsilon_i^j|\|x_i^{j,\lambda r}\|\|x_i^{n,\lambda r}\| + \Big(\sum_{i=1}^{N}\alpha_i^n(\|u_i^n\|)^2\Big)^{\frac12}\Big(\sum_{i=1}^{N}\alpha_i^n(\|x_i^{n,\lambda r}\|)^2\Big)^{\frac12},$$

which gives the estimate

$$\begin{aligned}\sum_{i=1}^{N}\alpha_i^n\delta_i^n(\|x_i^{n,\lambda r}\|)^2 &+ \sum_{i=1}^{N}\alpha_{i-1}^n(\|x_i^{n,\lambda r}-x_{i-1}^{n,\lambda r}\|)^2\\ &\le \Big[\sqrt{\alpha_N^n}\|b^n\| + \frac{1}{\sqrt{\alpha_1^n}}\|a^n\| + \Big(\sum_{i=1}^{N}\alpha_i^n(\|u_i^n\|)^2\Big)^{\frac12}\Big]\Big(\sum_{i=1}^{N}\alpha_i^n(\|x_i^{n,\lambda r}\|)^2\Big)^{\frac12}\\ &+ \sum_{i=1}^{N}\alpha_i^n\sum_{j\neq n}|\varepsilon_i^j|\|x_i^{j,\lambda r}\|\|x_i^{n,\lambda r}\| + (\|a^n\|)^2\\ &\le K_1^n\Big(\sum_{i=1}^{N}\alpha_i^n(\|x_i^{n,\lambda r}\|)^2\Big)^{\frac12} + \sum_{i=1}^{N}\alpha_i^n\sum_{j\neq n}|\varepsilon_i^j|\|x_i^{j,\lambda r}\|\|x_i^{n,\lambda r}\| + K_2^n,\end{aligned} \tag{4.9}$$

where

$$K_1^n := \sqrt{\alpha_N^n}\|b^n\| + (\alpha_1^n)^{-1/2}\|a^n\| + \Big(\sum_{i=1}^{N}\alpha_i^n(\|u_i^n\|)^2\Big)^{1/2}$$

and

$$K_2^n := (\|a^n\|)^2$$

are positive constants not depending on λ, r.

Now, from the obvious relation

$$\|x_m^{n,\lambda r}\| = \sum_{i=1}^{m} \left(\left\|x_i^{n,\lambda r}\right\| - \left\|x_{i-1}^{n,\lambda r}\right\| \right) + \|a^n\| \le \sum_{i=1}^{m} \left\|x_i^{n,\lambda r} - x_{i-1}^{\lambda r}\right\| + \|a^n\|$$
$$\le \left(\sum_{i=1}^{m} \frac{1}{\alpha_{i-1}^n} \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \alpha_{i-1}^n (\|x_i^{n,\lambda r} - x_{i-1}^{n,\lambda r}\|)^2 \right)^{\frac{1}{2}} + \|a^n\|,$$

we get

$$\alpha_m^n (\|x_m^{n,\lambda r}\|)^2 \le 2 \sum_{i=1}^{m} \frac{\alpha_m^n}{\alpha_{i-1}^n} \sum_{i=1}^{m} \alpha_{i-1}^n (\|x_i^{n,\lambda r} - x_{i-1}^{n,\lambda r}\|)^2 + 2\alpha_m^n (\|a^n\|)^2,$$

where we have used the elementary inequality $(p+q)^2 \le 2(p^2+q^2)$. Hence we obtain

$$\sum_{m=1}^{N} \alpha_m^n (\|x_m^{n,\lambda r}\|)^2 \le K_3^n \sum_{i=1}^{N} \alpha_{i-1}^n (\|x_i^{n,\lambda r} - x_{i-1}^{n,\lambda r}\|)^2 + K_4^n, \tag{4.10}$$

where the constants

$$K_3^n := 2 \sum_{m=1}^{N} \alpha_m^n \sum_{i=1}^{m} \frac{1}{\alpha_{i-1}^n} = \frac{\kappa^n}{2}$$

and

$$K_4^n := 2 \sum_{m=1}^{N} \alpha_m^n (\|a^n\|)^2$$

do not depend on the parameters λ, r. Combining relation (4.10) with (4.9) we obtain

$$\begin{aligned} K_3^n \sum_{i=1}^{N} \alpha_i^n \delta_i^n (\|x_i^{n,\lambda r}\|)^2 + \sum_{i=1}^{N} \alpha_i^n (\|x_i^{n,\lambda r}\|)^2 \\ - K_3^n \sum_{i=1}^{N} \alpha_i^n \sum_{j \ne n} |\varepsilon_i^j| \|x_i^{j,\lambda r}\| \|x_i^{n,\lambda r}\| \\ \le K_3^n K_1^n \Big(\sum_{i=1}^{N} \alpha_i^n (\|x_i^{n,\lambda r}\|)^2 \Big)^{\frac{1}{2}} + K_3^n K_2^n + K_4^n. \end{aligned} \tag{4.11}$$

