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A DISTRIBUTIONAL APPROACH TO FRAGMENTATION EQUATIONS

WILSON LAMB[1] AND ADAM C. MCBRIDE[2]

Department of Mathematics and Statistics, University of Strathclyde,
Glasgow, G1 1XH, UK.
E-mail: w.lamb@strath.ac.uk a.c.mcbride@strath.ac.uk

To Professor Jeff Webb on his retirement, with best wishes for the future.

ABSTRACT. We consider a linear integro-differential equation that models multiple fragmentation with inherent mass-loss. A systematic procedure is presented for constructing a space of generalised functions Z' in which initial-value problems involving singular initial conditions such as the Dirac delta distribution can be analysed. The procedure makes use of results on sun dual semigroups and quasi-equicontinuous semigroups on locally convex spaces. The existence and uniqueness of a distributional solution to an abstract version of the initial-value problem are established for any given initial data u_0 in Z'.

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1. INTRODUCTION

Fragmentation processes arise in a number of physical situations such as polymer degradation, liquid droplet breakup, combustion, and the crushing and grinding of rocks. In many cases, when modelling such processes, it is assumed that the total mass in the system is a conserved quantity. However, as pointed out in [1], there are situations where mass-loss can occur in a natural manner. Motivated by this, Edwards *et al* [1]–[3] introduced the linear rate equation

$$\partial_t u(x,t) = -a(x)u(x,t) + \int_x^\infty b(x|y)a(y)u(y,t)dy + \partial_x[r(x)u(x,t)], \quad u(x,0) = u_0(x) \tag{1.1}$$

to describe fragmentation with mass-loss. This equation involves a particle mass distribution function u, a fragmentation rate a, a continuous mass-loss rate r, and a non-negative measurable function b that describes the distribution of particle masses x spawned by the fragmentation of a particle of mass $y > x$. The continuous mass-loss rate r is defined so that $r(m(t)) = -dm/dt$ for a particle of time-dependent mass

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$m(t)$, while the normalizing condition

$$\int_0^y xb(x|y)\,dx = y - \lambda(y)y, \tag{1.2}$$

where $0 \le \lambda(y) \le 1$, allows for so-called discrete mass-loss to occur in the fragmentation process.

Initial investigations into (1.1) concentrated on finding exact and asymptotic solutions, usually for specific choices of a, b and r; see, for example [1]–[3]. More recently, in [4] and [5], the theory of semigroups of operators has been applied to an abstract formulation of equation (1.1) and this has led to existence and uniqueness results being established for solutions arising from initial data belonging to a physically relevant Banach space.

A common strategy used in several papers to obtain exact solutions to fragmentation equations is to begin by looking at the case of mono-disperse initial conditions which are modelled using Dirac delta functionals; see [6]–[8]. Formal arguments, in which the Dirac delta is treated as a conventional function, are used to find a solution of these singular initial-value problems and then a solution for general initial conditions is obtained by superposition. In two recent papers [9], [10], we have succeeded in presenting these arguments within the mathematically rigorous framework of distribution theory, in which the Dirac delta is treated properly as a singular distribution. Unfortunately, the method we used required us to have an explicit formula for the semigroup of operators associated with the fragmentation equation and, because of this, we were only able to cater for a restricted class of fragmentation rate functions, namely

$$a(x) = x^{\alpha+1}, \quad b(x|y) = (\nu+2)\left(\frac{x}{y}\right)^{\nu}\frac{1}{y}, \quad r(x) \equiv 0.$$

Our aim in the present paper is to show that an alternative method, which we developed in [11] and [12] to deal with photon transport problems involving point sources, can also be applied to a far more general class of fragmentation equations with mono-disperse initial conditions.

We begin in Section 2 by describing how the initial-value problem (1.1) can be expressed as an abstract Cauchy problem (ACP) posed in a physically relevant Banach space X, and for completeness, give a summary of existing Banach space results that we require later. In Section 3, we establish analogous results on the sun dual space $X^{\odot}$ and, in particular, define a subspace Y of $X^{\odot}$ that consists of suitably restricted continuous functions. A systematic procedure that is based on this Banach space Y is then used in Section 4 to produce a space of test-functions Z and corresponding space of generalised functions Z'. Finally, results on semigroups of operators on locally convex spaces are used in Section 5 to prove the existence and uniqueness of a solution $u : [0,\infty) \to Z'$ to an appropriate generalised ACP version of equation (1.1).

