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EXISTENCE OF EXTREMAL SOLUTIONS FOR SOME FOURTH ORDER FUNCTIONAL BVPS

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Dedicated to Professor J. Webb on the occasion of his retirement

ABSTRACT. In this work we present sufficient conditions for the existence of extremal solutions for the fourth order functional problem composed by the equation

$$-\left(\phi\left(u'''(x)\right)\right)' = f(x, u''(x), u'''(x), u, u', u''),$$

for *a.a.* $x \in]0,1[$, where ϕ is an increasing homeomorphism, $I := [0,1]$, and $f : I \times \mathbb{R}^2 \times (C(I))^3 \to \mathbb{R}$ is a L^1-Carathéodory function, and the boundary conditions

$$\begin{aligned}
0 &= L_1\left(u\left(a\right), u, u', u''\right)\\
0 &= L_2\left(u'\left(a\right), u, u', u''\right)\\
0 &= L_3\left(u''\left(a\right), u''\left(b\right), u'''\left(a\right), u'''\left(b\right), u, u', u''\right)\\
0 &= L_4\left(u''\left(a\right), u''\left(b\right)\right)
\end{aligned}$$

where L_i, $i = 1, 2, 3, 4$, are suitable functions with L_1 and L_2 not necessarily continuous, satisfying some monotonicity assumptions.

The arguments make use of lower and upper solutions technique, a version of Bolzano's theorem and existence of extremal fixed points for a suitable mapping.

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1. INTRODUCTION

This paper provides sufficient conditions for the existence of extremal solutions to the fourth order functional equation

$$-\left(\phi\left(u'''(x)\right)\right)' = f(x, u''(x), u'''(x), u, u', u''), \tag{1.1}$$

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for $a.a.\ x \in]0,1[$, with ϕ an increasing homeomorphism, $I := [0,1]$, and $f : I \times \mathbb{R}^2 \times (C(I))^3 \to \mathbb{R}$ a L^1-Carathéodory function, coupled with the boundary conditions

$$0 = L_1 \left(u\left(a\right), u, u', u''\right) \tag{1.2}$$
$$0 = L_2 \left(u'\left(a\right), u, u', u''\right), \tag{1.3}$$
$$0 = L_3 \left(u''\left(a\right), u''\left(b\right), u'''\left(a\right), u'''\left(b\right), u, u', u''\right) \tag{1.4}$$
$$0 = L_4 \left(u''\left(a\right), u''\left(b\right)\right), \tag{1.5}$$

where L_i, $i = 1,2,3,4$, are suitable functions, with L_1 and L_2 not necessarily continuous, satisfying some monotonicity assumptions to be specified.

Due to the functional dependences in the differential equation, which nonlinearity does not need to be continuous in the independent variable and in the functional part, and in the boundary conditions, (1.1)–(1.5) covers many types of boundary value problems, such as integro-differential, with advances, delays, deviated arguments, nonlinear, multi-point,... For a small sample of works in these fields we mention [1, 2, 10, 11, 12, 14, 15, 17, 18, 21, 22, 23] for nonlinear boundary conditions, and [3, 5, 6, 7, 19, 20] for functional problems. In the research for sufficient conditions to guarantee the existence of extremal solutions we refer, as example, [8, 16], for first and second order, and [4, 9], for higher orders.

The arguments used in this work follow the technique suggested by [9]. In short, it is considered a reduced order auxiliary problem together with two algebraic equations, the lower and upper solutions method, a sharp version of Bolzano's theorem and the existence of extremal fixed points for a suitable operator. However the new boundary functions assumed here, (1.2) and (1.3), require other types of monotonicity in the differential equation and in the boundary conditions, and, moreover, different definitions of lower and upper solutions with their first derivatives well-ordered. Therefore, (1.1)–(1.5) can be applied to different problems, not covered by the existent literature.

To emphasize these features, in the last section we consider an example that can not be solved by previous results, due to:

- the monotone growth not only in the equation but also in the boundary conditions;
- the monotonicity of the functional part is different in the two first boundary functions and in the third one;
- functions L_1 and L_2 in Definition 3.1 have opposite sign to L_3;
- lower and upper solutions are well ordered and their first and second derivatives as well.