Summing up both sides of inequality (4.11) for $n = 1, 2, \ldots, k$ we obtain

$$\sum_{i=1}^{N} \langle v_i^{\lambda r}, \tilde{C}_i v_i^{\lambda r} \rangle \le \rho_0 \Big(\sum_{i=1}^{N} (\|v_i^{\lambda r}\|)^2 \Big)^{\frac{1}{2}} + \rho_1,$$

where $v_i^{\lambda r}$ represents the vector with entries $\|x_i^{n,\lambda r}\|$ and the constants ρ_0, ρ_1 are given by

$$\rho_0 := \alpha \sum_{n=1}^{k} K_3^n K_1^n \ \text{ and } \ \rho_1 := \sum_{n=1}^{k} (K_3^n K_2^n + K_4^n),$$

where

$$\alpha := \max_{i,n} \alpha_i^n.$$

Now, we apply Lemma 2.1 and get inequality (4.5), with K being the real number $(2\mu)^{-1}(\rho_0 + [(\rho_0)^2 + 4\mu\rho_1]^{1/2})$, where μ is the least eigenvalue of all eigenvalues of the matrices $C_i(I_{k\times k} + \frac{1}{2}\mathcal{K}(\Theta_i - Z_i + G_i))$, $i = 1, 2, \ldots, N$. This fact, because of (4.4), guarantees that

$$\|T_\lambda^n x_i^{n,\lambda r}\| \le K_5, \tag{4.12}$$

for some constant K_5 not depending on i, n, λ, r. The lemma is proved. □

Lemma 4.4. *Assume that conditions* (C2) *and* (C3) *are satisfied and let* $(x_i^{n,\lambda r})$ *be a (finite) sequence satisfying equation* (4.4). *Then, given a sequence of real numbers* (λ_ν) *converging to zero, the strong limit*

$$\lim_{\nu} x_i^{n,\lambda_\nu r} =: x_i^{n,r}$$

exists and satisfies

$$x_{i+1}^{n,r} - (1+\theta_i^n)x_i^{n,r} + \zeta_i^n x_{i-1}^{n,r} + \sum_{j\neq n} \varepsilon_i^j x_i^{j,r} \in c_i^n T^n x_i^{n,r} + r x_i^{n,r} + u_i^n, \tag{4.13}$$

for all $n = 1, 2, \ldots, k$ *and*

$$x_0^{n,r} = a^n \ \text{ and } \ x_{N+1}^{n,r} = b^n. \tag{4.14}$$

Proof. First we notice that, since H is a reflexive Banach space, from the classical Banach-Alaoglu theorem, it is guaranteed that every bounded sequence in H has a weakly convergent subsequence. Therefore, due to (4.12), without loss of generality, we can assume that for the given sequence (λ_ν) there is a point $w_i^{n,r}$ such that

$$T_{\lambda_\nu}^n x_i^{n,\lambda_\nu r} \rightharpoonup w_i^{n,r}. \tag{4.15}$$

In order to prove the result, we observe that for two values of ν say ν_1, ν_2 by setting

$$y_i^n := x_i^{n,\lambda_{\nu_1} r} \quad \text{and} \quad z_i^n := x_i^{n,\lambda_{\nu_2} r}$$

from (4.4) it follows that

$$\sum_{i=1}^N \alpha_i^n \left\langle y_{i+1}^n - y_i^n - z_{i+1}^n + z_i^n, y_i^n - z_i^n \right\rangle - \sum_{i=1}^N \alpha_i^n \theta_i^n \left\langle y_i^n - z_i^n, y_i^n - z_i^n \right\rangle +$$
$$\sum_{i=1}^N \alpha_i^n \zeta_i^n \left\langle y_{i-1}^n - z_{i-1}^n, y_i^n - z_i^n \right\rangle + \sum_{i=1}^N \alpha_i^n \sum_{j\neq n} \varepsilon_i^j \left\langle y_i^j - z_i^j, y_i^n - z_i^n \right\rangle$$
$$= \sum_{i=1}^N \alpha_i^n c_i^n \left\langle T_{\lambda_{\nu_1}}^n y_i^n - T_{\lambda_{\nu_2}}^n z_i^n, y_i^n - z_i^n \right\rangle + r \sum_{i=1}^N \alpha_i^n (\|y_i^n - z_i^n\|)^2$$

or

$$\sum_{i=1}^N \alpha_i^n \left\langle y_{i+1}^n - y_i^n - z_{i+1}^n + z_i^n, y_i^n - z_i^n \right\rangle - \sum_{i=1}^N \alpha_i^n \delta_i^n (\|y_i^n - z_i^n\|)^2$$

$$
\begin{aligned}
&-\sum_{i=1}^{N}\alpha_{i-1}^{n}\left\langle y_i^n-y_{i-1}^n-z_i^n+z_{i-1}^n,y_{i-1}^n-z_{i-1}^n\right\rangle\\
&-\sum_{i=1}^{N}\alpha_{i-1}^{n}(\|y_i^n-y_{i-1}^n-z_i^n+z_{i-1}^n\|)^2+\sum_{i=1}^{N}\alpha_i^n\sum_{j\neq n}\varepsilon_i^j\left\langle y_i^j-z_i^j,y_i^n-z_i^n\right\rangle\\
&=\sum_{i=1}^{N}\alpha_i^n c_i^n\left\langle T_{\lambda_{\nu_1}}^n y_i^n-T_{\lambda_{\nu_2}}^n z_i^n,y_i^n-z_i^n\right\rangle+r\sum_{i=1}^{N}\alpha_i^n(\|y_i^n-z_i^n\|)^2.
\end{aligned}
$$