2. ABSTRACT FORMULATION OF THE PROBLEM

We shall make the following assumptions on the fragmentation model.

(A1) $a \in C(\mathbb{R}_+)$, where $\mathbb{R}_+ = [0, \infty)$, and $a(x) \geq 0 \ \forall x \geq 0$.

(A2) $b(x|y) \geq 0 \ \forall x, y \geq 0$, $b(x|y) = 0$ for $y \leq x$ and, for all $y \geq 0$,

$$\int_0^y b(x|y)\,dx \leq n_0, \qquad \int_0^y xb(x|y)\,dx = y(1 - \lambda(y)),$$

where n_0 is a positive constant and $0 \leq \lambda(y) \leq 1$.

(A3) $r(x) = r_0 \ \forall x \geq 0$, where r_0 is a positive constant.

(A4) The maximum size that a particle can attain is N, where N is a positive constant.

It follows from (A3) that we are assuming that the continuous mass-loss occurs at a constant rate r_0. We make this assumption only to keep technicalities to a minimum. The method we describe below will also cater for cases when the continuous mass-loss rate is size-dependent.

Under these assumptions, equation (1.1) becomes

$$\partial_t u(x,t) = -a(x)u(x,t) + \int_x^N b(x|y)a(y)u(y,t)dy + r_0 \partial_x u(x,t), \quad 0 < x < N, \tag{2.1}$$

with initial condition

$$u(x, 0) = u_0(x) \quad \forall x \in [0, N]. \tag{2.2}$$

Also, because of the maximum size restriction, it is appropriate to impose the boundary condition

$$u(N, t) = 0 \quad \forall t > 0. \tag{2.3}$$

Our aim is to show that the initial-boundary value problem (2.1)–(2.3) can be analysed in a rigorous manner when $u_0(x) = \delta(x - \ell)$, where $\ell > 0$ and δ is the Dirac delta distribution. First, however, we summarise what is known about the problem when u_0 is a conventional function. For this, we require the following notation.

Throughout X will denote the Banach space $L^1([0, N])$ with norm

$$\| \psi \|_X := \int_0^N |\psi(x)|\,dx.$$

The operators A, T_0, T and B are defined by

$$(A\psi)(x) := -a(x)\psi(x), \ D(A) := X, \tag{2.4}$$

$$(T_0\psi)(x) := r_0\psi'(x), \ D(T_0) := \{\psi \in X : \psi \in \mathrm{AC}([0, N]), \ \psi(N) = 0\}, \tag{2.5}$$

$$T := A + T_0, \ D(T) := D(T_0), \tag{2.6}$$

$$(B\psi)(x) := \int_x^N b(x|y)a(y)\psi(y)\,dy, \ D(B) := X. \tag{2.7}$$

Note that $\psi \in \mathrm{AC}([0,N])$ indicates that ψ is absolutely continuous on $[0,N]$. Also, routine calculations show that both A and B are bounded on X with

$$\| A\psi\|_X \le a_N \,\| \psi\|_X,\ \| B\psi\|_X \le n_0\, a_N\| \psi\|_X \quad \forall \psi \in X, \tag{2.8}$$

where $a_N = \max\{a(x) : 0 \le x \le N\}$.

Theorem 2.1. *The operator $(T, D(T))$ is the infinitesimal generator of a strongly continuous positive semigroup of contractions $\{\exp(tT)\}_{t\ge 0}$ on X. Moreover,*

$$(\exp(tT)\psi)(x) = \exp\left(-\frac{1}{r_0}\int_x^{x+r_0 t} a(s)\,ds\right)\psi(x+r_0 t)\,\chi_{I_N}(x+r_0 t), \tag{2.9}$$

where χ_{I_N} is the characteristic function of the interval $I_N = [0,N]$.

Proof. This follows from [5, Theorem 9.5 and Corollary 9.6]. □

Theorem 2.1 and the fact that B is a bounded positive operator on X lead to the following existence and uniqueness result for the problem (2.1)–(2.3).