2. AUXILIARY PROBLEM

Let us consider the nonlinear second order problem

$$-(\phi\,(y'(x))' = g(x, y(x), y'(x)) \quad \text{for a.a. } t \in I, \tag{2.1}$$

$$0 = l_1(y(a), y(b), y'(a), y'(b)), \tag{2.2}$$

$$0 = l_2(y(a), y(b)), \tag{2.3}$$

where $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing homeomorphism and $g : I \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ a Carathéodory function, i.e., $g(x, \cdot, \cdot)$ is a continuous function for a.a. $x \in I$, $g(\cdot, u, v)$ is measurable for all $(u, v) \in \mathbb{R}^2$, and for every $M > 0$ there exists a real-valued function $h_M \in L^1(I)$ such that for a.a. $x \in I$ and for every $(u, v) \in \mathbb{R}^2$ with $|u| \le M$ and $|v| \le M$ we have $|g(x, u, v)| \le h_M(t)$.

Moreover, the function $l_1 : \mathbb{R}^4 \to \mathbb{R}$ is continuous, nondecreasing in the third variable and nonincreasing in the fourth one, and $l_2 : \mathbb{R}^2 \to \mathbb{R}$ is continuous, nonincreasing with respect to its first variable and injective in the second argument.

We will denote by $AC(I)$ the set of absolutely continuous functions on I and by a solution of (2.1) we mean a function $\eta \in \mathcal{C}^1(I)$ such that $\phi\,(\eta') \in AC(I)$ and satisfying the differential equation almost everywhere on I.

Lemma 2.1 ([8, Theorem 4.1])**.** *Suppose that there exist $\alpha, \beta \in \mathcal{C}^1(I)$ such that $\alpha \le \beta$ on I, $\phi\,(\alpha')$, $\phi\,(\beta') \in AC(I)$, and*

$$-(\phi\,(\alpha'))'(x) \le g(x, \alpha(x), \alpha'(x)) \quad \textit{for a.a. } x \in I,$$

$$-(\phi\,(\beta'))'(x) \ge g(x, \beta(x), \beta'(x)) \quad \textit{for a.a. } x \in I,$$

$$l_1(\alpha(a), \alpha(b), \alpha'(a), \alpha'(b)) \ge 0 \ge l_1(\beta(a), \beta(b), \beta'(a), \beta'(b)),$$

$$l_2(\alpha(a), \alpha(b)) = 0 = l_2(\beta(a), \beta(b)).$$

Suppose that a Nagumo condition relative to α and β is satisfied, i.e., there exist functions $k \in L^p(I)$, $1 \le p \le \infty$, and $\theta : [0, +\infty) \longrightarrow (0, +\infty)$ continuous, such that, for a.a. $t \in I$,

$$|g(x, u, v)| \le k(x)\,\theta(|v|) \quad \textit{for all } u \in [\alpha(t), \beta(t)] \textit{ and all } v \in \mathbb{R},$$

and

$$\min\left\{\int_{\phi(\nu)}^{+\infty} \frac{|\phi^{-1}(u)|^{\frac{p-1}{p}}}{\theta(|\phi^{-1}(u)|)} du, \int_{-\infty}^{\phi(-\nu)} \frac{|\phi^{-1}(u)|^{\frac{p-1}{p}}}{\theta(|\phi^{-1}(u)|)} du\right\} > \mu^{\frac{p-1}{p}} \|k\|_p,$$

where

$$\mu = \max_{x \in I} \beta(x) - \min_{x \in I} \alpha(x),$$

$$\nu = \frac{\max\{|\alpha(a) - \beta(b)|, |\alpha(b) - \beta(a)|\}}{b - a},$$

$$\|k\|_p = \begin{cases} ess\ sup_{x\in I}|k(x)| & , \quad p=\infty \\ \left[\int_a^b |k(x)|^p dx\right]^{\frac{1}{p}} & , \quad 1\le p<\infty \end{cases}$$

where "ess sup" means essential supremum and considering $(p-1)/p \equiv 1$ *for* $p=\infty$.

Then the problem (2.1)–(2.3) *has extremal solutions in*

$$[\alpha,\beta] := \{\gamma \in \mathcal{C}^1(I) \,:\, \alpha \le \gamma \le \beta \quad on\ I\},$$

i.e., there exist a least and a greatest solution to the problem in the functional interval $[\alpha,\beta]$.