The latter, due to (2.3), gives that

$$
\begin{aligned}
&-\alpha_N^n(\|y_N^n-z_N^n\|)^2-\sum_{i=1}^{N}\alpha_i^n\delta_i^n(\|y_i^n-z_i^n\|)^2\\
&-\sum_{i=1}^{N}\alpha_{i-1}^{n}(\|y_i^n-y_{i-1}^n-z_i^n+z_{i-1}^n\|)^2+\sum_{i=1}^{N}\alpha_i^n\sum_{j\neq n}\varepsilon_i^j\left\langle y_i^j-z_i^j,y_i^n-z_i^n\right\rangle\\
&=\sum_{i=1}^{N}\alpha_i^n c_i^n\left\langle T_{\lambda_{\nu_1}}^n y_i^n-T_{\lambda_{\nu_2}}^n z_i^n,\lambda_{\nu_1}T_{\lambda_{\nu_1}}^n y_i^n-\lambda_{\nu_2}T_{\lambda_{\nu_2}}^n z_i^n\right\rangle\\
&+\sum_{i=1}^{N}\alpha_i^n c_i^n\left\langle T_{\lambda_{\nu_1}}^n y_i^n-T_{\lambda_{\nu_2}}^n z_i^n,J_{\lambda_{\nu_1}}^{T^n} y_i^n-J_{\lambda_{\nu_2}}^{T^n} z_i^n\right\rangle+r\sum_{i=1}^{N}\alpha_i^n(\|y_i^n-z_i^n\|)^2.
\end{aligned}
$$

Now, from (2.4), the monotonicity of T^n and (4.12) it follows that

$$
\begin{aligned}
&-\sum_{i=1}^{N}\alpha_i^n\delta_i^n(\|y_i^n-z_i^n\|)^2-\sum_{i=1}^{N}\alpha_{i-1}^{n}(\|y_i^n-y_{i-1}^n-z_i^n+z_{i-1}^n\|)^2\\
&+\sum_{i=1}^{N}\alpha_i^n\sum_{j\neq n}\varepsilon_i^j\left\langle y_i^j-z_i^j,y_i^n-z_i^n\right\rangle\\
&\geq\sum_{i=1}^{N}\alpha_i^n c_i^n\big(\lambda_{\nu_1}(\|T_{\lambda_{\nu_1}}^n y_i^n\|)^2+\lambda_{\nu_2}(\|T_{\lambda_{\nu_2}}^n z_i^n\|)^2\big)\\
&-(\lambda_{\nu_1}+\lambda_{\nu_2})\sum_{i=1}^{N}\alpha_i^n c_i^n\left\langle T_{\lambda_{\nu_1}}^n y_i^n,T_{\lambda_{\nu_2}}^n z_i^n\right\rangle+r\sum_{i=1}^{N}\alpha_i^n(\|y_i^n-z_i^n\|)^2\\
&\geq-(\lambda_{\nu_1}+\lambda_{\nu_2})(K_5)^2\sum_{i=1}^{N}\alpha_i^n c_i^n+r\sum_{i=1}^{N}\alpha_i^n(\|y_i^n-z_i^n\|)^2.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\sum_{i=1}^{N}\alpha_i^n(r+\delta_i^n)(\|y_i^n-z_i^n\|)^2-\sum_{i=1}^{N}\alpha_i^n\sum_{j\neq n}\varepsilon_i^j\left\langle y_i^j-z_i^j,y_i^n-z_i^n\right\rangle&\\
\leq(\lambda_{\nu_1}+\lambda_{\nu_2})(K_5)^2\sum_{i=1}^{N}\alpha_i^n c_i^n.&
\end{aligned}
$$

This implies that

$$\sum_{i=1}^{N}\alpha_i^n(r+\delta_i^n)(\|y_i^n-z_i^n\|)^2-\sum_{i=1}^{N}\alpha_i^n\sum_{j\neq n}|\varepsilon_i^j|\|y_i^j-z_i^j\|\|y_i^n-z_i^n\|$$
$$\leq(\lambda_{\nu_1}+\lambda_{\nu_2})(K_5)^2\sum_{i=1}^{N}\alpha_i^n c_i^n. \tag{4.16}$$