Theorem 2.2. *Let $K := T+B$ with $D(K) := D(T)$. Then the operator $(K, D(K))$ is the infinitesimal generator of a strongly continuous positive semigroup $\{\exp(tK)\}_{t\ge 0}$ on X. Consequently, the ACP*

$$\frac{d}{dt}u(t) = Ku(t)\ (t>0);\quad u(0) = u_0, \tag{2.10}$$

has a unique non-negative, strongly differentiable solution $u : [0,\infty) \to D(K)$ for all non-negative $u_0 \in D(K)$, given by

$$u(t) = \exp(tK)u_0, \quad t \ge 0. \tag{2.11}$$

Proof. The bounded perturbation theorem [5, Theorem 4.9] establishes that $K = T + B$ with $D(K) = D(T)$ is the infinitesimal generator of a strongly continuous semigroup $\{\exp(tK)\}_{t\ge 0}$ on X. Moreover, for non-negative $\psi \in D(T)$,

$$\int_0^N [(T\psi)(x) + (B\psi)(x)]\,dx \le (n_0 - 1)\int_0^N a(x)\psi(x)\,dx - r_0\psi(0) \tag{2.12}$$

and therefore, from the Kato-Voigt perturbation theorem [5, Corollary 5.17], we can deduce that the semigroup $\{\exp(-n_0 a_N t)\exp(tK)\}_{t\ge 0}$ is a strongly continuous positive semigroup of contractions on X, with generator $K - n_0 a_N I$. The positivity of $\{\exp(tK)\}_{t\ge 0}$ follows immediately. □

It should be noted that, by working in the space X, both the total number and the total mass of particles in the system at any time are controlled, since

$$\int_0^N x|u(x,t)|\,dx \le N\int_0^N |u(x,t)|\,dx < \infty \ \text{ if } u(\cdot,t) \in X.$$

3. THE ADJOINT TRANSPORT SEMIGROUP

In this section, our aim is to produce an operator, say L, defined in some Banach space Y of continuous functions on $[0, N]$ such that L generates a C_0-semigroup on Y and is adjoint to the operator $K = T + B$ of the previous section in the sense that

$$\int_0^N (L\eta)(x)\psi(x)\,dx = \int_0^N \eta(x)(K\psi)(x)\,dx, \quad \forall \eta \in D(L) \subset Y,\ \psi \in D(K) \subset X.$$

Due to the fact that X is not a reflexive Banach space, the approach we use involves the theory of sun dual semigroups [13, pp 62–63].

In the usual manner, the dual space X^* can be identified with $L^\infty([0, N])$ with norm

$$\|\eta\|_\infty := \operatorname*{ess\,sup}_{x\in[0,N]} |\eta(x)|\,,$$

via the duality pairing

$$(\eta, \psi) := \int_0^N \eta(x)\psi(x)\,dx,\ \eta \in L^\infty([0, N]),\ \psi \in X. \tag{3.1}$$

Let the family of operators $\{W(t)\}_{t\geq 0}$ be defined on X^* by

$$[W(t)\eta](x) := \exp\left(-\frac{1}{r_0}\int_{x-r_0\,t}^{x} a(s)\,ds\right)\eta(x - r_0\,t)\,\chi_{I_N}(x - r_0\,t). \tag{3.2}$$

Since

$$(\eta, \exp(tT)\psi) = (W(t)\eta, \psi),\ \forall \eta \in X^*,\ \psi \in X,$$

it follows that

$$W(t) = (\exp(tT))^*,\ \forall t \geq 0,$$

where $(\exp(tT))^*$ is the adjoint of the operator $\exp(tT)$. Standard duality arguments show that the algebraic semigroup properties of $\{\exp(tT)\}_{t\geq 0}$ on X are inherited by $\{W(t)\}_{t\geq 0}$ on X^*. However, to obtain a strongly continuous adjoint semigroup, each operator $W(t)$ must be restricted to the sun dual (or semigroup dual) $X^\odot \subset X^*$ of X defined by

$$X^\odot := \left\{\eta \in X^* : \lim_{t\downarrow 0} \|W(t)\eta - \eta\|_{X^*} = 0\right\}.$$

As discussed in [13, pp.62-63], the space $X^\odot = \overline{D(T^*)}$, and so $X^\odot$ is a closed subspace of X^*. Moreover, if we denote the restriction of $W(t)$ to $X^\odot$ by $\exp(tT^\odot)$, then $\{\exp(tT^\odot)\}_{t\geq 0}$ (the sun dual semigroup of $\{\exp(tT)\}_{t\geq 0}$) is a C_0-semigroup on $X^\odot$ and has generator $(T^\odot, D(T^\odot))$ defined by

$$T^\odot\eta = T^*\eta\,, \quad D(T^\odot) = \{\eta \in D(T^*) : T^*\eta \in X^\odot\}\,.$$

