*Communications in Applied Analysis* **15** (2011), no. 2, 3 and 4, 521–528

# POSITIVE SOLUTIONS OF SYSTEMS OF HAMMERSTEIN INTEGRAL EQUATIONS

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*Dedicated to Professor J. R. L. Webb's retirement*

**ABSTRACT.** New results on existence of positive solutions in $L^p(\Omega, \mathbb{R}^n)$ of systems of Hammerstein integral equations are obtained by using Leray-Schauder fixed point theorem. The principal eigenvalues of the corresponding linear Hammerstein integral equations are employed. Our results improve some previous results on existence of (not necessarily positive) solutions in $L^p(\Omega)$ of a single Hammerstein integral equation.

**AMS (MOS) Subject Classification.** Primary: 45G15; Secondary: 47H10, 47H30

**Key words.** systems of Hammerstein integral equations, positive solutions, Leray-Schauder fixed point theorem.

## 1. INTRODUCTION

We are interested in existence of positive solutions in $L^p(\Omega, \mathbb{R}^n)$ of systems of Hammerstein integral equations of the form

$$z_i(t) = g_i(t) + \int_\Omega k(t,s) f_i(s, \mathbf{z}(s))\, ds \quad \text{for a.e. } t \in \Omega \text{ and } i \in I_n, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^m$ and $I_n = \{1, \dots, n\}$.

When $n = 1$, existence of one (not necessarily positive) solution in $L^p(\Omega)$ of (1.1) is studied in [19], where $\Omega = [0, 1]$ and the nonlinear alternative theorem of Leray-Schauder type is used, and in [18], where $\Omega = [a, b]$ and Schaefer's fixed point theorem is used. None of these results use the principal eigenvalue of the corresponding linear Hammerstein integral equation. When $g_i(t) \equiv 0$, the existence of at least one solution in $L^p(\Omega)$ of (1.1) with $n = 1$ was studied by Krasnosel'skii [9] (also see [10, Chapter VI]) and existence of nonzero solutions is studied in [3, 6] under the superlinear conditions involving the principal eigenvalue of the corresponding linear integral equation. We refer to [16, 17, 20, 21] for the study of existence of solutions in $L^1[0, 1]$, where measures of noncompactness are involved.

Received September 29, 2010 

In this paper, by using Leray-Schauder theorem, we prove new results on existence of positive solutions in $L^p(\Omega, \mathbb{R}^n)$ of the system (1.1), where the principal eigenvalue of the corresponding linear Hammerstein integral equation is involved. Unlike the study on existence of positive solutions in $C(\Omega, \mathbb{R}^n)$, where a smaller cone than the standard cone $C(\Omega, \mathbb{R}^n_+)$ is considered (see [1, 4, 5, 8, 12, 13, 14, 15, 22]), only the standard cone $L^p(\Omega, \mathbb{R}^n_+)$ can be applied here. Therefore, there is difficulty to obtain results on existence of one or several nonzero positive solutions in $L^p(\Omega, \mathbb{R}^n_+)$ of (1.1). We refer to [11] for the study of nonzero positive solutions of systems of elliptic boundary value problems, where only the standard cone $C(\Omega, \mathbb{R}^n_+)$ is applied.

As illustrations of our results, we consider existence of positive solutions in $L^p(\Omega, \mathbb{R}^n_+)$ of the following systems

$$z_i(t) = g_i(t) + \int_\Omega k(t,s)[a_i(s) + u_i(s)|\mathbf{z}|_0^{\alpha_i}(s) + v_i(s)|\mathbf{z}|_0^{\beta_i}(s)]\,ds \quad \text{a.e. on } \Omega \text{ and } i \in I_n,$$

where $|\cdot|_0$ denotes a norm in $\mathbb{R}^n$. Specific functions $g_i$ and kernels $k$ are provided.

## 2. POSITIVE SOLUTIONS OF SYSTEMS OF HAMMERSTEIN INTEGRAL EQUATIONS

In this section, we study existence of positive solutions in $L^p(\Omega, \mathbb{R}^n)$ of systems of Hammerstein integral equations of the form

$$z_i(t) = g_i(t) + \int_\Omega k(t,s) f_i(s, \mathbf{z}(s))\,ds \qquad \text{for a.e. } t \in \Omega \text{ and } i \in I_n, \tag{2.1}$$

where $\mathbf{z}(s) = (z_1(s), \ldots, z_n(s))$ and $\Omega \subset \mathbb{R}^m$ is measurable with $0 < \text{meas}(\Omega) < \infty$.