Taking the sum of both sides of (4.16) for $n=1,2,\dots,k$ we let $v_i^{\nu_1,\nu_2}$ be the vector with arrows $\|y_i^j-z_i^j\|=\|x_i^{j,\lambda_{\nu_1}r}-x_i^{j,\lambda_{\nu_2}r}\|$. Then we obtain that

$$\sum_{i=1}^{N}\langle v_i^{\nu_1,\nu_2},C_i(rI_{k\times k}+\Theta_i-Z_i+G_i)v_i^{\nu_1,\nu_2}\rangle\leq(\lambda_{\nu_1}+\lambda_{\nu_2})K_6, \tag{4.17}$$

for a certain constant K_6 not depending on $\lambda_{\nu_1},\lambda_{\nu_2},r$ and n. Thus, due to Condition $(C2)$, we apply Lemma 2.2 to conclude that the sequence $(x_i^{n,\lambda_\nu r})_\nu$ converges strongly to some point $x_i^{n,r}$, say. From (4.12) and (2.3) it follows that

$$\lim\|J_{\lambda_\nu}^{T^n}x_i^{n,\lambda_\nu r}-x_i^{n,\lambda_\nu r}\|=0$$

and therefore

$$s-\lim J_{\lambda_\nu}^{T^n}x_i^{n,\lambda_\nu r}=x_i^{n,r}.$$

Now, since the operator T^n is maximal monotone, by Proposition 2.3, we know that it is $s-w$-demiclosed. This means that from the inclusion

$$T_{\lambda_\nu}^n x_i^{n,\lambda_\nu r}\in T^n(J_{\lambda_\nu}^{T^n}x_i^{n,\lambda_\nu r})$$

and (4.15), we may pass to the limit to get

$$x_i^{n,r}\in\mathcal{D}(T^n)\quad\text{and}\quad w_i^{n,r}\in T^n(x_i^{n,r}).$$

Considering (4.4) for the values λ_ν and taking into account the demiclosedness, we, finally, obtain that $(x_i^{j,r})$ is a solution of the problem (4.13)–(4.14). ☐

5. PROOF OF THEOREM 3.3

In this section we shall use the previous tools (lemmas etc) to give the proof of the main theorem of this work.

Proof. Consider, as previously, the Yosida approximation T_λ^j of each maximal monotone operator T^j and define the operator $\mathcal{T}_\lambda:\mathcal{L}\to\mathcal{L}$ by

$$(\mathcal{T}_\lambda(u_i^j))_m^n:=c_m^n T_\lambda^n u_m^n,\ \ m=1,2,\dots,N,\ \ n=1,2,\dots,k.$$

It is easy to see that the operator $\mathcal{T}_\lambda$ is maximal operator, since each of its components has the same property.

Then, due to Lemmas 4.2 and 2.4, the operator $\Omega(.;A,B)+\mathcal{T}_\lambda$ is maximal monotone (see, e.g., [17, 165–167, p. 177, exerc. 14 and p. 178 exerc. 18]). This implies that

$$\mathcal{R}(\Omega(.;A,B)+\mathcal{T}_\lambda+rI)=\mathcal{L},$$

namely for a given $(u_i^j) \in \mathcal{L}$, there is a point $(x_i^{j,\lambda r}) \in \mathcal{L}$, satisfying the problem (4.4). Then taking a sequence λ_ν converging to 0, from Lemma 4.4 we conclude that there is a point $(x_i^{j,r}) \in \mathcal{L}$, satisfying relation (4.13) and (4.14).

We claim that given a sequence (r_ν) converging to zero the strong limit of the sequence (x_i^{j,r_ν}) exists and it is a vector (x_i^j) satisfying the boundary value problem (1.1)–(1.2).

Indeed, first of all, from (4.5) and Banach-Alaoglou theorem it follows that there exists some (x_i^j) such that

$$w - \lim x_i^{j,r_\nu} = x_i^j, \;\; i = 1, 2, \ldots, N, \;\; j = 1, 2, \ldots, k. \tag{5.1}$$

Next consider equation (4.13) for two values r_{ν_1} and r_{ν_2} and set

$$\phi_{i,m}^j := x_i^{j,r_{\nu_m}}, \;\; m = 1, 2.$$

There are points

$$u_{i,m}^n \in T^n x_i^{n,r_{\nu_m}}, \;\; m = 1, 2$$

such that

$$\phi_{i+1,m}^n - (1 + \theta_i^n)\phi_{i,m}^n + \zeta_i^n \phi_{i-1,m}^n + \sum_{j \neq n} \varepsilon_i^j \phi_{i,m}^j = c_i^n u_{i,m}^n + r_{\nu_m} \phi_{i,m}^n + u_i^n,$$

for $m = 1, 2$. Subtracting the two equations for $m = 1$ and $m = 2$ we obtain

$$\begin{aligned} \phi_{i+1,1}^n - \phi_{i+1,2}^n - (1 + \theta_i^n)[\phi_{i,1}^n - \phi_{i,2}^n] + \zeta_i^n [\phi_{i-1,1}^n - \phi_{i-1,2}^n] \\ + \sum_{j \neq n} \varepsilon_i^j [\phi_{i,1}^j - \phi_{i,2}^j] = c_i^n [u_{i,1}^n - u_{i,2}^n] + r_{\nu_1} \phi_{i,1}^n - r_{\nu_2} \phi_{i,2}^n, \end{aligned} \tag{5.2}$$