We now define the space Y by

$$Y := \{\eta \in C([0, N]) : \eta(0) = 0\}. \tag{3.3}$$

This space is clearly a closed subspace of X^* and, since $(W(t)\eta)(x) = 0$ for all $x \in [0, N]$, $t > N/r_0$ and, for $t \le N/r_0$,

$$[W(t)\eta](x) = \begin{cases} \exp[-\frac{1}{r_0}\int_{x-r_0 t}^{x} a(s)\,ds]\,\eta(x - r_0\,t) & r_0\,t \le x \le N, \\ 0 & x < r_0\,t, \end{cases}$$

it follows that $W(t)\eta \in Y$ for each $t \ge 0$ and $\eta \in Y$. Moreover, for $\eta \in Y$ and $t \le N/r_0$,

$$\begin{aligned} \|W(t)\eta - \eta\|_{X^*} &= \max_{0\le x\le N} |(W(t)\eta)(x) - \eta(x)| \\ &= \max_{r_0\,t\le x\le N} \left| \exp[-\frac{1}{r_0}\int_0^{r_0\,t} a(x-s)\,ds]\,\eta(x - r_0\,t) - \eta(x - r_0\,t)\right| \\ &+ \max_{0\le x\le N} |\eta(x - r_0\,t)\chi_{I_N}(x - r_0\,t) - \eta(x)| \\ &\le \|\eta\|_{X^*} \max_{r_0\,t\le x\le N} \left|\exp[-\frac{1}{r_0}\int_0^{r_0\,t} a(x-s)\,ds] - 1\right| \qquad (3.4) \\ &+ \max_{r_0\,t\le x\le N} |\eta(x - r_0\,t) - \eta(x)| + \max_{0\le x\le r_0\,t} |\eta(x)|. \qquad (3.5) \end{aligned}$$

Since a, η are continuous, and $\eta(0) = 0$, the three terms in (3.4) and (3.5) all tend to zero as $t \to 0^+$ and therefore Y is also a closed subspace of $X^\odot$.

The fact that Y is a closed subspace of $X^\odot$ means that the restrictions, say $\exp(tG)$, of $\exp(tT^\odot)$ to Y form a C_0-semigroup on Y with generator given by

$$G\eta := T^\odot\eta,\; D(G) := \{\eta \in D(T^\odot) \cap Y : T^\odot\eta \in Y\} \subset Y. \qquad (3.6)$$

As G is a restriction of T^*, it follows that

$$(G\eta, \psi) = (\eta, T\psi) \quad \forall\eta \in Y,\, \psi \in D(T). \qquad (3.7)$$

We now examine the effect of the operator $B \in B(X)$ defined by (2.7). As the adjoint operator B^* is given by

$$(B^*\eta)(x) := a(x)\int_0^x b(y|x)\eta(y)\,dy, \quad \eta \in X^*, \qquad (3.8)$$

the restriction $B^*_{|_Y}$ of B^* to Y is a bounded operator on Y. Consequently, $(L, D(L))$, where

$$L = G + B^*_{|_Y},\; D(L) = D(G), \qquad (3.9)$$

is the infinitesimal generator of a C_0-semigroup $\{e^{tL}\}_{t\ge 0}$ on Y. The following theorem shows that this semigroup is of type $M(1, n_0a_N)$.

Theorem 3.1. *Let*

$$S(t) := \exp(-n_0a_Nt)\exp(tL), \quad t \ge 0. \qquad (3.10)$$

Then $\{S(t)\}_{t\ge 0}$ is a strongly continuous semigroup of contractions on Y.