Remark 2.2. The Nagumo condition guarantees that the first derivative is *a priori* bounded, i.e., there exists $N>0$, depending only on α, β, k, θ, ϕ and p, such that every solution $y\in[\alpha,\beta]$ of (2.1)–(2.3) satisfies $|y'(t)|\le N$ for all $t\in I$.

3. EXISTENCE OF EXTREMAL SOLUTIONS

In the following, a mapping $\omega : \mathcal{C}(I) \to \mathbb{R}$ is nondecreasing if $\omega(\gamma)\le\omega(\delta)$ whenever $\gamma(x)\le\delta(x)$ for all $x\in I$, and ω is nonincreasing if $\omega(\gamma)\ge\omega(\delta)$ whenever $\gamma(x)\le\delta(x)$ for all $x\in I$).

Let us consider now the initial problem (1.1)–(1.5) with the following assumptions:

(E) $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing homeomorphism and $f : I\times\mathbb{R}^2\times(\mathcal{C}(I))^3 \longrightarrow \mathbb{R}$ satisfying:
- (a) For all $(u,v,\gamma,\delta,\varepsilon)\in\mathbb{R}^2\times(\mathcal{C}(I))^3$, $f(\cdot,u,v,\gamma,\delta,\varepsilon)$ is measurable;
- (b) For a.a. $x\in I$ and all $(u,v,\gamma,\delta,\varepsilon)\in\mathbb{R}^2\times(\mathcal{C}(I))^3$, $f(x,u,v,\cdot,\delta,\varepsilon)$, $f(x,u,v,\gamma,\cdot,\varepsilon)$ and $f(t,u,v,\gamma,\delta,\cdot)$ are nondecreasing;
- (c) For a.a. $x\in I$ and all $(\gamma,\delta,\varepsilon)\in(\mathcal{C}(I))^3$, $f(x,\cdot,\cdot,\gamma,\delta,\varepsilon)$ is continuous on $\mathbb{R}^2$;
- (d) For every $M>0$ there exists a real-valued function $h_M\in L^1(I)$ such that for a.a. $x\in I$ and for every $(u,v,\gamma,\delta,\varepsilon)\in\mathbb{R}^2\times(\mathcal{C}(I))^3$ with

$$|u|+|v|+\max_{x\in I}|\gamma(x)|+\max_{x\in I}|\delta(x)|+\max_{x\in I}|\varepsilon(x)|\le M$$

  we have $|f(x,u,v,\gamma,\delta,\varepsilon)|\le h_M(t)$.

$(L1)$ For $i=1,2$, for all $\gamma,\delta,\varepsilon\in\mathcal{C}(I)$, and for all $t\in\mathbb{R}$, we have

$$\limsup_{y\to t^-} L_i(y,\gamma,\delta,\varepsilon)\le L_i(t,\gamma,\delta,\varepsilon)\le \liminf_{y\to t^+} L_i(y,\gamma,\delta,\varepsilon)$$

and the mappings L_i are nonincreasing in the second, third and fourth arguments.

$(L2)$ For every $\gamma,\delta,\varepsilon\in\mathcal{C}(I)$ the mappings

$$l_1 : (t,y,u,v)\in\mathbb{R}^4 \longmapsto l_1(t,y,u,v) := L_3(t,y,u,v,\gamma,\delta,\varepsilon)$$

and L_4 satisfy the conditions assumed for l_1 and l_2 in the previous section. Moveover, the operator L_3 is nondecreasing in the fifth, sixth and seventh arguments.

Definition 3.1. A function $\alpha \in \mathcal{C}^3(I)$ is a lower solution of (1.1)–(1.5) if $\phi(\alpha''') \in AC(I)$ and

$$\begin{aligned} -(\phi(\alpha'''))'(x) &\le f(x, \alpha''(x), \alpha'''(x), \alpha, \alpha', \alpha'') \quad \text{for a.a. } x \in I = [a,b], \\ 0 &\ge L_1(\alpha(a), \alpha, \alpha', \alpha''), \\ 0 &\ge L_2(\alpha'(a), \alpha, \alpha', \alpha''), \\ 0 &\le L_3(\alpha''(a), \alpha''(b), \alpha'''(a), \alpha'''(b), \alpha, \alpha', \alpha''), \\ 0 &= L_4(\alpha''(a), \alpha''(b)). \end{aligned}$$

An upper solution is defined analogously with the reverse inequalities.