from which it follows that

$$\begin{aligned} &\sum_{i=1}^N \alpha_i^n \left\langle \phi_{i+1,1}^n - \phi_{i,1}^n - \phi_{i+1,2}^n + \phi_{i,2}^n, \phi_{i,1}^n - \phi_{i,2}^n \right\rangle \\ &- \sum_{i=1}^N \alpha_i^n \theta_i^n \left\langle \phi_{i,1}^n - \phi_{i,2}^n, \phi_{i,1}^n - \phi_{i,2}^n \right\rangle + \sum_{i=1}^N \alpha_i^n \zeta_i^n \left\langle \phi_{i-1,1}^n - \phi_{i-1,2}^n, \phi_{i,1}^n - \phi_{i,2}^n \right\rangle \\ &+ \sum_{i=1}^N \alpha_i^n \sum_{j \neq n} \varepsilon_i^j \left\langle \phi_{i,1}^j - \phi_{i,2}^j, \phi_{i,1}^n - \phi_{i,2}^n \right\rangle \\ &= \sum_{i=1}^N \alpha_i^n c_i^n \left\langle u_{i,1}^n - u_{i,2}^n, \phi_{i,1}^n - \phi_{i,2}^n \right\rangle + \sum_{i=1}^N \alpha_i^n \left\langle r_{\nu_1} \phi_{i,1}^n - r_{\nu_2} \phi_{i,2}^n, \phi_{i,1}^n - \phi_{i,2}^n \right\rangle. \end{aligned}$$

Therefore we get

$$\begin{aligned} &\sum_{i=1}^N \alpha_i^n \left\langle \phi_{i+1,1}^n - \phi_{i,1}^n - \phi_{i+1,2}^n + \phi_{i,2}^n, \phi_{i,1}^n - \phi_{i,2}^n \right\rangle - \sum_{i=1}^N \alpha_i^n \delta_i^n (\|\phi_{i,1}^n - \phi_{i,2}^n\|)^2 \\ &\qquad - \sum_{i=1}^N \alpha_{i-1}^n \left\langle \phi_{i,1}^n - \phi_{i-1,1}^n - \phi_{i,2}^n + \phi_{i-1,2}^n, \phi_{i-1,1}^n - \phi_{i-1,2}^n \right\rangle \end{aligned}$$

$$-\sum_{i=1}^{N}\alpha_{i-1}^{n}(\|\phi_{i,1}^{n}-\phi_{i-1,1}^{n}-\phi_{i,2}^{n}+\phi_{i-1,2}^{n}\|)^2$$
$$+\sum_{i=1}^{N}\alpha_i^n\sum_{j\neq n}\varepsilon_i^j\left\langle\phi_{i,1}^{j}-\phi_{i,2}^{j},\phi_{i,1}^{n}-\phi_{i,2}^{n}\right\rangle$$
$$=\sum_{i=1}^{N}\alpha_i^n c_i^n\left\langle u_{i,1}^{n}-u_{i,2}^{n},\phi_{i,1}^{n}-\phi_{i,2}^{n}\right\rangle+\sum_{i=1}^{N}\alpha_i^n r_{\nu_1}(\|\phi_{i,1}^{n}\|)^2$$
$$+\sum_{i=1}^{N}\alpha_i^n r_{\nu_2}(\|\phi_{i,2}^{n}\|)^2-\sum_{i=1}^{N}\alpha_i^n(r_{\nu_1}+r_{\nu_2})\langle\phi_{i,1}^{n},\phi_{i,2}^{n}\rangle$$
$$\geq\sum_{i=1}^{N}\alpha_i^n c_i^n\left\langle u_{i,1}^{n}-u_{i,2}^{n},\phi_{i,1}^{n}-\phi_{i,2}^{n}\right\rangle-\sum_{i=1}^{N}\alpha_i^n(r_{\nu_1}+r_{\nu_2})\langle\phi_{i,1}^{n},\phi_{i,2}^{n}\rangle.$$

Hence, taking into account the boundedness of $(x_i^{n,\lambda r})$, we obtain

$$\sum_{n=1}^{k}\Big[\sum_{i=1}^{N}\alpha_i^n\delta_i^n(\|\phi_{i,1}^{n}-\phi_{i,2}^{n}\|)^2-\sum_{i=1}^{N}\alpha_i^n\sum_{j\neq n}\varepsilon_i^j\left\langle\phi_{i,1}^{j}-\phi_{i,2}^{j},\phi_{i,1}^{n}-\phi_{i,2}^{n}\right\rangle\Big]$$
$$+\sum_{n=1}^{k}\sum_{i=1}^{N}\alpha_{i-1}^{n}(\|\phi_{i,1}^{n}-\phi_{i-1,1}^{n}-\phi_{i,2}^{n}+\phi_{i-1,2}^{n}\|)^2$$
$$\leq\sum_{n=1}^{k}\sum_{i=1}^{N}\alpha_i^n(r_{\nu_1}+r_{\nu_2})\|\phi_{i,1}^{n}\|\|\phi_{i,2}^{n}\|\leq K_7(r_{\nu_1}+r_{\nu_2}),$$