We use the following maximum norm in $\mathbb{R}^n$:

$$|\mathbf{z}| = \max\{|z_i| : i \in I_n\}, \tag{2.2}$$

where $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$. We define

$$(\mathbb{R}^n_+)_I = \{\mathbf{z} \in \mathbb{R}^n_+ : |\mathbf{z}| \in I\}, \tag{2.3}$$

where $I = [a, b]$ if $a, b \in [0, \infty)$ with $a \le b$ and $I = [a, b)$ if $a, b \in [0, \infty]$ with $a < b$.

Let $p, q \in (1, \infty)$ be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. We list the following conditions:

$(h_1)$ For each $i \in I_n$, $g_i \in L^p(\Omega)$.

$(h_2)$ $k \in L^p(\Omega \times \Omega)$.

$(h_3)$ For each $i \in I_n$, $f_i : \Omega \times \mathbb{R}^n_+ \to \mathbb{R}_+$ satisfies Carathéodory conditions, that is, $f(\cdot, \mathbf{z})$ is measurable on $\Omega$ for each fixed $\mathbf{z} \in \mathbb{R}^n_+$ and $f(s, \cdot)$ is continuous on $\mathbb{R}^n_+$ for a.e. $s \in \Omega$, and there exist $a_i \in L^q_+(\Omega)$ and $b_i > 0$ such that

$$f_i(s, \mathbf{z}) \le a_i(s) + b_i|\mathbf{z}|^{p-1} \quad \text{for a.e. } s \in \Omega \text{ and all } \mathbf{z} \in \mathbb{R}^n_+. \tag{2.4}$$

$(h_4)$ For each $i \in I_n$, $f_i : \Omega \times \mathbb{R}^n_+ \to \mathbb{R}_+$ satisfies Carathéodory conditions, and for each $r > 0$ there exist $a_{i,r} \in L^q_+(\Omega)$ such that

$$f_i(s, \mathbf{z}) \le a_{i,r}(s) \quad \text{for a.e. } s \in \Omega \text{ and all } \mathbf{z} \in (\mathbb{R}^n_+)_{[0,r]}. \tag{2.5}$$

When $n = 1$, $(h_1)$-$(h_3)$ are used in [18, 19]. It is obvious that $(h_3)$ implies $(h_4)$. We shall see that $(h_4)$ together with some additional conditions implies $(h_3)$ (see Theorem 2.4 below).

We can write (2.1) into the following fixed point equation:

$$\mathbf{z}(t) = (g_1(t), \ldots, g_n(t)) + (L(F_1\mathbf{z})(t), \ldots, L(F_n\mathbf{z})(t)) := A\mathbf{z}(t) \quad \text{a.e. on } \Omega, \tag{2.6}$$

where

$$(F_i\mathbf{z})(t) = f_i(t, \mathbf{z}(t)) \quad \text{for } i \in I_n \tag{2.7}$$

and

$$(Lu)(t) = \int_\Omega k(t,s)u(s)\, ds. \tag{2.8}$$

We write $L^p(\Omega) = L^p(\Omega, \mathbb{R})$, $L^p_+(\Omega) = L^p(\Omega, \mathbb{R}_+)$ and $\|\cdot\| = \|\cdot\|_{L^p(\Omega)}$. We use the following norm in $L^p(\Omega, \mathbb{R}^n)$: for $\mathbf{z} = (z_1, \ldots, z_n) \in L^p(\Omega, \mathbb{R}^n)$, let

$$\|\mathbf{z}\| = \max\{\|z_i\|_{L^p(\Omega)} : i \in I_n\}.$$

Let $P = L^p(\Omega, \mathbb{R}^n_+)$ be the standard positive cone in $L^p(\Omega, \mathbb{R}^n)$.

The following results show that the linear operator $L$ defined in (2.8) and the map $A$ defined in (2.6) are compact.