In the sequel we will use the following notation: for a couple of functions $\gamma, \delta \in \mathcal{C}(I)$ such that $\gamma \le \delta$ on I, we define

$$[\gamma, \delta] := \{\, \xi \in \mathcal{C}(I) \,:\, \gamma \le \xi \le \delta \text{ on } I \}.$$

Definition 3.2. Let $\alpha,\ \beta \in \mathcal{C}^3(I)$ be such that $\alpha^{(i)} \le \beta^{(i)}$ on I for $i = 0, 1, 2$. We say that $f : I \times \mathbb{R}^2 \times (\mathcal{C}(I))^3 \to \mathbb{R}$ satisfies a Nagumo condition relative to α and β if there exist functions $k \in L^p(I)$, $\ 1 \le p \le \infty$, and $\theta : [0, +\infty) \longrightarrow (0, +\infty)$ continuous, such that, for a.e. $x \in I$, for all $u \in [\alpha''(t), \beta''(t)]$ and for all $(\gamma, \delta, \varepsilon) \in [\alpha, \beta] \times [\beta', \alpha'] \times [\alpha'', \beta'']$, we have

$$|f(x, u, v, \gamma, \delta, \varepsilon)| \le k(x)\, \theta(|v|) \quad \text{for all } v \in \mathbb{R},$$

and

$$\min\left\{ \int_{\phi(\nu)}^{+\infty} \frac{|\phi^{-1}(u)|^{\frac{p-1}{p}}}{\theta(|\phi^{-1}(u)|)} du,\ \int_{-\infty}^{\phi(-\nu)} \frac{|\phi^{-1}(u)|^{\frac{p-1}{p}}}{\theta(|\phi^{-1}(u)|)} du \right\} > \mu^{\frac{p-1}{p}} \|k\|_p,$$

where

$$\mu = \max_{x \in I} \beta''(x) - \min_{x \in I} \alpha''(x)$$

and

$$\nu = \frac{\max\{|\alpha''(a) - \beta''(b)|, |\alpha''(b) - \beta''(a)|\}}{b - a}.$$

The following version of Bolzano's theorem plays a key role in the proof of the main result:

Lemma 3.3 ([13, Lemma 2.3])**.** *Let $a, b \in \mathbb{R}$, $a \le b$, and let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be such that either $h(a) \ge 0 \ge h(b)$ and*

$$\limsup_{z \to x^-} h(z) \le h(x) \le \liminf_{z \to x^+} h(z) \quad \textit{for all } x \in [a, b],$$

or $h(a) \le 0 \le h(b)$ *and*

$$\liminf_{z\to x^-} h(z) \ge h(x) \ge \limsup_{z\to x^+} h(z) \quad \textit{for all } x \in [a,b].$$

Then there exist $c_1, c_2 \in [a,b]$ *such that* $h(c_1) = 0 = h(c_2)$ *and if* $h(c) = 0$ *for some* $c \in [a,b]$ *then* $c_1 \le c \le c_2$*, i.e.,* c_1 *and* c_2 *are, respectively, the least and the greatest of the zeros of* h *in* $[a,b]$*.*

For the reader's convenience let us introduce some additional notation which allows more concise statements.

In $\mathcal{C}^2(I)$ we consider the standard partial ordering: Given $\gamma, \delta \in \mathcal{C}^2(I)$,

$$\gamma \widetilde{\le} \delta \text{ if and only if } \gamma^{(i)} \le \delta^{(i)} \text{ on } I \text{ for } i = 0,1,2.$$

Notice that $\mathcal{C}^2(I)$ is an ordered metric space when equipped with this partial ordering together with the usual metric, in the sense that for every $\gamma \in \mathcal{C}^2(I)$ the intervals

$$[\gamma)_{\widetilde{\le}} = \{\delta \in \mathcal{C}^2(I) \,:\, \gamma \widetilde{\le} \delta\} \quad \text{and} \quad (\gamma]_{\widetilde{\le}} = \{\delta \in \mathcal{C}^2(I) \,:\, \delta \widetilde{\le} \gamma\},$$

are closed in the corresponding topology. More details about ordered metric spaces can be seen in [16].

