Communications in Applied Analysis **15** (2011), no. 2, 3 and 4, 569–588

LOCATING CERAMI SEQUENCES IN A MOUNTAIN PASS GEOMETRY

C. A. STUART

Section de Mathématiques, Station 8, EPFL
Lausanne, CH 1015 Switzerland
E-mail: charles.stuart@epfl.ch

To J.R.L. Webb on his retirement, congratulations and best wishes

ABSTRACT. Let X be a real Banach space and $\Phi \in C^1(X, \mathbb{R})$ a function with a mountain pass geometry. This ensures the existence of a Palais-Smale, and even a Cerami, sequence $\{u_n\}$ of approximate critical points for the mountain pass level. We obtain information about the location of such a sequence by estimating the distance of u_n from S for certain types of set S as $n \to \infty$. Under our hypotheses we can find a Palais-Smale sequence for the mountain pass level with $d(u_n, S) \to 0$, but in general there is no Cerami sequence with this property and our result yields $d(u_n, S)/(1 + \|u_n\|) \to 0$. Our results extend to Cerami sequences the earlier work on localization of Palais-Smale sequences due to Kuzin-Pohozaev and Ghoussoub-Preiss.

AMS (MOS) Subject Classification. 58E05, 46T05.

1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a real Banach space and $\Phi \in C^1(X, \mathbb{R})$. The search for critical points of Φ at a given level $c \in \mathbb{R}$ is often split into two steps. First, one finds a sequence $\{u_n\}$ of approximate critical points for the level c and then one shows that this sequence has a convergent subsequence. Since the seminal work by Ambrosetti and Rabinowitz, [2], it is well-known that the so called mountain pass geometry provides a particularly useful setting in which the first step can be accomplished. The second step usually involves a compactness property of Φ such as the Palais-Smale condition, [21]. Of course, for this $\{u_n\}$ has to have a bounded subsequence but, in infinite dimensions, this is not sufficient and other features of Φ come into play. In trying to establish either of these properties it may be useful to have some information about the location of the sequence $\{u_n\}$ and this paper provides some results in this direction. To be more precise, let us recall some basic terminology.

A Palais-Smale sequence of approximate critical points of Φ for the level $c \in \mathbb{R}$ is a sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \to c$ and $\|\Phi'(u_n)\|_* \to 0$ where $\|\cdot\|_*$ is the norm on the dual space X^*.

Received October 27, 2010

To deal better with the issue of boundedness, Cerami [4, 5] introduced the following more restrictive notion and its usefulness is now well-established, [6, 20, 10, 25, 23, 22, 24] for example.

A Cerami sequence of approximate critical points of Φ for the level $c \in \mathbb{R}$ is a sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \to c$ and $(1+\|u_n\|)\|\Phi'(u_n)\|_* \to 0$. Every Cerami sequence is a (PS) sequence, but the extra requirement that $\|u_n\|\|\Phi'(u_n)\|_* \to 0$ has proved to be useful when trying to show that there is a bounded subsequence. In particular, it implies that $|\Phi'(u_n)u_n| \to 0$.

The functional Φ is said to have a strong form of the mountain pass geometry when (SMPG) there exist $e \in X\backslash\{0\}$ and $r \in (0, \|e\|)$ such that

$$\max\{\Phi(0), \Phi(e)\} < \inf_{\|u\|=r} \Phi(u).$$

Supposing that (SMPG) holds, the corresponding mountain pass critical level c is defined by

$$c = \inf_{f\in\Gamma} \max_{t\in[0,1]} \Phi(f(t)) \text{ where } \Gamma = \{f \in C([0,1], X) : f(0) = 0 \text{ and } f(1) = e\}.$$

The following result is well-known and has many interesting applications.

Mountain Pass Lemma If $\Phi \in C^1(X, \mathbb{R})$ and satisfies (SMPG), then there exists a Cerami sequence $\{u_n\}$ for the mountain pass level.

The mountain pass theorem is obtained by adding the hypothesis that Φ satisfies the compactness condition: every Cerami sequence for the mountain pass level c contains a convergent subsequence. This yields an element $u \in X$ with $\Phi(u) = c$ and $\nabla\Phi(u) = 0$. (In the original version, this hypothesis was replaced by the stronger assumption that every Palais-Smale sequence for the level c contains a convergent subsequence.) The mountain pass theorem has become a corner stone of nonlinear analysis with many important applications, particularly in the field of elliptic partial differential equations, [1, 13, 19, 21, 26] for example.

As its title suggests, this paper focusses on trying to provide information about the position of the elements of the sequence given by the mountain pass lemma. This can be useful in proving that a convergent subsequence exists and, when this does occur, we have information about where the resulting critical point is located. Historically, this seems to have been approached in two ways which now summarize and relate to the main result of this paper, Theorem 1.1.

1.1 Sequences near almost optimal paths. When (SMPG) holds, a sequence $\{f_n\} \subset \Gamma$ such that $\max_{t\in[0,1]} \Phi(f_n(t)) \to c = \inf_{f\in\Gamma} \max_{t\in[0,1]} \Phi(f(t))$ is called an optimal sequence of paths. In this case, Kuzin and Pohozaev ([17], Theorem E.5)[1] have shown that there exists a (PS) sequence, $\{u_n\}$, for the level c such that $d(u_n, f_n([0,1]) \to$

[1]See the note added in proof.

0 as $n \to \infty$. Of course, if S is a set containing an optimal sequence of paths (in the sense that $f_n([0,1]) \subset S$ for all $n \in \mathbb{N}$) this yields a (PS) sequence $\{u_n\}$ for the level c such that $d(u_n, S) \to 0$.

Since (SMPG) implies the existence of a Cerami sequence at level c, one might expect to find a Cerami sequence satisfying $d(u_n, f_n([0,1]) \to 0$, too. However this is not the case. Indeed in Section 2 we give an example of a function $\Phi \in C^\infty(\mathbb{R}^3, \mathbb{R})$ which satisfies (SMPG) and a set S containing an optimal sequence of paths such that $d(u_n, S) \to \infty$ for every Cerami sequence for the mountain pass level c. Nonetheless some information about the location of a Cerami sequence can still be obtained. In Corollary 1.2 we show that, if (SMPG) holds and $\{f_n\}$ is an optimal sequence of paths, then there exists a Cerami sequence $\{u_n\}$ such that $d(u_n, f_n([0,1])/(1 + \|u_n\|) \to 0$ as $n \to \infty$. Although the example in Section 2 shows that it may happen that $d(u_n, f_n([0,1]) \to \infty$ for a given choice of optimal sequence f_n and every Cerami sequence for the mountain pass level, Theorem 1.1 shows that there do exist a Cerami sequence for this level and an optimal sequence of paths $\{h_n\} \subset \Gamma$ such that $u_n \in h_n([0,1])$ for all n.