**Lemma 2.1.** $(i)$ *Under the hypothesis* $(h_2)$ *the linear operator* $L$ *defined in* (2.8) *maps* $L^q(\Omega)$ *into* $L^p(\Omega)$ *and is compact. Moreover,* $L(L^q_+(\Omega)) \subset L^p_+(\Omega)$.

$(ii)$ *Under the hypotheses* $(h_1)$–$(h_3)$, *the map* $A$ *defined in* (2.6) *maps* $P$ *into* $P$ *and is compact.*

*Proof.* $(i)$ Since $\text{meas}(\Omega) \in (0, \infty)$, it follows from $(h_2)$ and a result mentioned in [10, page 19] that $L : L^q(\Omega) \to L^p(\Omega)$ is compact. The result $(i)$ follows.

$(ii)$ By [10, Theorem 2.3], for each $i \in I_n$, $F_i : P \to L^p_+(\Omega)$ is continuous. This, together with the result $(i)$ implies that the result $(ii)$ holds. □

Let $\rho > 0$ and let $P_\rho = \{x \in P : \|x\| < \rho\}$, $\partial P_\rho = \{x \in P : \|x\| = \rho\}$ and $\overline{P}_\rho = \{x \in P : \|x\| \le \rho\}$.

We need the following Leray-Schauder fixed point theorem (see [2]).

**Lemma 2.2.** $(i)$ *Assume that* $A : \overline{P}_\rho \to P$ *is a compact map and satisfies the following Leray-Schauder condition:*

$(LS)$ $z \neq \varrho Az$ *for* $x \in \partial P_\rho$ *and* $\varrho \in (0, 1]$.

*Then* $A$ *has a fixed point in* $P_\rho$.

$(ii)$ *Assume that* $A : P \to P$ *is a compact map and satisfies*

$$\lim_{\|x\| \to \infty} \frac{\|Ax\|}{\|x\|} < 1.$$

*Then* $A$ *has a fixed point in* $P$.

We first give the following result on existence of positive solutions in $L^p(\Omega, \mathbb{R}^n)$ of (2.1) when $p \in (1, 2]$.

**Theorem 2.3.** *Assume that $p \in (1,2]$, $(h_1)$–$(h_3)$ hold and one of the following conditions hold.*

*(i)* $1 < p < 2$.

*(ii)* $p = 2$ *and* $b\|k\| < 1$,

*where* $b = \max\{b_i : i \in I_n\}$ *and* $\|k\| = \left(\int_\Omega \int_\Omega (k(t,s))^p \, ds dt\right)^{\frac{1}{p}}$.

*Then* (2.1) *has a positive solution in* $L^p(\Omega, \mathbb{R}^n)$.

*Proof.* By (2.4), we have

$$\|A_i \mathbf{z}\| \leq \|g_i\|_{L^p(\Omega)} + \|La_i\|_{L^p(\Omega)} + b_i \|k\| \|\mathbf{z}\|^{p-1}$$

and

$$\|A\mathbf{z}\| \leq \|g\| + \omega + b\|k\| \|\mathbf{z}\|^{p-1}, \tag{2.9}$$

where $\|g\| = \max\{\|g_i\|_{L^p(\Omega)} : i \in I_n\}$ and $\omega = \max\{\|La_i\|_{L^p(\Omega)} : i \in I_n\}$.

If $1 < p < 2$, then by (2.9) we have

$$\lim_{\|\mathbf{z}\| \to \infty} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|} \leq \lim_{\|\mathbf{z}\| \to \infty} \frac{\|g\|}{\|\mathbf{z}\|} + \lim_{\|\mathbf{z}\| \to \infty} \frac{\omega}{\|\mathbf{z}\|} + \lim_{\|\mathbf{z}\| \to \infty} \frac{b\|k\|}{\|\mathbf{z}\|^{2-p}} = 0.$$

If $p = 2$, then by (2.9) we have

$$\lim_{\|\mathbf{z}\| \to \infty} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|} \leq \lim_{\|\mathbf{z}\| \to \infty} \frac{\|g\|}{\|\mathbf{z}\|} + \lim_{\|\mathbf{z}\| \to \infty} \frac{\omega}{\|\mathbf{z}\|} + b\|k\| = b\|k\| < 1.$$

The result follows from Lemma 2.2 *(ii)*. □

When $n = 1$ and $\Omega = [a,b]$, existence of (not necessarily positive) solutions of (2.1) is obtained in [18, Theorem 6 ].