for some constant K_7 not depending on ν_1 and ν_2. This relation implies that

$$\sum_{n=1}^{k}\sum_{i=1}^{N}\alpha_{i-1}^{n}(\|\phi_{i,1}^{n}-\phi_{i-1,1}^{n}-\phi_{i,2}^{n}+\phi_{i-1,2}^{n}\|)^2+\sum_{i=1}^{N}\langle v_i,C_i(\Theta_i-Z_i+G_i)v_i\rangle \leq K_8(r_{\nu_1}+r_{\nu_2}), \tag{5.3}$$

where v_i is the vector with entries $\|\phi_{i,1}^{n}-\phi_{i,2}^{n}\|$. From (5.3) it follows that the sequence

$$(x_i^{j,r_\nu}-x_{i-1}^{j,r_\nu})_\nu$$

converges strongly to v_i^j, say. This and (5.1) guarantee the fact that the limit

$$s-\lim x_1^{j,r_\nu}=s-\lim(x_1^{j,r_\nu}-x_0^{j,r_\nu})+a^j=v_1^j+a^j$$

exists, thus it is equal to x_1^j. Hence, the limit

$$s-\lim x_2^{j,r_\nu}=s-\lim[(x_2^{j,r_\nu}-x_1^{j,r_\nu})+x_1^{j,r_\nu}]=v_2^j+x_1^j,$$

exists and it is equal to x_2^j. Inductively we conclude that the (strong) limit

$$s-\lim x_i^{j,r_\nu}$$

exists and it is equal to x_i^j, for all $i=1,2,\dots,N$ and $j=1,2,\dots,k$. This fact, the $s-w$-demiclosedness of operator T^n and relation (4.13) imply that x_i^j satisfies equation (1.1) associated with the conditions (1.2).

It remains to show uniqueness of the solutions. To this end, assume that x_i^j and y_i^j are two solutions of equation (1.1) satisfying the conditions (1.2). Then the quantity $z_i^j := x_i^j - x_i^j$ satisfies the relation

$$z_{i+1}^n - (1+\theta_i^n)z_i^n + \zeta_i^n z_{i-1}^n + \sum_{j\neq n} \varepsilon_i^j z_i^j \in c_i^n(T^n x_i^n - T^n y_i^n) \tag{5.4}$$

for all $n = 1, 2, \ldots, k$, as well as

$$z_0^n = 0, \quad z_{N+1}^n = 0. \tag{5.5}$$

Multiply both sides of (5.4) with $\alpha_i^n z_i^n$ and reformulate it to get

$$\begin{aligned} \alpha_i^n \langle z_{i+1}^n - z_i^n, z_i^n\rangle - \alpha_i^n(\theta_i^n - \zeta_i^n)(\|z_i^n\|)^2 - \alpha_i^n \zeta_i^n \langle z_i^n - z_{i-1}^n, z_{i-1}^n\rangle \\ - \alpha_i^n\zeta_i^n(\|z_i^n - z_{i-1}^n\|)^2 + \sum_{j\neq n} \alpha_i^n \varepsilon_i^j \langle z_i^j, z_i^n\rangle = \alpha_i^n c_i^n \langle v_{x,i}^n - v_{y,i}^n, x_i^n - y_i^n\rangle, \end{aligned} \tag{5.6}$$

where $v_{x,i}^n$ and $v_{y,i}^n$ are selections of $T^n x_i^n$ and $T^n y_i^n$ respectively. We use the monotonicity of the operators T^n and take the sum of both sides of (5.6) from $i = 1$ up to $i = N$ to obtain

$$\begin{aligned} \sum_{i=1}^N \alpha_i^n \langle z_{i+1}^n - z_i^n, z_i^n\rangle - \sum_{i=1}^N \alpha_{i-1}^n \langle z_i^n - z_{i-1}^n, z_{i-1}^n\rangle - \sum_{i=1}^N \alpha_i^n(\theta_i^n - \zeta_i^n)(\|z_i^n\|)^2 \\ - \sum_{i=1}^N \alpha_{i-1}^n(\|z_i^n - z_{i-1}^n\|)^2 + \sum_{i=1}^N \sum_{j\neq n} \alpha_i^n \varepsilon_i^j \langle z_i^j, z_i^n\rangle \geq 0, \end{aligned}$$

or, due to (5.5),

$$\begin{aligned} -\alpha_N^n(\|z_N^n\|)^2 - \sum_{i=1}^N \alpha_i^n(\theta_i^n - \zeta_i^n)(\|z_i^n\|)^2 - \sum_{i=1}^N \alpha_{i-1}^n(\|z_i^n - z_{i-1}^n\|)^2 \\ + \sum_{i=1}^N \sum_{j\neq n} \alpha_i^n \varepsilon_i^j \langle z_i^j, z_i^n\rangle \geq 0. \end{aligned} \tag{5.7}$$

Next, take the sum of both sides in (5.7) from $n = 1$ up to k and get

$$\begin{aligned} A := \sum_{n=1}^k \alpha_N^n(\|z_N^n\|)^2 + \sum_{n=1}^k \sum_{i=1}^N \alpha_i^n(\theta_i^n - \zeta_i^n)(\|z_i^n\|)^2 \\ + \sum_{n=1}^k \sum_{i=1}^N \alpha_{i-1}^n(\|z_i^n - z_{i-1}^n\|)^2 - \sum_{n=1}^k \sum_{i=1}^N \sum_{j\neq n} \alpha_i^n \varepsilon_i^j \langle z_i^j, z_i^n\rangle \leq 0. \end{aligned} \tag{5.8}$$