In fact, only a weaker form of (SMPG) is required and the full conclusion is that, for every $k \in [0,1]$, there exists a sequence $\{u_n^k\}_{n=0}^\infty \subset X$ such that

$$\Phi(u_n^k) \to c, (1 + \|u_n^k\|)^k \Phi'(u_n^k) \to 0 \text{ and } \frac{d(u_n^k, f_n([0,1]))}{(1 + \|u_n^k\|)^k} \to 0 \text{ as } n \to \infty. \tag{1.1}$$

For $k = 0$ we recover a (PS) sequence of the type found by Kuzin and Pohozaev and for $k = 1$ we have a Cerami sequence. If a set $S \subset X$ contains an optimal sequence of paths, then (1.1) implies that, for every $k \in [0,1]$, there exists a sequence $\{u_n^k\}_{n=0}^\infty \subset X$ such that

$$\Phi(u_n^k) \to c, (1 + \|u_n^k\|)^k \Phi'(u_n^k) \to 0 \text{ and } \frac{d(u_n^k, S)}{(1 + \|u_n^k\|)^k} \to 0 \text{ as } n \to \infty. \tag{1.2}$$

This trivial observation is useful since, although it may be difficult to find explicitly an optimal sequence of paths, it may be easy to describe a set containing such a sequence. For example, if $X = H_0^1(\Omega)$ and if Φ satisfies (SMPG) and has the property that $\Phi(u) = \Phi(|u|)$ for all $u \in H_0^1(\Omega)$, then the positive cone $S = \{u \in H_0^1(\Omega) : u \geq 0 \text{ a.e. in } \Omega\}$ contains an optimal sequence of paths. See Section 5.

1.2 Sequences near a set separating 0 and e.

In [11], Ghoussoub and Preiss introduced a different way of localizing a (PS) sequence and, in the spirit of earlier work by Pucci and Serrin, they did not require (SMPG) but only the following weaker condition. For $\Phi \in C^1(X, \mathbb{R})$ and $e \in X \backslash \{0\}$, let Γ and c be defined as above. Let W be a closed subset of X such that

$$\textbf{(I)}\ \{0, e\} \cap W \cap \Phi_c = \emptyset \text{ and } \textbf{(II)} \text{ for all } f \in \Gamma, f([0,1]) \cap W \cap \Phi_c \neq \emptyset,$$

where $\Phi_c = \{u \in X : \Phi(u) \geq c\}$. In the terminology of Ghoussoub and Preiss, (I) and (II) mean that $W \cap \Phi_c$ separates 0 and e in X (i.e. 0 and e belong to different connected components of $X \backslash [W \cap \Phi_c]$) and this does not require (SMPG) to hold. In Theorem (1) of [11] they proved that (I) and (II) imply the existence of a (PS) sequence for the level c such that $d(u_n, W) \to 0$ as $n \to \infty$. Subsequently, in [8] (Chapter IV.1, Theorem 6), Ekeland showed that, under these conditions, there exists a Cerami sequence for the level c such that $\delta(u_n, W) \to 0$ where δ is the metric on X defined by

$$\delta(u, v) = \inf\{\int_0^1 \frac{\|h'(t)\|}{1 + \|h(t)\|} dt : h \in C^1([0,1], X) \text{ with } h(0) = u \text{ and } h(1) = v\}.$$

(See [7, 10, 13] for further work using other metrics like δ.) In contrast our Theorem 1.1 involves only the usual metric d associated with the norm of X and yields a Cerami sequence for the level c such that $d(u_n, W)/(1 + \|u_n\|) \to 0$ as $n \to \infty$. However, when W is unbounded, which as we observe below is the situation where our results seem new, the relation between our conclusion that $d(u_n, W)/(1 + \|u_n\|) \to 0$ and statements involving other metrics such as $\delta(u_n, W) \to 0$ needs to be clarified and we hope to return to this.[2]

Under the assumptions (I) and (II), we also we have the more general conclusion that, for every $k \in [0, 1]$, there exists a sequence $\{u_n^k\}_{n=0}^\infty \subset X$ such that

$$\Phi(u_n^k) \to c, (1 + \|u_n^k\|)^k \Phi'(u_n^k) \to 0 \text{ and } \frac{d(u_n^k, W)}{(1 + \|u_n^k\|)^k} \to 0 \text{ as } n \to \infty. \tag{1.3}$$

For $k = 0$ we recover the original result of Ghoussoub and Preiss concerning the existence of a (PS) sequence approaching W. Localization of a (PS) sequence has also been established by Willem in a rather general setting. See, for example, Theorem 2.20 in his book [26].

1.3 The main result. Having outlined the context for our work and mentioned some special cases, we now state the main result. By treating both an optimal sequence and an appropriate set W it covers both of the situations presented above in Sections 1.1 and 1.2. For the case of (PS) sequences, this kind of formulation appears in Theorem 3.1 of [9], for example.

For $d \in \mathbb{R}$, recall that $\Phi_d = \{u \in X : \Phi(u) \geq d\}$.

Theorem 1.1. *Let $\Phi \in C^1(X, \mathbb{R})$ and $e \in X \backslash \{0\}$. Set*
$\Gamma = \{f \in C([0,1], X) : f(0) = 0$ and $f(1) = e\}$ and $c = \inf_{f \in \Gamma} \max_{t \in [0,1]} \Phi(f(t))$.
Let $\{f_n\}$ be an optimal sequence of paths: $M_n = \max_{t \in [0,1]} \Phi(f_n(t)) \to c$.
Let W be a closed subset of X such that

(I) *$\{0, e\} \cap W \cap \Phi_c = \emptyset$ and* ***(II)*** *for all $f \in \Gamma, f([0,1]) \cap W \cap \Phi_c \neq \emptyset$.*

[2] See the note added in proof.

Then, for every $k \in [0,1]$, there exists a sequence $\{u_n^k\} \subset X$ such that

$$\Phi(u_n^k) \to c, (1+\|u_n^k\|)^k \Phi'(u_n^k) \to 0, \frac{d(u_n^k, W \cap \Phi_c)}{(1+\|u_n^k\|)^k} \text{ and}$$
$$\frac{d(u_n^k, f_n([0,1]))}{(1+\|u_n^k\|)^k} \to 0 \text{ as } n \to \infty. \tag{1.4}$$

Furthermore, if in addition,

$$(MPG) \qquad \max\{\Phi(0), \Phi(e)\} < c$$

holds, then for every $k \in [0,1]$, there exists an optimal sequence of paths $\{h_n^k\} \subset \Gamma$ such that $u_n^k \in h_n^k([0,1])$ for all n.

Remark 1.1 If any of these sequences has a convergent subsequence we obtain a critical point u of Φ with $\Phi(u) = c$ and $u \in W \cap \overline{S}$ where S is any subset of X containing an optimal sequence of paths.

Note that if there exists $f \in \Gamma$ such that $\max_{t\in[0,1]} \Phi(f(t)) = c$, then $f_n = f$ for all $n \in \mathbb{N}$ is an optimal sequence of paths and by (1.4), $d(u_n^0, f([0,1])) \to 0$. Since $f([0,1])$ is compact, it follows easily that $\{u_n^0\}$ has a subsequence converging to an element $u \in f([0,1])$. Hence, in this case, Φ has a critical point on the optimal path f.

Remark 1.2 The conditions (I) and (II) are much weaker than (SMPG). Indeed, taking $W = X$, the definition of c implies that (II) holds and so (I) and (II) are satisfied if and only if (MPG) holds.

Clearly (SMPG) implies that $c \geq \inf_{\|u\|=r} \Phi(u)$ and hence that (MPG) holds. However when dim $X < \infty$, (SMPG) implies that the ball $\{u \in X : \|u\| < r\}$ contains a local minimum of Φ whereas (MPG) imposes no such restriction as is shown by the example $\Phi(x,y) = x^2 - (y-1)^2$ with $e = (0,2)$ and $c = 0$.