Now, we give new results on existence of positive solutions in $L^p(\Omega, \mathbb{R}^n)$ of (2.1) when $p \in [2, \infty)$.

Recall that the radius of the spectrum of the linear operator $L$, denoted by $r(L)$, is given by the well-known spectral radius formula

$$r(L) = \lim_{m \to \infty} \sqrt[m]{\|L\|^m}, \tag{2.10}$$

where $\|L\|$ is the norm of $L$. We write $\mu_1 = 1/r(L)$.

**Notation:** Let $E$ be a fixed subset of $[0,1]$ of measure zero and let

$$\overline{f_i}(\mathbf{z}) = \sup_{s \in \Omega \setminus E} f_i(s, \mathbf{z}) \quad \text{and } (f_i)^\infty = \limsup_{|\mathbf{z}| \to \infty} \overline{f_i}(\mathbf{z})/|\mathbf{z}|.$$

**Theorem 2.4.** *Assume that $p \in [2, \infty)$, $(h_1)$, $(h_2)$, $(h_4)$ hold, $r(L) > 0$ and the following condition holds.*

$$(f_i)^\infty < \mu_1 \quad \text{for each } i \in I_n.$$

*Then* (2.1) *has a positive solution in* $L^p(\Omega, \mathbb{R}^n)$.

*Proof.* Since $(f_i)^\infty < \mu_1$, there exist $\varepsilon > 0$ and $\rho_1 > 0$ such that for each $i \in I_n$,

$$f_i(s, \mathbf{z}) \le (\mu_1 - \varepsilon)|\mathbf{z}| \quad \text{for a.e. } s \in \Omega \text{ and all } \mathbf{z} \in (\mathbb{R}^n_+)_{[\rho_1,\infty)}. \tag{2.11}$$

Let $\rho_2 > \max\{1, \rho_1\}$. Then when $p \in [2, \infty)$, we have

$$|\mathbf{z}| \le |\mathbf{z}|^{p-1} \quad \text{for } \mathbf{z} \in (\mathbb{R}^n_+)_{[\rho_2,\infty)}.$$

This, together with (2.11), implies

$$f_i(s, \mathbf{z}) \le (\mu_1 - \varepsilon)|\mathbf{z}| \le (\mu_1 - \varepsilon)|\mathbf{z}|^{p-1} \quad \text{for a.e. } s \in \Omega \text{ and all } \mathbf{z} \in (\mathbb{R}^n_+)_{[\rho_2,\infty)}. \tag{2.12}$$

Let $u_0(s) = \max\{a_{i,\rho_2}(s) : i \in I_n\}$. Then $u_0 \in L^q_+(\Omega)$. By $(h_4)$ and (2.12), we have

$$f_i(s, \mathbf{z}) \le u_0(s) + (\mu_1 - \varepsilon)|\mathbf{z}|^{p-1} \quad \text{for a.e. } s \in \Omega \text{ and all } \mathbf{z} \in \mathbb{R}^n_+$$

and $(h_3)$ holds.

Let $u(s) = \max\{a_{i,\rho_1}(s) : i \in I_n\}$. By $(h_4)$ and (2.11), we have

$$f_i(s, \mathbf{z}) \le u(s) + (\mu_1 - \varepsilon)|\mathbf{z}| \quad \text{for a.e. } s \in \Omega \text{ and all } \mathbf{z} \in \mathbb{R}^n_+. \tag{2.13}$$

Since $r((\mu_1 - \varepsilon)L) = (\mu_1 - \varepsilon)r(L) < 1$, $(I - (\mu_1 - \varepsilon)L)^{-1}$ exists and is bounded and satisfies

$$(I - (\mu_1 - \varepsilon)L)^{-1}(L^p_+(\Omega)) \subset L^p_+(\Omega). \tag{2.14}$$

Let $g(s) = \max\{g_i(s) : i \in I_n\}$. Then $g \in L^p_+(\Omega)$. Let

$$\rho^* = \|((I - (\mu_1 - \varepsilon)L)^{-1}(g + Lu))\|$$

and $\rho > \rho^*$. We prove

$$\mathbf{z} \ne \varrho A\mathbf{z} \quad \text{for } \mathbf{z} \in \partial P_\rho \text{ and } \varrho \in [0, 1]. \tag{2.15}$$