For $\gamma, \delta \in \mathcal{C}^2(I)$ such that $\gamma \widetilde{\le} \delta$ define

$$[\gamma, \delta]_{\widetilde{\le}} := \{\, \xi \in \mathcal{C}^2(I) \,:\, \gamma \widetilde{\le} \xi \widetilde{\le} \delta\}.$$

The function γ^* is the $\widetilde{\le}$-greatest solution of (1.1)–(1.5) in $[\gamma, \delta]_{\widetilde{\le}}$ if γ^* is a solution of (1.1)–(1.5) which belongs to $[\gamma, \delta]_{\widetilde{\le}}$ and such that for any other solution $\gamma \in [\gamma, \delta]_{\widetilde{\le}}$ we have $\gamma \le \gamma^*$. The $\widetilde{\le}$-least solution of (1.1)–(1.5) in $[\gamma, \delta]_{\widetilde{\le}}$ is defined analogously. If the $\widetilde{\le}$-least and $\widetilde{\le}$-greatest solutions of (1.1)–(1.5) in $[\gamma, \delta]_{\widetilde{\le}}$ exist we call them $\widetilde{\le}$-extremal solutions of (1.1)–(1.5) in $[\gamma, \delta]_{\widetilde{\le}}$.

The following fixed point theorem is also useful:

Lemma 3.4 ([16, Theorem 1.2.2])**.** *Let* Y *be a subset of an ordered metric space* $(X, \le)$*,* $[a,b]$ *a nonempty order interval in* Y*, and* $G : [a,b] \to [a,b]$ *a nondecreasing mapping. If* $\{Gx_n\}_n$ *converges in* Y *whenever* $\{x_n\}_n$ *is a monotone sequence in* $[a,b]$*, then there exists* x_* *the least fixed point of* G *in* $[a,b]$ *and* x^* *is the greatest one. Moreover*

$$x_* = \min\{y \,|\, Gy \le y\} \quad \textit{and}\hat{E} \quad x^* = \max\{y \,|\, y \le Gy\}.$$

The main result for problem (1.1)–(1.5) is the following:

Theorem 3.5. *Suppose that conditions* (E)*,* $(L1)$ *and* $(L2)$ *hold, and the problem* (1.1)–(1.5) *has a lower solution* α *and an upper solution* β *such that*

$$\alpha(a) \le \beta(a), \quad \alpha'(a) \le \beta'(a) \quad \textit{and} \quad \alpha'' \le \beta'' \textit{ on } I. \tag{3.1}$$

If f satisfies a Nagumo condition with respect to α and β then the problem (1.1)–(1.5) *has $\widetilde{\leq}$-extremal solutions in $[\alpha,\beta]_{\widetilde{\leq}}$.*

Remark 3.6. The relations (3.1) imply that $\alpha \leq \beta$, by successive integrations between a and $x \in]a,b]$.

Proof. For every $\gamma \in [\alpha,\beta]_{\widetilde{\leq}}$ fixed, consider the nonlinear second-order problem

$$(P_\gamma) \qquad \begin{cases} -(\phi(y'))'(x) &= f(x,y(t),y'(t),\gamma,\gamma',\gamma'') \quad \text{for a.a. } t \in I, \\ 0 &= L_3(y(a),y(b),y'(a),y'(b),\gamma,\gamma',\gamma''), \\ 0 &= L_4(y(a),y(b)), \end{cases}$$

together with the two equations

$$0 = L_1(w,\gamma,\gamma',\gamma''), \tag{3.2}$$

$$0 = L_2(w,\gamma,\gamma',\gamma''). \tag{3.3}$$

By the assumptions, α'' and β'' are, respectively, lower and upper solutions of (P_γ), according to the definitions given in Lemma 2.1. Moreover, as the remaining conditions in Lemma 2.1 are satisfied, there exists the greatest solution of (P_γ) in $[\alpha'',\beta'']$, which will be denoted by y_γ.

According to Remark 2.2, there exists $N > 0$ such that

$$|y'_\gamma(x)| \leq N \quad \text{for all } \gamma \in [\alpha,\beta]_{\widetilde{\leq}} \text{ and all } x \in I. \tag{3.4}$$

On the other hand, we have

$$0 \geq L_1(\alpha(a),\alpha,\alpha',\alpha'') \geq L_1(\alpha(a),\gamma,\gamma',\gamma''),$$

and, similarly, $0 \leq L_1(\beta(a),\gamma,\gamma',\gamma'')$. Thus, by Lemma 3.3, the equation (3.2) has a greatest solution $u_a = u_a(\gamma)$ in $[\alpha(a),\beta(a)]$.