By the definition of c we always have $\max\{\Phi(0), \Phi(e)\} \leq c$. It is easy to find examples where (MPG) fails but (I) and (II) are still satisfied for an appropriate subset W if X, see [18]. A function which is constant on X provides the most trivial example.

Remark 1.3 For any set S containing $\cup f_n([0,1])$ for an optimal sequence $\{f_n\}$, (1.4) implies that (1.2) holds. Now suppose that at least one of the following conditions is satisfied.

(i) There is bounded set S containing $\cup f_n([0,1])$ for an optimal sequence of paths.

(ii) The set $W \cap \Phi_c$ is bounded.

Then the Palais-Smale sequence $\{u_n^0\}$ given by (1.4) is bounded and so it has "a fortiori" all the requirements for $k \in (0,1]$ as well. Conversely, if $\|u_n\| \to \infty$, then $d(u_n, S)/(1+\|u_n\|) \to 1$ when (i) holds and $d(u_n, W)/(1+\|u_n\|) \to 1$ when (ii) holds. Hence the Cerami sequence $\{u_n^1\}$ given by (1.4) must be bounded and so it has also all the requirements for $k \in [0, 1)$. Thus we see that, with respect to the earlier work of Kuzin-Pohozaev and Ghoussoub-Preiss, the novelty of Theorem 1.1 lies in situations where W is unbounded and no “a priori ”bound for a sequence of optimal paths is available.

Remark 1.4 In the case where $f_n([0, 1]) \subset S$ for all $n \in \mathbb{N}$ and $S = tS$ for all $t > 0$, the information that $\frac{d(u_n,S)}{1+\|u_n\|} \to 0$ can be exploited in the following way. Let $T > 0$ be fixed and set $z_n = t_n u_n$ where $t_n = T/\|u_n\|$. Then, for $u_n \neq 0$,

$$d(u_n, S) = \inf_{v\in S} \|u_n - v\| = \frac{1}{t_n} \inf_{v\in S} \|z_n - t_n v\| = \frac{1}{t_n} \inf_{z\in t_n S=S} \|z_n - z\| = \frac{\|u_n\|}{T} d(z_n, S).$$

Suppose that, for a subsequence, $\|u_n\| \to \infty$. Then

$$d(z_n, S) = \frac{T(1 + \|u_n\|)}{\|u_n\|} \frac{d(u_n, S)}{1 + \|u_n\|} \to 0.$$

If in addition, X is reflexive and S is a cone (i.e. a closed convex subset of X such that $tS = S$ for all $t > 0$), we can go further by passing to a subsequence such that $z_n \rightharpoonup z$ weakly in X. Since $d(z_n, S) \to 0$, there exist $s_n \in S$ and $r_n \in X$ such that $z_n = s_n + r_n$ and $\|r_n\| \to 0$. This implies that $s_n \rightharpoonup z$ and, since S is closed and convex, we conclude that $z \in S$. This information, together with the property that $(1 + \|u_n\|)\|\Phi'(u_n)\|_* \to 0$, may lead to a contradiction and hence establish the boundedness of $\{u_n\}$. See [24] for an example of this procedure using the cone S of positive elements in the space $H^1(\mathbb{R}^N)$ as discussed in Section 5. Indeed the original motivation for the present work was to prove the following corollary to Theorem 1.1 which was used in [24].

Corollary 1.2. *Let $\Phi \in C^1(X, \mathbb{R})$ satisfy (MPG) and let $\{f_n\}$ be an optimal sequence of paths: $M_n = \max_{t\in[0,1]} \Phi(f_n(t)) \to c$ where c is the mountain pass level.*

Then, for every $k \in [0, 1]$, there exists a sequence $\{u_n^k\}_{n=0}^{\infty} \subset X$ such that

$$\Phi(u_n^k) \to c, (1 + \|u_n^k\|)^k \Phi'(u_n^k) \to 0 \text{ and } \frac{d(u_n^k, f_n([0, 1]))}{(1 + \|u_n^k\|)^k} \to 0$$

as $n \to \infty$.

Hence, if S is a subset of X such that $f_n([0, 1]) \subset S$ for all n, then $\{u_n^k\}$ satisfies (1.2).

Proof In Remark 1.2 we have already observed that when (MPG) holds the hypotheses of Theorem 1.1 are satisfied for the set $W = X$. □

The proof of Theorem 1.1 is given in Section 4. It is based on a deformation lemma which is proved in Section 3, using the approach found in [3], but adapting it so as to incorporate information about the factor $(1 + \|u\|)^k$.

As has been already mentioned, this work was stimulated by the need to overcome difficulties in proving that a Cerami sequence is bounded during the preparation of

[24]. Of course, localization is not the only way of overcoming such obstacles, notably Jeanjean's elaboration [14] of Struwe's monotonicity trick has proved remarkably successful. See [15] for an early example of this and [16] for ramifications.

2. EXAMPLES

Our main aim here is to show that, contrary to what one might expect, the hypotheses of Corollary 1.2 (and hence the more general Theorem 1.1) do not ensure the existence of a Cerami sequence $\{u_n\}$ for the level c with $d(u_n, S) \to 0$ for a set S which contains an optimal sequence of paths. In fact, we construct and example where $X = \mathbb{R}^3$ and $d(u_n, S) \to \infty$ for all Cerami sequences for this level. The starting point is a well-known example due to Brézis and Nirenberg of a function on $\mathbb{R}^2$ which does not satisfy the Palais-Smale condition at level c.

2.1 Cerami sequences for the B-N example. In [3], Brézis and Nirenberg use the following example to illustrate the failure of the Mountain Pass Theorem in the absence of the Palais-Smale condition.

$$\Psi : \mathbb{R}^2 \to \mathbb{R},\ \Psi(x, y) = x^2(1 + y)^3 + y^2 \tag{2.1}$$

It is easy to see that $\Psi(0, 0) = 0$ and that Ψ has a strict local minimum at $(0, 0)$. Also $\Psi(3, -4) < 0$ and so Ψ satisfies (SMPG). Let

$$G = \{f \in C([0, 1], \mathbb{R}^2) : f(0) = (0, 0) \text{ and } f(1) = (3, -4)\} \tag{2.2}$$

and

$$c = \inf_{f\in G} \max_{t\in[0,1]} \Psi(f(t)). \tag{2.3}$$

A study of the level sets of Ψ shows that $c = 1$. From the Mountain Pass Lemma (or Corollary 1.2 with $X = S = \mathbb{R}^2$ and $k = 1$) it follows that there exists a Cerami sequence $\{(x_n, y_n)\}$:

$$\Psi(x_n, y_n) \to 1 \text{ and } (1 + \|(x_n, y_n)\|)\|\nabla\Psi(x_n, y_n)\| \to 0 \text{ as } n \to \infty. \tag{2.4}$$

However, it is easily checked that $(0, 0)$ is the only critical point of Ψ. This implies that $\|(x_n, y_n)\| \to \infty$ for every sequence having the properties (2.4), showing that Ψ does not satisfy the Palais-Smale condition at level $c = 1$. Of course, one can find a sequence satisfying (2.4) by elementary calculations. Here is an example, which we shall use later,

$$(x_n, y_n) = (n\sqrt{\frac{2}{3}(1 - \frac{1}{n})}, -1 + \frac{1}{n}).$$

Clearly $\Psi(x_n, y_n) \to 1$ and

$$\partial_x\Psi(x_n, y_n) = 2x_n(1 + y_n)^3 = \frac{2}{n^2}\sqrt{\frac{2}{3}(1 - \frac{1}{n})} \text{ whereas } \partial_y\Psi(x_n, y_n) = 0.$$

Hence $\|(x_n, y_n)\|/n \to \sqrt{\frac{2}{3}}$ and $n^2\|\nabla\Psi(x_n, y_n)\| \to 2\sqrt{\frac{2}{3}}$ from which it follows that $\|(x_n, y_n)\|\|\nabla\Psi(x_n, y_n)\| \to 0$.