Indeed, if not, there exist $\mathbf{z} \in \partial P_\rho$ and $\varrho \in [0, 1]$ such that $\mathbf{z} = \varrho A\mathbf{z}$. By (2.13), we have for each $i \in I_n$,

$$z_i(s) \le g(s) + Lu(s) + (\mu_1 - \varepsilon)(L|\mathbf{z}|)(s) \quad \text{for a.e. } s \in \Omega,$$

where $|\mathbf{z}|(s) = \max\{|z_i(s)| : i \in I_n\}$. Taking the maximum in the above inequality implies

$$|\mathbf{z}|(s) \le g(s) + Lu(s) + (\mu_1 - \varepsilon)(L|\mathbf{z}|)(s) \quad \text{for a.e. } s \in \Omega$$

and $(I - (\mu_1 - \varepsilon)L)|\mathbf{z}|(s) \le g(s) + Lu(s)$ for a.e. $s \in \Omega$. This, together with (2.14), implies

$$|\mathbf{z}|(s) \le ((I - (\mu_1 - \varepsilon)L)^{-1}(g + Lu))(s) \quad \text{for a.e. } s \in \Omega.$$

Hence, we have

$$\rho = \|\mathbf{z}\| \le \||\mathbf{z}|\| \le \|((I - (\mu_1 - \varepsilon)L)^{-1}(g + Lu)))\| = \rho^* < \rho,$$

a contradiction. By (2.15) and Lemma 2.2 $(i)$, (2.1) has a positive solution in $P_\rho$. $\square$

Note that both Theorems 2.3 and 2.4 contain the case when $p = 2$. However, in some cases, they are different. In fact, since $r(L) \le \|L\| \le \|k\|$, when $p = 2$ and $a_i \in L^\infty_+(\Omega)$, then $(h_3)$ implies

$$(f_i)^\infty \le b_i < \frac{1}{\|k\|} \le \mu_1 \quad \text{for each } i \in I_n.$$

Hence, if $p = 2$ and $a_i \in L^\infty_+(\Omega)$, then Theorem 2.3 $(ii)$ is a special case of Theorem 2.4. However, if $a_i \notin L^\infty_+(\Omega)$, Theorems 2.3 and 2.4 may not be same.

In Theorem 2.4, $r(L) > 0$ is required. In the following, we show that if $p \in [2, \infty)$ and $k$ is symmetric, then $r(L) > 0$.

**Lemma 2.5.** *Assume that $p \in [2, \infty)$ and $k$ satisfies $(h_2)$ and the following condition:*

$(S)$ *$k(t, s) = k(s, t)$ for $t, s \in \Omega$ and $k(t, s) \ne 0$ a.e. on $\Omega \times \Omega$.*

*Then $r(L) \in (0, \infty)$.*

*Proof.* Since $p \in [2, \infty)$ and $\text{meas}(\Omega) \in (0, \infty)$, we have

$$L^p(\Omega, \mathbb{R}^n) \subset L^2(\Omega, \mathbb{R}^n) \subset L^q(\Omega, \mathbb{R}^n) \tag{2.16}$$

and by $(h_2)$ we obtain

$$\int_\Omega \int_\Omega |k(t, s)|^2 \, ds dt < \infty.$$

It follows from $(S)$ that $L|_{L^2(\Omega)} : L^2(\Omega) \to L^2(\Omega)$ is a compact self-adjoint linear operator and $L|_{L^2(\Omega,\mathbb{R})}$ has a nonzero real eigenvalue denoted by $\lambda_0$. Hence,

$$r(L|_{L^2(\Omega)}) \ge |\lambda_0| > 0.$$

Since $r(L) \ge r(L|_{L^2(\Omega)})$, we obtain $r(L) > 0$. ☐

By Lemma 2.5 and Theorem 2.4, we obtain the following result.

**Corollary 2.6.** *If the condition $r(L) > 0$ in Theorem 2.4 is replaced by the condition $(S)$ of Lemma 2.5, then (2.1) has a positive solution in $L^p(\Omega, \mathbb{R}^n)$.*

We refer to [3, Theorem 5.8] and [7, Theorem 3.4.1] for results on nonzero positive solutions under superlinear conditions, where $n = 1$.