Analogously, the greatest solution of (3.3) in $[\alpha'(a),\beta'(a)]$ exists and it will be denoted by $u'_a = u'_a(\gamma)$.

Define, for each $x \in I$, the functional operator $G : [\alpha,\beta]_{\widetilde{\leq}} \to [\alpha,\beta]_{\widetilde{\leq}}$ by

$$G\gamma(x) := u_a + u'_a(x-a) + \int_a^x \int_a^s y_\gamma(r)drds.$$

In order to prove that G is nondecreasing for the ordering $\widetilde{\leq}$ in $[\alpha,\beta]_{\widetilde{\leq}}$, consider $\gamma_i \in [\alpha,\beta]_{\widetilde{\leq}}$ for $i = 1,2$ such that $\gamma_1 \leq \gamma_2$. The function y_{γ_1} is a lower solution of (P_{γ_2}), and so Lemma 2.1 implies that (P_{γ_2}) has extremal solutions in $[y_{\gamma_1},\beta'']$. In particular, the greatest solution of (P_{γ_2}) between α'' and β'' must be greater than y_{γ_1}, i.e., $y_{\gamma_2} \geq y_{\gamma_1}$ on I.

Furthermore we have

$$0 = L_1(u_a(\gamma_1),\gamma_1,\gamma_1',\gamma_1'') \geq L_1(u_a(\gamma_1),\gamma_2,\gamma_2',\gamma_2''),$$

and, as $\gamma_2 \in [\alpha,\beta]_{\widetilde{\leq}}$ then, by the definition of upper solution, $0 \leq L_1(\beta(a),\gamma_2,\gamma_2',\gamma_2'')$. Hence Lemma 3.3 guarantees that the equation $0 = L_1(w,\gamma_2,\gamma_2',\gamma_2'')$ has extremal solutions in $[u_a(\gamma_1),\, \beta(a)]$. In particular, its greatest solution between $\alpha(a)$ and $\beta(a)$ must be greater than or equal to $u_a(\gamma_1)$, i.e., $u_a(\gamma_2) \geq u_a(\gamma_1)$. In a similar way we deduce that $u_a'(\gamma_2) \geq u_a'(\gamma_1)$ and, therefore, $G\gamma_1 \leq G\gamma_2$.

Let $\{\gamma_n\}_n$ be a $\widetilde{\leq}$-monotone sequence in $[\alpha,\beta]_{\widetilde{\leq}}$. Since G is nondecreasing, the sequence $\{G\gamma_n\}_n$ is also $\widetilde{\leq}$-monotone and, moreover, $G\gamma_n \in [\alpha,\beta]_{\widetilde{\leq}}$ for all $n \in \mathbb{N}$ and $\{G\gamma_n\}_n$ is bounded in $\mathcal{C}^2(I)$.

For all $n \in \mathbb{N}$ and all $x \in I$ it can be verified that

$$(G\gamma_n)'''(x) = y_{\gamma_n}'(x),$$

and, by (3.4), $\{(G\gamma_n)''\}_n$ is equicontinuous on I. So, from Ascoli-Arzelá's theorem $\{(G\gamma_n)''\}_n$ is convergent in $\mathcal{C}^2(I)$. Therefore G applies $\widetilde{\leq}$-monotone sequences into convergent sequences and, by Lemma 3.4, G has a $\widetilde{\leq}$-greatest fixed point in $[\alpha,\beta]_{\widetilde{\leq}}$, denoted by γ^*, such that

$$\gamma^* = \max\{\gamma \in [\alpha,\beta]_{\widetilde{\leq}} \,:\, \gamma \widetilde{\leq} G\gamma\}. \tag{3.5}$$

As γ^* is a solution of (1.1)–(1.5) in $[\alpha,\beta]_{\widetilde{\leq}}$, we will show that γ^* is the $\widetilde{\leq}$-greatest solution of (1.1)–(1.5) in $[\alpha,\beta]_{\widetilde{\leq}}$. Let γ be an arbitrary solution of (1.1)–(1.5) in $[\alpha,\beta]_{\widetilde{\leq}}$. Notice that the relations (1.2) and (1.3), with u replaced by γ, imply that $\gamma(a) \leq u_a(\gamma)$ and $\gamma'(a) \leq u_a'(\gamma)$. Moreover, conditions (1.1), (1.4) and (1.5), with u replaced by γ, imply that $\gamma'' \leq y_\gamma$. Therefore $\gamma \widetilde{\leq} G\gamma$ which, together with (3.5), yields $\gamma \widetilde{\leq} \gamma^*$, so γ^* is the $\widetilde{\leq}$-greatest solution to (1.1)–(1.5) in $[\alpha,\beta]_{\widetilde{\leq}}$.