Proof. This is an immediate consequence of the fact that

$$\{\exp(-n_0 a_N t)\exp(tK)\}_{t\geq 0}$$

is a strongly continuous positive semigroup of contractions on X. $\square$

Note that the operator L has the desired property of being adjoint to K since

$$\begin{aligned}(L\eta,\psi) &= (G\eta + B^*_{|_Y}\eta,\psi)\\ &= (T^*\eta + B^*_{|_Y}\eta,\psi)\\ &= (\eta, T\psi + B\psi) = (\eta, K\psi) \quad \forall\, \eta\in D(L),\ \psi\in D(K)\,.\end{aligned}$$

4. TEST FUNCTIONS AND GENERALISED FUNCTIONS

We use the Banach space Y and operator L, given by (3.3) and (3.9) respectively, to produce a space of test functions Z, and corresponding space of generalised functions Z', in the following manner. The space Z is defined to be

$$Z := \bigcap_{k=0}^{\infty} D(L^k)\ (\subset Y)\,, \tag{4.1}$$

where L^0 is interpreted as the identity operator. The topology in Z is generated by the countable collection of seminorms $\{\alpha_k\}_{k=0}^{\infty}$ where

$$\alpha_k(\phi) := \|L^k\phi\|_Y \ = \|L^k\phi\|_\infty, \quad \phi\in Z.$$

Since the operator L is the infinitesimal generator of a strongly continuous semigroup on Y, Z is a Fréchet space and is also a dense subspace of Y; see [13, p.53] and [14, Theorem 2.3]. Moreover, it is clear from the topology on Z that $\mathcal{L}$, the restriction of L to Z, is a continuous linear mapping from Z into Z. Note that an equivalent topology can be defined on Z via

$$|\phi|_k := \sum_{j=0}^{k} \alpha_j(\phi)\,,\ \phi\in Z\,,\ k=0,1,2,\ldots,$$

and with this topology, Z can also be interpreted as a countably normed space. With this additional structure we can express Z in the form

$$Z = \bigcap_{n=0}^{\infty} Z_n, \text{ where } \quad Z_n := (D(L^n), |\cdot|_n) = \overline{(Z, |\cdot|_n)}.$$

If, for each n, we denote the restriction of $\exp(tL)$ to Z_n by $S_n(t)$, then it can also be shown that $\{S_n(t)\}_{t\geq 0}$ is a strongly continuous semigroup on the Banach space Z_n with generator given by the part of L in Z_n; see [12, Section 5] for details.

We define the dual space, Z', of Z, equipped with the weak*-topology, to be the associated space of generalised functions. Note that, from [15, p.34], $f \in Z'$ if and only if there exists some non-negative integer r and constant C such that

$$|\langle f, \phi \rangle| \leq C|\phi|_r \ \forall \phi \in Z ,$$

where $\langle f, \phi \rangle$ denotes the action of $f \in Z'$ on $\phi \in Z$. The least such r is called the order of the functional f. Given $\psi \in X$ if we define $\widetilde{\psi}$ on Z by

$$\langle \widetilde{\psi}, \phi \rangle := \int_0^N \psi(x)\phi(x)\, dx, \quad \phi \in Z,$$

then

$$|\langle \widetilde{\psi}, \phi \rangle| \leq \|\psi\|_X \, \|\phi\|_Y = \|\psi\|_X \, |\phi|_0 \, , \ \forall \phi \in Z \, ,$$

and so ψ generates a regular generalised function of order 0 in Z'. Similarly,

$$|\langle \delta(x-\ell), \phi \rangle| = |\phi(\ell)| \leq |\phi|_0 \ \forall \phi \in Z,$$

showing that the Dirac delta functional $\delta(x-\ell)$ is also a generalised function of order 0 in Z'.

It is now an easy matter to obtain an extension $\widetilde{K}$, of the operator K, defined on Z'. For each $\psi \in D(K)$, we require

$$\begin{aligned} \langle \widetilde{K}\widetilde{\psi}, \phi \rangle &:= \langle \widetilde{K\psi}, \phi \rangle \\ &= (\phi, K\psi) = (\mathcal{L}\phi, \psi) = \langle \widetilde{\psi}, \mathcal{L}\phi \rangle, \ \forall \phi \in Z. \end{aligned}$$

This leads to the definition

$$\langle \widetilde{K} f, \phi \rangle := \langle f, \mathcal{L}\phi \rangle \quad \forall f \in Z', \phi \in Z. \tag{4.2}$$

Thus $\widetilde{K} = \mathcal{L}'$, where $\mathcal{L}'$ denotes the adjoint of $\mathcal{L}$ on the complete countably normed space Z, and therefore, from standard results on adjoints, $\widetilde{K}$ is well-defined as a continuous linear mapping from Z' into Z'.