As illustration, we consider existence of positive solutions in $L^p(\Omega, \mathbb{R}^n)$ of the following system

$$z_i(t) = g_i(t) + \int_\Omega k(t, s)[a_i(s) + u_i(s)|\mathbf{z}|_0^{\alpha_i}(s) + v_i(s)|\mathbf{z}|_0^{\beta_i}(s)] \, ds \quad \text{a.e. on } \Omega \text{ and } i \in I_n, \tag{2.17}$$

where $|\cdot|_0$ denotes a norm in $\mathbb{R}^n$.

Since $|\cdot|$ and $|\cdot|_0$ are norms in $\mathbb{R}^n$, there exists $\sigma > 0$ such that

$$|\mathbf{z}|_0 \le \sigma |\mathbf{z}|. \tag{2.18}$$

**Theorem 2.7.** *Assume that $p \in [2, \infty)$, $(h_1)$, $(h_2)$ and the condition $(S)$ of Lemma 2.5 hold and for each $i \in I_n$, the following conditions hold.*

$(i)$ $a_i \in L^\infty_+(\Omega)$.

*(ii)* $0 < \alpha_i < 1$ *and* $u_i, v_i \in L^\infty_+(\Omega)$.

*(iii)* *One of the following conditions holds:*

$(C_1)$ $0 < \beta_i < 1$.

$(C_2)$ $\beta_i = 1$ *and* $\|v_i\|_{C(\Omega)} < \mu_1/\sigma$, *where* $\sigma$ *is same as in* (2.18).

*Then* (2.17) *has a positive solution in* $L^p(\Omega, \mathbb{R}^n)$.

*Proof.* For each $i \in I_n$, we define a function $f_i : \Omega \times \mathbb{R}^n_+ \to \mathbb{R}_+$ by

$$f_i(s, \mathbf{z}) = a_i(s) + u_i(s)|\mathbf{z}|_0^{\alpha_i} + v_i(s)|\mathbf{z}|_0^{\beta_i}.$$

Then $f_i$ satisfies Carathéodory conditions. For each $r > 0$, let

$$a_{i,r}(s) = a_i(s) + u_i(s)\sigma^{\alpha_i} r^{\alpha_i} + v_i(s)\sigma^{\beta_i} r^{\beta_i}.$$

Then $a_{i,r} \in L^\infty_+(\Omega) \subset L^q_+(\Omega)$ and we have for a.e. $s \in \Omega$ and all $\mathbf{z} \in (\mathbb{R}^n_+)_{[0,r]}$,

$$f_i(s, \mathbf{z}) \le a_i(s) + u_i(s)\sigma^{\alpha_i}|\mathbf{z}|^{\alpha_i} + v_i(s)\sigma^{\beta_i}|\mathbf{z}|^{\beta_i} \le a_{i,r}(s). \tag{2.19}$$

Hence, $(h_4)$ holds. By (2.19), we obtain

$$\overline{f_i}(\mathbf{z}) \le \|a_i\|_{L^\infty(\Omega)} + \|u_i\|_{L^\infty(\Omega)}\sigma^{\alpha_i}|\mathbf{z}|^{\alpha_i} + \|v_i\|_{L^\infty(\Omega)}\sigma^{\beta_i}|\mathbf{z}|^{\beta_i} \quad \text{for } i \in I_n. \tag{2.20}$$

If $(C_1)$ holds, then by (2.20) we have

$$(f_i)^\infty = \limsup_{|\mathbf{z}|\to\infty} \overline{f_i}(\mathbf{z})/|\mathbf{z}| = 0 < \mu_1.$$

If $(C_2)$ holds, then by (2.20) we have

$$(f_i)^\infty = \limsup_{|\mathbf{z}|\to\infty} \overline{f_i}(\mathbf{z})/|\mathbf{z}| \le \|v_i\|\sigma < \mu_1.$$

The result follows from Corollary 2.6. □

**Remark 2.8.** There are a lot of functions $g_i$ and kernels $k$ which satisfy the conditions of Theorem 2.7. For example, for each $i \in I_n$, let $\alpha_i \in (0, 1/p)$, $g_i(t) = 1/t^{\alpha_i}$ for $t \in (0, 1)$ and

$$k(t, s) = \begin{cases} s(1-t), & \text{if } 0 \le s \le t \le 1, \\ t(1-s), & \text{if } 0 \le t < s \le 1, \end{cases} \tag{2.21}$$

then $g_i \in L^p(0, 1)$ and by [22, Theorem 5.1], we obtain $\mu_1 = \pi^2$. The kernel $k$ can be replaced by a more general kernel arising from the separeted boundary conditions given in [22, section 5].