The existence of the $\widetilde{\leq}$-least solution of (1.1)–(1.5) in $[\alpha,\beta]_{\widetilde{\leq}}$ can be proven by analogous arguments and obvious changes in the definition of the operator G. $\square$

4. EXAMPLE

The example below does not pretend to illustrate some real phenomena, but only to show the applicability of the functional components in the equation and in the boundary conditions. Notice that, like it was referred before, this problem is not covered by the existent results.

Consider the fourth order functional differential equation

$$\begin{aligned} -\frac{u^{(iv)}(x)}{1+(u'''(x))^2} &= -(u''(x))^3 + |u'''(x)+1|^\xi + \max_{x\in I]} u'(x) + \int_0^x u(t)dt \\ &\quad + h(x)\; g\left(\max_{x\in I} u''(x)\right) \end{aligned} \tag{4.1}$$

where $0 \leq \xi \leq 2$, $I := [0,1]$, $h \in L^{\infty}(I,[0,+\infty))$ and $g : \mathbb{R} \to \mathbb{R}$ a nondecreasing function, with the boundary conditions

$$\begin{aligned} A\ (u(0))^{2p+1} &= -\max_{x\in I} u(x) - \sum_{j=1}^{+\infty} a_j\ u'(\xi_j)\,, \\ B\ \sqrt[3]{u'(0)} &= e^{-\max_{x\in I} u''(x)}, \\ C\ u'''(1) &= u'(\max\{0, x-\tau\})\,, \\ u''(0) &= u''(1), \end{aligned} \tag{4.2}$$

where $A, B, C \in \mathbb{R}$, $0 \leq \xi_j \leq 1$, $\forall j \in \mathbb{N}$, $p \in \mathbb{N}$ and $\sum_{j=1}^{+\infty} a_j$ is a nonnegative and convergent series with sum $\overline{a}$.

This problem is a particular case of (1.1)–(1.5), where $\phi(z) = \arctan z$ (remark that $\phi(\mathbb{R}) \neq \mathbb{R}$),

$$\begin{aligned} f(x,y,v,\gamma,\delta,\varepsilon) &= -y^3 + |v+1|^{\xi} + \max_{x\in I} \delta(x) + \int_0^x \gamma(t)dt + h(x)\ g\left(\max_{x\in I} \varepsilon(x)\right), \\ L_1(t,\gamma,\delta,\varepsilon) &= -A\ t^{2p+1} - \max_{x\in I} \gamma(x) - \sum_{j=1}^{+\infty} a_j\ \delta(\xi_j)\,, \\ L_2(t,\gamma,\delta,\varepsilon) &= e^{-\max_{x\in I} \varepsilon(x)} - B\sqrt[3]{t}, \\ L_3(t,y,z,v,w,\gamma,\delta,\varepsilon) &= \delta(\max\{0,x-\tau\}) - C\ w, \\ L_4(t,y) &= y - t. \end{aligned}$$

The functions $\alpha(x) = -x^2 - 2x - 1$ and $\beta(x) = x^2 + 2x + 1$ are, respectively, lower and upper solutions of the problem (4.1)–(4.2) for

$$-\frac{37}{6} \leq h(x)\ g(-2) \leq h(x)\ g(2) \leq \frac{13}{6},\ \forall x \in [0,1],$$

$A \leq -3 - 3\overline{a}$, $B \leq -e^2$ and $C \geq \frac{3}{2}$.

Moreover, the homeomorphism ϕ and the nonlinearity f verify condition (E) and the Nagumo condition given by Definition 3.2 with

$$k(x) \equiv 14,\ \theta(v) = |v+1|^{\xi},\ \mu = 4,\ \upsilon = 4.$$

The boundary functions L_i, $i = 1,2,3,4$, satisfy the assumptions $(L1)$ and $(L2)$. So, by Theorem 3.5, there are $\widetilde{\leq}-$extremal solutions of (4.1)–(4.2) in $[\alpha,\beta]_{\widetilde{\leq}}$.

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