Hence, from the Schwarz inequality, we have

$$B \leq A \leq 0, \tag{5.9}$$

where

$$B := \sum_{n=1}^{k} \alpha_N^n(\|z_N^n\|)^2 + \sum_{n=1}^{k}\sum_{i=1}^{N} \alpha_{i-1}^n(\|z_i^n - z_{i-1}^n\|)^2$$
$$+ \sum_{n=1}^{k}\sum_{i=1}^{N} \alpha_i^n(\theta_i^n - \zeta_i^n)(\|z_i^n\|)^2 - \sum_{n=1}^{k}\sum_{i=1}^{N}\sum_{j\neq n} \alpha_i^n|\varepsilon_i^j|\|z_i^j\|\|z_i^n\|.$$

The sum of the last two terms of B can be written as

$$\sum_{i=1}^{N}\langle v_i, C_i(\Theta_i - Z_i + G_i)v_i\rangle,$$

where v_i is the k-vector with elements $\|z_i^j\|$. Thus, if we assume that z_i^j is not zero for at least one j, then, due to (5.9), we obtain

$$0 < \sum_{n=1}^{k} \alpha_N^n(\|z_N^n\|)^2 + \sum_{n=1}^{k}\sum_{i=1}^{N} \alpha_{i-1}^n(\|z_i^n - z_{i-1}^n\|)^2$$
$$+ \sum_{i=1}^{N}\langle v_i, C_i(\Theta_i - Z_i + G_i)v_i\rangle = B \leq A \leq 0,$$

which is a contradiction. The proof is complete. □

6. AN APPLICATION

We close the present work by giving an example which illustrates the results. Also by this example we show that at least condition $(C2)$ can not be ommited for the existence of solutions.

Consider the system of equations

$$\begin{aligned} &x_{i+1} - (1+\theta)x_i + \zeta x_{i-1} + \varepsilon y_i \in c_i^1 T^1 x_i + u_i^1, \;\; i = 1,2,\dots,5, \\ &y_{i+1} - (1+\theta)y_i + \zeta y_{i-1} + \varepsilon x_i \in c_i^2 T^2 y_i + u_i^2, \;\; i = 1,2,\dots,5, \end{aligned} \tag{6.1}$$

with $\zeta, c_i^j > 0$, for all i, j, associated with the conditions

$$x_0 =: a^1, \;\; x_6 =: b^1, \;\; y_0 =: a^2, \;\; y_6 =: b^2. \tag{6.2}$$

We set $\sigma := 2 + \theta$ and formulate the matrices

$$\Theta := \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} = \theta I, \;\; Z := \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} = \zeta I, \;\; E := -\varepsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =: -\varepsilon J$$

and we let $\bar{E} := \sigma I - \varepsilon J$. Here I is the 2×2 identity matrix and the matrix J satisfies $JJ = I$. Then the matrix M is given by

$$M := \begin{pmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{pmatrix} := \times_{i=0}^{3} \begin{pmatrix} \bar{E} & -\zeta I \\ I & 0 \end{pmatrix} = \begin{pmatrix} \bar{E} & -\zeta I \\ I & 0 \end{pmatrix}^4.$$

It is not hard to see that[2] $M^{11} = \bar{E}^4 - 3\zeta\bar{E}^2 + \zeta^2 I$ and $M^{12} = -\zeta\bar{E}^3 + 2\zeta^2\bar{E}$.

[2] From now on the symbol a^n will denote the power with exponent n and base a.

Hence we have

$$
\begin{aligned}
M^{11}\bar{E}+M^{12} &= \bar{E}^5-4\zeta\bar{E}^3+3\zeta^2\bar{E}\\
&= (\sigma I-\varepsilon J)^5-4\zeta(\sigma I-\varepsilon J)^3+3\zeta^2(\sigma I-\varepsilon J)\\
&= \sigma^5 I-5\sigma^4\varepsilon J+10\sigma^3\varepsilon^2 I-10\sigma^2\varepsilon^3 J+5\sigma\varepsilon^4 I-\varepsilon^5 J\\
&\quad -4\zeta\sigma^3 I+12\zeta\sigma^2\varepsilon J-12\zeta\sigma\varepsilon^2 I+4\zeta\varepsilon^3 J+3\zeta^2\sigma I-3\zeta^2\varepsilon J\\
&= \Big(\sigma^5+10\sigma^3\varepsilon^2+5\sigma\varepsilon^4-4\zeta\sigma^3-12\zeta\sigma\varepsilon^2+3\zeta^2\sigma\Big)I\\
&\quad -\Big(5\sigma^4\varepsilon+10\sigma^2\varepsilon^3+\varepsilon^5-12\zeta\sigma^2\varepsilon-4\zeta\varepsilon^3+3\zeta^2\varepsilon\Big)J
\end{aligned}
$$