## ACKNOWLEDGMENTS

The author was supported in part by the Natural Sciences and Engineering Research Council (NSERC) of Canada.

## REFERENCES


[1] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, Constant-sign solutions of a system of Fredholm integral equations, *Acta Appl. Math.* **80** (2004), 57-94.

[2] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM. Rev.* **18** (1976), 620-709.

[3] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis* **14**(1973), 349-381.

[4] L. Erbe, Eigenvalue criteria for existence of positive solutions to nonlinear boundary value problems, *Math. Computer Modelling* **32** (2000), 529–539.

[5] D. Franco, G. Infante and D. O'Regan, Nontrivial solutions in abstract cones for Hammerstein integral systems, *Dyn. Contin. Discrete Impuls. Syst.* Ser. A **14** (2007), 837-850.

[6] D. Guo, The number of nontrivial solutions of Hammerstein nonlinear integral equations, *Chinese Ann. Math.* Ser. B **7** (1986) (2), 191-204.

[7] D. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, San Diego, 1988.

[8] G. Infante and P. Pietramala, Existence and multiplicity of non-negative solutions for systems of perturbed Hammerstein integral equations, *Nonlinear Anal.* **71** (2009), 1301-1310.

[9] M. A. Krasnosel'skii, Positive Solutions of Operator Equations, P. Noordhoff, Groningen, The Netherlands, 1964.

[10] M. A. Krasnosel'skii, Topological Methods in the Theory of Nonlinear Integral Equations, Macmillan, New York, 1964

[11] K. Q. Lan, Nonzero positive solutions of systems of elliptic boundary value problems, accepted for publication in *Proc. Amer. Math. Soc.*

[12] K.Q. Lan, Multiple positive solutions of semilinear differential equations with singularities, *J. London Math. Soc.*, **63** (2001) (2), 690-704.

[13] K. Q. Lan, Multiple positive solutions of Hammerstein integral equations with singularities, *Differential Equations and Dynamical Systems* **8** (2000), 175-192.

[14] K. Q. Lan and W. Lin, Multiple positive solutions of systems of Hammerstein integral equations with applications to fractional differential equations, accepted for publication in *J. London Math. Soc.*

[15] K. Q. Lan and J. R. L. Webb, Positive solutions of semilinear differential equations with singularities, *J. Differential Equations* **148** (1998), 407-421.

[16] J. Banaś, Integrable solutions of Hammerstein and Urysohn integral equations, *J. Austral. Math. Soc.* Ser. A **46** (1989), 61-68.

[17] J. Banaś and W. G. Ei-Sayed, Measures of noncompactness and solvability of an integral equations in the class of functions of locally bounded variation, *J. Math. Anal. Appl.* **167** (1992), 133-151.

[18] A. Karoui and A. Jawahdou, Existence and approximate $L^p$ and continuous solutions of nonlinear integral equations of the Hammerstein and Volterra types, *Appl. Math. Comput.* **216**(2010), 2077-2091.

[19] D. O'Regan, A note on solutions in $L^1[0,1]$ to Hammerstein integral equations, *J. Integral Equations* **9** (1997) (2), 165-178.

[20] G. Emmanuele, Integrable solutions of a functional-integral equation, *J. Integraal Equations Appl.* **4** (1992), 89-94.

[21] G. Emmanuele, Integrable solutions of Hammerstein integral equations, Appl. Anal. **50** (1993), 277-284.

[22] J. R. L. Webb and K. Q. Lan, Eigenvalue criteria for existence of multiple positive solutions of nonlinear boundary value problems of local and nonlocal type, *Topol. Methods Nonlinear Anal.* **27** (2006)(1), 91-116.