5. SOLUTION OF FRAGMENTATION EQUATIONS WITH SINGULAR INITIAL CONDITIONS

The mathematical framework is now in place for treating mass-loss fragmentation processes with mono-disperse initial conditions in a rigorous manner. The abstract formulation of the problem we consider is

$$\frac{d}{dt}u(t) = \widetilde{K}u(t); \quad u(0) = u_0 \in Z', \tag{5.1}$$

where the time derivative is interpreted in the weak*-sense in Z' and a solution $u : [0, \infty) \to Z'$ is sought. We shall establish that a unique weak*-differentiable solution to (5.1) exists for any $u_0 \in Z'$ and hence must also exist for the particular case when $u_0 = \delta(x - \ell)$.

Theorem 5.1. *The operator $\mathcal{L}$ is the infinitesimal generator of a uniquely defined quasi-equicontinuous semigroup $\{\exp(t\mathcal{L})\}_{t\geq 0}$ of class C_0 on Z. Moreover*

$$\exp(t\mathcal{L})\phi = \exp(tL)\phi \quad \forall \phi \in Z,$$

where $\{\exp(tL)\}_{t\geq 0}$ is the semigroup on Y generated by L.

Proof. To establish that $\{\exp(t\mathcal{L})\}_{t\geq 0}$ is quasi-equicontinuous on Z we must find some positive constant ρ such that $\{\exp(-\rho\, t)\exp(t\mathcal{L})\}_{t\geq 0}$ is equicontinuous on Z. Let $\{S(t)\}_{t\geq 0}$ be the strongly continuous semigroup of contractions on Y defined by (3.10). The infinitesimal generator of $\{S(t)\}_{t\geq 0}$ is $G := L - n_0 a_N I$, with $D(G) := D(L)$. If we denote the restriction of G to Z by $\mathcal{G}$, then for each $k = 0, 1, 2, \ldots$ and $\phi \in Z$,

$$\alpha_k([I - n^{-1}\mathcal{G}]^{-1}\phi) = \alpha_k(n[nI - \mathcal{G}]^{-1}\phi) \leq \alpha_k(\phi).$$

Consequently

$$\alpha_k([I - n^{-1}\mathcal{G}]^{-m}\phi) \leq \alpha_k(\phi)\,,\ \forall n = 1, 2, 3, \ldots,\ m = 0, 1, 2, \ldots,$$

and so, from [16, p. 246], $\mathcal{G}$ is the infinitesimal generator of an equicontinuous semigroup $\{e^{t\mathcal{G}}\}_{t\geq 0}$ of class C_0 on Z. Moreover, from [16, p. 248],

$$e^{t\mathcal{G}}\phi = \lim_{n\to\infty} \exp(t\mathcal{G}[I - n^{-1}\mathcal{G}]^{-1})\phi\,,\ \forall \phi \in Z.$$

Since $\mathcal{G}$ is the restriction of G to Z and convergence in Z implies convergence in Y, we obtain

$$e^{t\mathcal{G}}\phi = \lim_{n\to\infty} \exp(tG[I - n^{-1}G]^{-1})\phi = S(t)\phi\,.$$

The stated result now follows from the fact that $\mathcal{L} = \mathcal{G} + n_0 a_N I$ is the infinitesimal generator of the semigroup $\{\exp(n_0 a_N t)\exp(t\mathcal{G})\}_{t\geq 0}$. □

Theorem 5.2. *The abstract Cauchy problem* (5.1) *has a unique, weak*-differentiable solution $u : \mathbb{R}_+ \to Z'$ for any $u_0 \in Z'$.*

Proof. The operator $\widetilde{K} = \mathcal{L}'$ is the infinitesimal generator of the weak*-continuous semigroup of operators $\{\exp(\mathcal{L}'t)\}_{t\geq 0}$ on Z' defined by

$$\langle \exp(t\mathcal{L}')f, \phi\rangle := \langle f, \exp(t\mathcal{L})\phi\rangle \quad \forall f \in Z', \phi \in Z.$$

Standard results concerning semigroups on complete countably normed spaces can now be applied to deduce that the unique solution of (5.1) is given by

$$u(t) = \exp(t\mathcal{L}')u_0, \quad \forall t \geq 0;$$

for example, see [12, Theorem 4.2]. □

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