and therefore

$$
\begin{aligned}
det\Big[M^{11}\bar{E}+M^{12}\Big] &= \Big(\sigma^5+10\sigma^3\varepsilon^2+5\sigma\varepsilon^4-4\zeta\sigma^3-12\zeta\sigma\varepsilon^2+3\zeta^2\sigma\Big)^2\\
&\quad -\Big(5\sigma^4\varepsilon+10\sigma^2\varepsilon^3+\varepsilon^5-12\zeta\sigma^2\varepsilon-4\zeta\varepsilon^3+3\zeta^2\varepsilon\Big)^2\\
&= \Big(\sigma^5+10\sigma^3\varepsilon^2+5\sigma\varepsilon^4-4\zeta\sigma^3-12\zeta\sigma\varepsilon^2+3\zeta^2\sigma+5\sigma^4\varepsilon\\
&\quad +10\sigma^2\varepsilon^3+\varepsilon^5-12\zeta\sigma^2\varepsilon-4\zeta\varepsilon^3+3\zeta^2\varepsilon\Big)\\
&\quad \times\Big(\sigma^5+10\sigma^3\varepsilon^2+5\sigma\varepsilon^4-4\zeta\sigma^3-12\zeta\sigma\varepsilon^2+3\zeta^2\sigma-5\sigma^4\varepsilon\\
&\quad -10\sigma^2\varepsilon^3-\varepsilon^5+12\zeta\sigma^2\varepsilon+4\zeta\varepsilon^3-3\zeta^2\varepsilon\Big)\\
&= \Big((\sigma+\varepsilon)^5-4\zeta(\sigma+\varepsilon)^3+3\zeta^2(\sigma+\varepsilon)\Big)\\
&\quad \times\Big((\sigma-\varepsilon)^5-4\zeta(\sigma-\varepsilon)^3+3\zeta^2(\sigma-\varepsilon)\Big)\\
&= (\sigma^2-\varepsilon^2)\Big((\sigma+\varepsilon)^2-\zeta\Big)\Big((\sigma+\varepsilon)^2-3\zeta\Big)\\
&\quad \times\Big((\sigma-\varepsilon)^2-\zeta\Big)\Big((\sigma-\varepsilon)^2-3\zeta\Big).
\end{aligned}
$$

This means that condition $(C1)$ is satisfied if and only if

$$
|\sigma|\neq|\varepsilon| \text{ and } \sigma\pm\varepsilon\neq\pm\sqrt{\zeta},\ \pm\sqrt{3\zeta}. \tag{6.3}
$$

Also, we can obtain that Condition $(C2)$ is satisfied if

$$
|\varepsilon|\leq\theta-\zeta \tag{6.4}
$$

and Condition $(C3)$ is satisfied if

$$
|\varepsilon|\leq\frac{\zeta^5}{2(5\zeta^4+4\zeta^3+3\zeta^2+2\zeta+1)}+\theta-\zeta. \tag{6.5}
$$

Obviously, (6.4) implies both conditions (6.3) and (6.5). Consequently, if the coefficients ζ and c_i^j are positive, and the operators T^1, T^2 are maximal monotone, then under the condition (6.4), Theorem 3.3 is applicable and therefore the problem (6.1)-(6.2) admits a unique solution.

Remark 6.1. Condition (6.4) is not only sufficient for the existence of solutions of the problem. Indeed, as the following example shows, if this condition does not hold, then solutions of the problem may not exist.

EXAMPLE. For fixed real numbers α, θ, consider the system

$$\begin{aligned} x_{i+1} - (1+\theta)x_i + x_{i-1} + (1+\theta)y_i \in \partial|x_i| + \sin^2(\alpha) \\ y_{i+1} - (1+\theta)y_i + y_{i-1} + (1+\theta)x_i \in \partial|y_i| + \cos^2(\alpha) \end{aligned} \tag{6.6}$$

for $i = 1, 2, \ldots, 5$, associated with the boundary conditions

$$x_0 = a_1, \;\; y_0 = a_2, \;\; x_6 = b_1, \;\; y_6 = b_2. \tag{6.7}$$

This problem is of the form (6.1) and its coefficients do not satisfy (6.4). Assume that it admits a solution $x_i, \;\; y_i, \;\; i = 1, 2, \ldots, 5$. Then the (finite) sequence $u_i := x_i + y_i, \;\; i = 1, 2, \ldots, 5$ solves the discrete boundary value problem

$$u_{i+1} + u_{i-1} \in \partial|x_i| + \partial|y_i| + 1, \;\; u_0 = a_1 + a_2, \;\; u_6 = b_1 + b_2$$

and therefore we have

$$b_1 + b_2 + a_1 + a_2 \in \partial|x_1| - \partial|x_3| + \partial|x_5| + \partial|y_1| - \partial|y_3| + \partial|y_5| + 1.$$

The right side of this inclusion is a subset of the real line belonging to the interval $[-5, 7]$. (Recall that the range of the sub-differential operator is a subset of the interval $[-1.1]$.) Therefore, if, for instance, the boundary conditions satisfy $|b_1+b_2+a_1+a_2| > 7$, then there is no solution of the problem (6.6)–(6.7).

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