2.2 An example where $d(u_n, S) \to \infty$ for every Cerami sequence.

We begin by defining a function $\Phi : \mathbb{R}^3 \to \mathbb{R}$ which satisfies the hypotheses of Corollary 1.2 with $S = \mathbb{R}^2 \times \{0\}$. This definition requires the introduction of some auxiliary functions:

- $\varphi \in C^\infty([0,\infty))$ with $\varphi(r) = 1$ for $0 \le r \le 1/2$, $\varphi(r) = 0$ for $r \ge 1$ and $\varphi'(r) < 0$ for $r \in (1/2, 1)$
- $\psi \in C^\infty([0,\infty))$ with $\psi(r) = 0$ for $0 \le r \le 8$, $\psi(r) = 1$ for $r \ge 9$ and $\psi'(r) > 0$ for $r \in (8, 9)$
- $g \in C^\infty(\mathbb{R})$ with $g(-z) = -g(z)$ and $g'(z) > 0$ for all z and $\lim_{z\to\infty} g(z) = 1$.

Now define Φ by

$$\Phi(x, y, z) = \Psi(x, y) + \varphi(r)z^2 + \psi(r)g(z)(1 + y)\{x(1 + y) + 2\} \tag{2.5}$$

where Ψ is given by (2.1) and $r = \sqrt{x^2 + y^2}$.

Clearly $\Phi \in C^\infty(\mathbb{R}^3)$ and $\Phi(x, y, 0) = \Psi(x, y)$ since $g(0) = 0$. Hence $\Phi(0, 0, 0) = 0$ and $\Phi(3, -4, 0) < 0$.

Furthermore, Φ has the following properties.

(A) $(0, 0, 0)$ **is the only critical point of** Φ **and it is a strict local minimum.**

(B) If $(1 + \|(x_n, y_n, z_n)\|)\|\nabla\Phi(x_n, y_n, z_n)\| \to 0$ **and** $\{z_n\}$ **is bounded, then** $(x_n, y_n, z_n) \to (0, 0, 0)$ **and** $\Phi(x_n, y_n, z_n) \to 0$.

(C) The function Φ **satisfies the hypotheses of Corollary 1.2 with** $e = (3, -4, 0)$ **and** $c = 1$. **The set** $S = \mathbb{R}^2 \times \{0\}$ **contains an optimal sequence of paths and there is a (PS) sequence for the level** $c = 1$ **with** $d(u_n, S) \to 0$**, but** $d(u_n, S) \to \infty$ **for any Cerami sequence for the level** $c = 1$.

By (C), there exists a Cerami sequence $\{u_n^1 = (x_n^1, y_n^1, z_n^1)\}$ for the level $c = 1$ such that $d(u_n^1, S)/(1 + \|u_n^1\|) \to 0$. Since $\Phi(u_n^1) \to 1$, it follows from (B) that $|z_n^1| = d(u_n^1, S) \to \infty$ and the property $d(u_n^1, S)/(1 + \|u_n^1\|) \to 0$ is equivalent to $z_n^1/r_n^1 \to 0$ where $r_n^1 = \sqrt{(x_n^1)^2 + (y_n^1)^2}$.

Note that (B) implies that $|z_n| \to \infty$ if (x_n, y_n, z_n) is a Cerami sequence for any level $d \ne 0$.

Checking (A): For $r < 1/2$, $y > -1/2$ and so

$$\Phi(x, y, z) = x^2(1 + y)^3 + y^2 + z^2 > \frac{1}{8}x^2 + y^2 + z^2,$$

showing that Φ has a strict local minimum at $(0, 0, 0)$.

Suppose now that $\nabla\Phi(x,y,z)=(0,0,0)$. If $z=0$, this implies that $\nabla\Psi(x,y)=(0,0)$ and hence $(x,y)=(0,0)$. Suppose henceforth that $z\neq 0$. Concerning r, there are three cases:

(a) $r<1$, (b) $1\leq r\leq 8$, (c) $r>8$.

(a) Since $\psi(r)=0$, $\partial_z\Phi(x,y,z)=2\varphi(r)z$ where $\varphi(r)>0$ and $z\neq 0$. Hence $\partial_z\Phi(x,y,z)\neq 0$, contradicting $\nabla\Phi(x,y,z)=(0,0,0)$.

(b) We have $\varphi(r)=\varphi'(r)=\psi(r)=\psi'(r)=0$ and so $0=\partial_x\Phi(x,y,z)=\partial_x\Psi(x,y)$ and $0=\partial_y\Phi(x,y,z)=\partial_y\Psi(x,y)$. Hence $\nabla\Psi(x,y)=(0,0)$, which implies $r=0$, a contradiction.

(c) Now $\varphi(r)=0$ and $0=\partial_z\Phi(x,y,z)=\psi(r)g'(z)(1+y)\{x(1+y)+2\}$, where $\psi(r)\neq 0$ and $g'(z)\neq 0$. Hence, either $y=-1$ or $x=-2/(1+y)$.

If $y=-1$, we find that $\partial_y\Phi(x,y,z)=\partial_y\Psi(x,-1)+2g(z)\psi(r)=-2+2g(z)\psi(r)<0$ since $0<\psi(r)\leq 1$ and $g(z)<1$, contradicting $\nabla\Phi(x,y,z)=(0,0,0)$. Hence $1+y\neq 0$ and $x=-2/(1+y)$. But, in this case, $\partial_x\Phi(x,y,z)=2x(1+y)^3+\psi(r)g(z)(1+y)^2=(1+y)^2\{-4+\psi(r)g(z)\}<0$ again since $0<\psi(r)\leq 1$ and $g(z)<1$. Thus we have a contradiction and there are no critical points with $z\neq 0$.

Checking (B): Suppose that $(1+\|(x_n,y_n,z_n)\|)\|\nabla\Phi(x_n,y_n,z_n)\|\to 0$ and that $\{z_n\}$ is bounded. Suppose that there exist $\delta>0$ and a subsequence $\{(x_{n_k},y_{n_k},z_{n_k})\}$ such that $\|(x_{n_k},y_{n_k},z_{n_k})\|\geq\delta$ for all n_k. By passing to a further subsequence, we can suppose that $z_{n_k}\to z$ and either (a) $(x_{n_k},y_{n_k})\to(x,y)\in\mathbb{R}^2$ or (b) $r_{n_k}\to\infty$ where $r_{n_k}=\sqrt{x_{n_k}^2+y_{n_k}^2}$.

Case (a) We have that $(1+\|(x,y,z)\|)\|\nabla\Phi(x,y,z)\|=0$ and so $(x,y,z)=(0,0,0)$ by property (A). But we also have $\|(x,y,z)\|\geq\delta$, a contradiction.

Case (b) We may suppose that $r_{n_k}>9$ for all n_k. Then

$$\partial_y\Phi(u_k)=3w_k^2+2y_{n_k}+g(z_{n_k})2[w_k+1] \text{ and } \partial_z\Phi(u_k)=g'(z_{n_k})(1+y_{n_k})[w_k+2]$$

where we have set $u_k=(x_{n_k},y_{n_k},z_{n_k})$ and $x_{n_k}(1+y_{n_k})=w_k$ for convenience. Since $x_{n_k}\partial_z\Phi(u_k)\to 0$ and $g'(z_{n_k})\to g'(z)\neq 0$, we must have $w_k[w_k+2]\to 0$ and hence $w_k\to w_\infty$, where $w_\infty=0$ or -2. But we also have that $\partial_y\Phi(u_k)\to 0$ from which it follows that

$$y_{n_k}\to y_\infty=-\frac{1}{2}\{3w_\infty^2+g(z)2[w_\infty+1]\}.$$

For $w_\infty=0$, this yields $y_\infty=-g(z)>-1$ and so $x_{n_k}=w_k/(1+y_{n_k})\to 0$. Hence $r_{n_k}\to|g(z)|<1$, a contradiction.

For $w_\infty=-2$, $y_\infty=g(z)-6\in(-7,-5)$ and so $x_{n_k}\to -2/(1+y_\infty)$, again contradicting the fact that $r_{n_k}\to\infty$. Hence case (b) cannot occur either and we have established (B).

Checking (C): Let

$$\Gamma=\{h\in C([0,1],\mathbb{R}^3):h(0)=(0,0,0) \text{ and } h(1)=(3,-4,0)\}$$

and observe that, for all $f \in G$ as defined by (2.2), $h = (f_1, f_2, 0) \in \Gamma$. It follows that

$$\inf_{h\in\Gamma} \max_{t\in[0,1]} \Phi(h(t)) \leq \inf_{f\in G} \max_{t\in[0,1]} \Phi(f_1(t), f_2(t), 0) = \inf_{f\in G} \max_{t\in[0,1]} \Psi(f(t)) = 1.$$

On the other hand, $\Phi(x, -1, z) = \Psi(x, -1) = 1$ for all $x, z \in \mathbb{R}$ and, for all $h \in \Gamma$, there exists a $t \in [0, 1]$ such that $h_2(t) = -1$ since $h_2 \in C([0, 1])$ with $h_2(0) = 0$ and $h_2(1) = -4$. Hence, for all $h \in \Gamma$, $\max_{t\in[0,1]} \Phi(h(t)) \geq 1$, so in fact,

$$c = \inf_{h\in\Gamma} \max_{t\in[0,1]} \Phi(h(t)) = 1.$$

Clearly $0 = \Phi(0, 0, 0) > \Phi(e) = \Psi(3, -4)$ so by (A) we see that Φ satisfies (SMPG) and hence (MPG). Furthermore, if $\{f_n\}$ is an optimal sequence of paths for Ψ and G, then $\{h_n = (f_n^1, f_n^2, 0)\}$ is an optimal sequence of paths for Φ and Γ, showing that S contains an optimal sequence. By Corollary 1.2, Φ has a (PS) sequence $\{u_n^0\}$ for the level $c = 1$ such that $d(u_n^0, S) \to 0$. In fact, we give an example below where we even have $u_n^0 \in S$ for all n. However, it follows from (B) that $d(u_n, S) \to \infty$ for any Cerami sequence for the level $c = 1$.

Examples of sequences

For the cases $0 \leq k < 1$ in (1.2), we can even obtain $u_n^k \in S$ since we can use

$$u_n^k = (x_n^k, y_n^k, z_n^k) = (n\sqrt{\frac{2}{3}(1 - \frac{1}{n})}, -1 + \frac{1}{n}, 0).$$

Indeed, we have already noted in Section 2.1 that for this sequence,

$$\|u_n^k\|/n = \|(x_n^k, y_n^k)\|/n \to \sqrt{\frac{2}{3}} \text{ and } \Phi(u_n^k) = \Psi(x_n^k, y_n^k) \to 1 \text{ as } n \to \infty.$$

Furthermore, $\partial_x\Phi(u_n^k) = \partial_x\Psi(x_n^k, y_n^k)$ and $\partial_y\Phi(u_n^k) = \partial_y\Psi(x_n^k, y_n^k)$ so

$$(1 + \|u_n^k\|)\|(\partial_x\Phi(u_n^k), \partial_y\Phi(u_n^k))\| = (1 + \|(x_n^k, y_n^k)\|)\|\nabla\Psi(x_n^k, y_n^k)\| \to 0$$

since $\{(x_n^k, y_n^k)\}$ is a Cerami sequence for Ψ. Clearly $d(u_n^k, S) = 0$ so it only remains to study $\partial_z\Phi(u_n^k)$ as $n \to \infty$. Since $\|(x_n^k, y_n^k)\| \to \infty$ we have that, for n large,

$$\partial_z\Phi(u_n^k) = g'(0)(1 + y_n^k)\{x_n^k(1 + y_n^k) + 2\} = \frac{g'(0)}{n}\{\sqrt{\frac{2}{3}(1 - \frac{1}{n})} + 2\}$$

and

$$(1 + \|u_n^k\|)^k\partial_z\Phi(u_n^k) \to 0 \text{ since } \|u_n^k\|/n \to \sqrt{2/3} \text{ and } 0 \leq k < 1.$$

Hence we see that, for $0 \leq k < 1$, the sequence $\{(n\sqrt{\frac{2}{3}(1 - \frac{1}{n})}, -1 + \frac{1}{n}, 0)\}$ has all the properties required for (1.2). But it is not as Cerami sequence for Φ since $\|u_n^k\|\partial_z\Phi(u_n^k) \to g'(0)\sqrt{\frac{2}{3}}\{\sqrt{\frac{2}{3}} + 2\} \neq 0$ and consequently it does not satisfy (1.2) for $k = 1$.

To write down a suitable Cerami sequence, we need more precise information about the asymptotic behaviour of g, so we make the additional assumption that

$$\text{for some } \alpha > 1 \text{ and } z_0 > 0, g(z) = 1 - z^{-\alpha} \text{ for all } z > z_0.$$

Now setting $u_n = (n, -1 - \frac{2}{3n}, (\frac{n}{2})^{1/\alpha})$, we find that,

$$\Phi(u_n) \to 1, \|u_n\|/n \to 1, n^2\|\nabla\Phi(u_n)\| \to 16/27 \text{ and } d(u_n, S) = (\frac{n}{2})^{1/\alpha}.$$

Hence this sequence has all the properties required by (1.2) for any $k \in (\frac{1}{\alpha}, 1]$.

3. A DEFORMATION LEMMA

We shall use the following well-known and elementary results from the theory of differential equations.

Lemma 3.1 (Gronwall). *Suppose that $a \in \mathbb{R}$ and $b, h \in C([0,\infty))$ satisfy the inequalities*

$$b(t) \geq 0 \text{ and } h(t) \leq a + \int_0^t b(s)h(s)ds \text{ for all } t \geq 0.$$

Then $h(t) \leq ae^{\int_0^t b(s)ds}$ for all $t \geq 0$.

Proposition 3.2. *Let $f : X \to X$ be locally Lipschitz continuous and suppose that there exists a constant A such that*

$$\|f(u)\| \leq A(1 + \|u\|) \text{ for all } u \in X. \tag{3.1}$$

Then, for every $u_0 \in X$, the initial value problem

$$\begin{cases} u'(t) = f(u(t)) & \text{for } t > 0 \\ u(0) = u_0 \end{cases}$$

has a unique solution $\eta(\cdot, u_0) \in C^1([0,\infty), X)$. Furthermore,

1. *$\eta \in C([0,\infty) \times X, X)$,*
2. *$\eta(t, \cdot) : X \to X$ is a homeomorphism for all $t \geq 0$,*
3. *$\eta(t, \eta(s, u)) = \eta(t + s, u)$ for all $t, s \geq 0$ and $u \in X$.*

The main result of this section is an appropriate version of what is usually referred to as a deformation lemma. We begin by recalling the definition and existence of a pseudo-gradient, [26].

Let $F \in C^1(X, \mathbb{R})$ and let $\Omega = \{u \in X : F'(u) \neq 0\}$. There is a locally Lipschitz continuous function $p : \Omega \to X$ such that, for every $u \in \Omega$,

$$\|p(u)\| \leq 2\|F'(u)\|_* \text{ and } \langle F'(u), p(u)\rangle \geq \|F'(u)\|_*^2.$$

Such a mapping p is called a pseudo-gradient for F. Note that $\|p(u)\| \geq \|F'(u)\|_*$ for all $u \in \Omega$.

We now sharpen the deformation lemma proved in [3] in various ways required for the proof of Theorem 1.1.

Theorem 3.3. *Let $F \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$, $k \in [0, 1]$ and $\delta > 0$.*

Set $N = \{u \in X : |F(u) - c| < 2\delta$ and $(1 + \|u\|)^k \|F'(u)\|_* > \sqrt{\delta}\}$.

There exists $\eta : [0, \infty) \times X \to X$ such that

(d1): *$\eta \in C([0, \infty) \times X, X)$ and $\eta(t, \cdot) : X \to X$ is a homeomorphism for all $t \geq 0$.*
(d2): *$\eta(0, u) = u$ and $\eta(t - s, \eta(s, u)) = \eta(t, u)$ for all $u \in X$ and $t \geq s \geq 0$.*
(d3): *$\eta(t, u) = u$ for all $t \geq 0$ if $u \notin N$.*
(d4): *$F(\eta(t, u)) \leq F(\eta(s, u)) \leq F(u)$ for all $t \geq s \geq 0$ and $u \in X$.*
(d5): *$F(u) - F(\eta(t, u)) \leq 4\delta$ for all $t \geq 0$ and $u \in X$.*
(d6): *$\|\eta(t, u) - \eta(s, u)\| \leq 16\sqrt{\delta} e^{16k\sqrt{\delta}} (1 + \|\eta(s, u)\|)^k$ for all $t \geq s \geq 0$ and $u \in X$*
(d7): *$\|\eta(t, u) - \eta(s, u)\| \leq 32\sqrt{\delta} e^{16k\sqrt{\delta}} (1 + \|\eta(t, u)\|)^k$ for all $t \geq s \geq 0$ and $u \in X$ provided that $16\sqrt{\delta} e^{16k\sqrt{\delta}} \leq \frac{1}{2}$.*
(d8): *Suppose that $F(u) < c + \delta$. Then, for all $t \geq 8\delta$, either $F(\eta(t, u)) \leq c - \delta$ or there exists $\tau \in [0, t]$ such that $(1 + \|\eta(\tau, u)\|)^k \|F'(\eta(\tau, u))\|_* < 2\sqrt{\delta}$.*

Proof Let $M = \{u \in X : |F(u) - c| \leq \delta$ and $(1 + \|u\|)^k \|F'(u)\|_* \geq 2\sqrt{\delta}\}$. We have $M \subset N$, M and $N^c = X \backslash N$ are closed, $M \cap N^c = \emptyset$. Set

$$\psi(u) = \frac{d(u, N^c)}{d(u, M) + d(u, N^c)}$$

Then $\psi : X \to [0, 1]$ is locally Lipschitz continuous with $\psi(u) = \begin{cases} 1 & \text{for } u \in M \\ 0 & \text{for } u \in N^c. \end{cases}$

Let $p : \Omega = \{u \in X : F'(u) \neq 0\} \to X$ by a pseudo-gradient field for F.

Noting that $N \subset \Omega$, we define $f : X \to X$ by $f(u) = \begin{cases} -\frac{\psi(u) p(u)}{\|p(u)\|^2} & \text{for } u \in N \\ 0 & \text{for } u \notin N. \end{cases}$
Then $f : X \to X$ is locally Lipschitz continuous and, for $u \in N$,

$\|f(u)\| \leq \frac{1}{\|p(u)\|} \leq \frac{1}{\|F'(u)\|_*} \leq \frac{(1+\|u\|)^k}{\sqrt{\delta}}$.

Since $f(u) = 0$ on N^c, we have that $\|f(u)\| \leq \frac{(1+\|u\|)^k}{\sqrt{\delta}} \leq \frac{1}{\sqrt{\delta}}(1 + \|u\|)$ for all $u \in X$.

Let $\eta(t, u)$ be the flow defined by the unique solution of initial value problem

$$\eta'(t) = f(\eta(t)) \text{ for } t > 0, \eta(0) = u.$$

By Theorem 3.2, $\eta \in C([0,\infty)\times X, X)$ and (d1),(d2) and (d3) are satisfied. Also, for $t>0$ and $\eta=\eta(t,u)$,

$$\frac{d}{dt}F(\eta) = \langle F'(\eta), f(\eta)\rangle = \begin{cases} -\frac{\langle F'(\eta),\psi(\eta)p(\eta)\rangle}{\|p(\eta)\|^2} & \text{for } \eta\in N \\ 0 & \text{for } \eta\notin N \end{cases}$$

$$\le \begin{cases} -\psi(\eta)\frac{\|F'(\eta)\|_*^2}{\|p(\eta)\|^2} & \text{for } \eta\in N \\ 0 & \text{for } \eta\notin N \end{cases} \le \begin{cases} -\frac{1}{4}\psi(\eta) & \text{for } \eta\in N \\ 0 & \text{for } \eta\notin N \end{cases} = -\frac{1}{4}\psi(\eta)\le 0,$$

proving (d4) and showing that

$$F(u)-F(\eta(t,u)) \ge \frac{1}{4}\int_0^t \psi(\eta(s,u))ds \text{ for all } t\ge 0 \text{ and } u\in X. \tag{3.2}$$

By (d3), (d5) is trivial for $u\notin N$. Considering $u\in N$, we have $F(u)-c<2\delta$ and hence $F(u)-F(\eta(t,u)) < c+2\delta-F(\eta(t,u)) \le 4\delta$ if $F(\eta(t,u))\ge c-2\delta$. But $F(\eta(0,u))=F(u)>c-2\delta$ since $u\in N$ and so, if $F(\eta(t,u))<c-2\delta$, there exists $s\in(0,t)$ such that $F(\eta(s,u))=c-2\delta$ and we have $F(u)-F(\eta(s,u))\le 4\delta$. Since $\eta(s,u)\notin N$, $\eta(t,u)=\eta(t-s,\eta(s,u))=\eta(s,u)$ for all $t\ge s\ge 0$ by (d2) and (d3). Thus $F(u)-F(\eta(t,u))=F(u)-F(\eta(s,u))\le 4\delta$ in this case too. This proves (d5).

Combining (3.2) and (d5) we get

$$\int_0^t \psi(\eta(s,u))ds \le 16\delta \text{ for all } t\ge 0 \text{ and } u\in X. \tag{3.3}$$

To prove (d6) we consider $t>0$ and $u\in X$. Let $A(t,u)=(0,t)\cap\{s:\eta(s,u)\in N\}$. Then

$$\begin{aligned}
&\|\eta(t,u)-u\| \le \int_0^t \|\frac{d}{ds}\eta(s,u)\|ds = \int_0^t \|f(\eta(s,u))\|ds \\
&\le \int_{A(t,u)} \|f(\eta(s,u))\|ds = \int_{A(t,u)} \frac{\psi(\eta(s,u))}{\|p(\eta(s,u))\|}ds \le \int_{A(t,u)} \frac{\psi(\eta(s,u))}{\|F'(\eta(s,u))\|_*}ds \\
&\le \int_{A(t,u)} \psi(\eta(s,u))\frac{(1+\|\eta(s,u)\|)^k}{\sqrt{\delta}}ds \le \int_0^t \psi(\eta(s,u))\frac{(1+\|\eta(s,u)\|)^k}{\sqrt{\delta}}ds
\end{aligned} \tag{3.4}$$

Hence

$$\begin{aligned}
1+\|\eta(t,u)\| &\le 1+\|u\|+\|\eta(t,u)-u\| \\
&\le 1+\|u\|+\frac{1}{\sqrt{\delta}}\int_0^t \psi(\eta(s,u))(1+\|\eta(s,u)\|)ds
\end{aligned}$$

since $k\ge 0$ and this can be written as $h(t)\le a(t)+\int_0^t b(s)h(s)ds$ where

$$h(t)=\|1+\eta(t,u)\|, a(t)=1+\|u\|, b(t)=\frac{\psi(\eta(t,u))}{\sqrt{\delta}}.$$

The Gronwall inequality yields

$$1+\|\eta(t,u)\| \le (1+\|u\|)e^{\int_0^t b(s)ds}$$

where $\int_0^t b(s)ds \leq 16\sqrt{\delta}$ by (3.3).

Hence $\|1+\eta(t,u)\| \leq (1+\|u\|)e^{16\sqrt{\delta}}$ for all $t \geq 0$ and $u \in X$ and consequently (3.4) yields

$$\|\eta(t,u)-u\| \leq \int_0^t \psi(\eta(s,u))\frac{(1+\|u\|)^k e^{16k\sqrt{\delta}}}{\sqrt{\delta}}ds.$$

Using (3.3) we now have that

$$\|\eta(t,u)-u\| \leq \frac{(1+\|u\|)^k e^{16k\sqrt{\delta}}}{\sqrt{\delta}}\int_0^t \psi(\eta(s,u))ds \leq 16\sqrt{\delta}e^{16k\sqrt{\delta}}(1+\|u\|)^k$$

for all $t \geq 0$ and $u \in X$, proving (d6) for $s=0$. We get the complete conclusion with $s \neq 0$ by using (2).

To deduce (d7) from (d6), we observe that, for all $v, w \in X$ and for $k \in [0,1]$,

$$\left(\frac{1+\|v\|}{1+\|w\|}\right)^k \leq \left(1+\frac{\|w-v\|}{1+\|w\|}\right)^k \leq 1+\frac{\|w-v\|}{1+\|w\|} \leq 1+\frac{\|w-v\|}{(1+\|w\|)^k}.$$

Returning to (d6) and putting $v=\eta(s,u)$ and $w=\eta(t,u)$, we obtain

$$\begin{aligned}\frac{\|\eta(t,u)-\eta(s,u)\|}{(1+\|\eta(t,u)\|)^k} &\leq 16\sqrt{\delta}e^{16k\sqrt{\delta}}\left(\frac{1+\|\eta(s,u)\|}{1+\|\eta(t,u)\|}\right)^k \\ &\leq 16\sqrt{\delta}e^{16k\sqrt{\delta}}\{1+\frac{\|\eta(t,u)-\eta(s,u)\|}{(1+\|\eta(t,u)\|)^k}\},\end{aligned}$$

from which (d7) follows easily.

For (d8), we consider $t \geq 8\delta$ and u such that $F(u) < c+\delta$. Suppose that $F(\eta(t,u)) > c-\delta$. By (d4), this implies that $c-\delta < F(\eta(s,u)) \leq F(u) < c+\delta$ for all $s \in [0,t]$. If $(1+\|\eta(s,u)\|)^k\|F'(\eta(s,u))\|_* \geq 2\sqrt{\delta}$ for all $s \in [0,t]$, we have that $\eta(s,u) \in M$ for all $s \in [0,t]$ and so, by (3.2) and the definition of ψ,

$$F(u)-F(\eta(t,u)) \geq \frac{1}{4}\int_0^t ds = \frac{t}{4}.$$

Hence $c-\delta < F(\eta(t,u)) \leq F(u)-\frac{t}{4} < c+\delta-\frac{t}{4}$ and so $8\delta > t$. □

4. THE PROOF OF THEOREM 1.1

In addition to Theorem 3.3 the following simple lemma will be used.

Lemma 4.1. *Let S be a closed subset of X. For some $k \in [0,1]$ and $\varepsilon > 0$, let*

$$S_\varepsilon = \{v \in X : \frac{\|v-u\|}{(1+\|u\|)^k} \leq \varepsilon \text{ for some } u \in S\}.$$

(i) If $w \notin S$, then $w \notin S_\varepsilon$ for $\varepsilon < \min\{\frac{1}{2}, \frac{d(w,S)}{[2(1+\|w\|)]^k}\}$.

(ii) For all $w \in X$, $d(w,S_\varepsilon)+d(w,X\backslash S_{2\varepsilon}) > 0$.

Proof (i) For $u \in S$ with $\|u\| \le 2\|w\| + 1$, $\frac{\|w-u\|}{(1+\|u\|)^k} \ge \frac{d(w,S)}{(2+2\|w\|)^k}$.
whereas for $u \in S$ with $\|u\| \ge 2\|w\| + 1$,
$\frac{\|w-u\|}{(1+\|u\|)^k} \ge \frac{\|u\|-\|w\|}{(1+\|u\|)^k} \ge \frac{1}{2}(1+\|u\|)^{1-k} \ge \frac{1}{2}$.

(ii) Suppose that $d(w, S_\varepsilon) = 0$. Then there exists $\{v_n\} \subset S_\varepsilon$ such that $\|w - v_n\| \to 0$. But, for each n, there exists $u_n \in S$ such that $\frac{\|v_n-u_n\|}{(1+\|u_n\|)^k} \le \varepsilon$ and so

$$\frac{\|w-u_n\|}{(1+\|u_n\|)^k} \le \frac{\|v_n-u_n\| + \|w-v_n\|}{(1+\|u_n\|)^k} \le \varepsilon + \|w - v_n\|.$$

For any $\delta > \varepsilon$, we can choose n such that $\varepsilon + \|w - v_n\| < \delta$, showing that $w \in S_\delta$ for all $\delta > \varepsilon$.

On the other hand, if $d(w, X \backslash S_{2\varepsilon}) = 0$, there exists a sequence $\{z_n\}$ such that $\|w - z_n\| \to 0$ and $\frac{\|z_n-u\|}{(1+\|u\|)^k} > 2\varepsilon$ for all $u \in S$. Hence, for all $u \in S$,

$$\frac{\|w-u\|}{(1+\|u\|)^k} \ge \frac{\|u-z_n\| - \|w-z_n\|}{(1+\|u\|)^k} > 2\varepsilon - \|w - z_n\|.$$

For any $\delta < 2\varepsilon$, we can choose n such that $2\varepsilon - \|w - z_n\| > \delta$, showing that $w \notin S_\delta$.

If $d(w, S_\varepsilon) = 0$ and $d(w, S_{2\varepsilon}) = 0$, we would have $w \in S_{\frac{3}{2}\varepsilon} \cap (X \backslash S_{\frac{3}{2}\varepsilon})$. □

Remark The proof of (ii) shows that $\overline{S_\varepsilon} \subset S_\delta$ for all $\delta > \varepsilon$ and that $\overline{X \backslash S_\varepsilon} \subset X \backslash S_\delta$ for all $\delta < \varepsilon$.

Proof of Theorem 1.1 Set $S = W \cap \Phi_c$.

Choose and fix $k \in [0, 1]$. For $\delta > 0$, let $\varepsilon(\delta) = 32\sqrt{\delta}e^{16k\sqrt{\delta}}$ and observe that $\varepsilon(\delta)$ is a strictly increasing function of δ with $\lim_{\delta\to 0} \varepsilon(\delta) = 0$. Since S is closed and $0, e \notin S$ by the hypothesis (I), $d(0, S) > 0$ and $d(e, S) > 0$. By Lemma 4.1(i), there exists $\delta_0 > 0$ such that $\{0, e\} \cap S_{2\varepsilon(\delta)} = \emptyset$ for all $\delta \in (0, \delta_0)$. By reducing δ_0 if necessary, we can also obtain $\delta_0 < 1/8$ and $16\sqrt{\delta_0}e^{16k\sqrt{\delta_0}} < 1/2$.

Consider $\delta \in (0, \delta_0)$. By Lemma 4.1(ii) we can define a function $T_\delta : X \to [0, 1]$ by

$$T_\delta(u) = \frac{d(u, X \backslash S_{2\varepsilon(\delta)})}{d(u, S_{\varepsilon(\delta)}) + d(u, X \backslash S_{2\varepsilon(\delta)})}$$

and it follows that T_δ is locally Lipschitz continuous with $T_\delta(u) = 0$ for all $u \notin S_{2\varepsilon(\delta)}$ and $T_\delta(u) = 1$ for all $u \in S_{\varepsilon(\delta)}$.

By (II), $M_n \ge c$ for all n and, since $\{f_n\}$ is an optimal sequence, there exists $n_0 \in \mathbb{N}$ such that $M_n < c + \delta_0$ for all $n \ge n_0$. For each $n \ge n_0$, choose some $\delta_n \in (M_n - c, \delta_0)$ in such a way that $\delta_n \to 0$ as $n \to 0$. For example, we could use $\delta_n = M_n - c + \frac{1}{n}$ for all large enough n. Then, to simplify the notation, set $T^n = T_{\delta_n}$.

Let N_n and η_n be the set and the deformation given by Theorem 3.3 for $F = \Phi$ and these values of c, k and $\delta = \delta_n$ where $n \ge n_0$. Define $g_n : [0, 1] \to X$ by

$$g_n(t) = \eta_n(T^n(f_n(t)), f_n(t)) \text{ for } t \in [0, 1].$$

Clearly $g_n \in C([0,1],X)$ and, since $T^n(0) = T^n(e) = 0$, we have that $g_n \in \Gamma$. Hence by hypothesis (II), there exists $t_n \in (0,1)$ such that $g_n(t_n) \in S$. Setting $w_n = f_n(t_n)$, we have that $g_n(t_n) = \eta_n(T^n(w_n), w_n)$. By property (d7) of η_n, it follows that

$$\frac{\|g_n(t_n) - f_n(t_n)\|}{(1+\|g(t_n)\|)^k} = \frac{\|\eta_n(T^n(w_n), w_n) - w_n\|}{(1+\|\eta_n(T^n(w_n), w_n)\|)^k} \leq 32\sqrt{\delta_n}e^{16k\sqrt{\delta_n}}$$

and so $w_n = f_n(t_n) \in S_{\varepsilon(\delta_n)}$. Thus $T^n(w_n) = 1$ and $g_n(t_n) = \eta(1, w_n)$. But $g_n(t_n) \in S \subset \Phi_c$ which implies that

$$c \leq \Phi(\eta_n(1, w_n)) \leq \Phi(w_n) = \Phi(f_n(t_n)) \leq M_n < c + \delta_n.$$

Using property (d8) of η_n (with $t = 1$ and recalling that $0 < \delta_n < \delta_0 < 1/8$), there exists $\tau_n \in [0,1]$ such that

$$(1 + \|\eta_n(\tau_n, w_n)\|)^k \|\Phi'(\eta_n(\tau_n, w_n))\|_* < 2\sqrt{\delta_n}.$$

By property (d6) of η_n we also have that

$$\frac{\|\eta_n(1, w_n) - \eta_n(\tau_n, w_n)\|}{(1+\|\eta_n(\tau_n, w_n)\|)^k} \leq 16\sqrt{\delta_n}e^{16k\sqrt{\delta_n}}$$

and so

$$\frac{d(\eta_n(\tau_n, w_n), S)}{(1+\|\eta_n(\tau_n, w_n)\|)^k} \leq 16\sqrt{\delta_n}e^{16k\sqrt{\delta_n}} \text{ since } \eta_n(1, w_n) = g_n(t_n) \in S.$$

On the other hand, by property (d7) of η_n,

$$\frac{d(\eta_n(\tau_n, w_n), f_n([0,1]))}{(1+\|\eta_n(\tau_n, w_n)\|)^k} \leq \frac{\|\eta_n(\tau_n, w_n) - f_n(t_n)\|}{(1+\|\eta_n(\tau_n, w_n)\|)^k} = \frac{\|\eta_n(\tau_n, w_n) - w_n\|}{(1+\|\eta_n(\tau_n, w_n)\|)^k}$$
$$\leq 32\sqrt{\delta_n}e^{16k\sqrt{\delta_n}}.$$

Finally we observe that

$$c \leq \Phi(\eta_n(1, w_n)) \leq \Phi(\eta_n(\tau_n, w_n)) \leq \Phi(w_n) < c + \delta_n$$

and that $\delta_n \to 0$ as $n \to \infty$.

Setting $u_n^k = \eta_n(\tau_n, w_n)$, we see that the sequence $\{u_n^k\}$ has all the required properties for the value of k chosen at the beginning of the proof.

Suppose now that (MPG) holds. In this case, we can add to the conditions on δ_0 imposed at the beginning of the proof the requirement that $c - \max\{\Phi(0), \Phi(e)\} > 2\delta_0$. This ensures that $\{0, e\} \cap N_n = \emptyset$ for all n.

Recalling that $u_n^k = \eta_n(\tau_n, w_n)$ where η_n, τ_n and $w_n = f_n(t_n)$ also depend on k although this was not indicated explicitly, consider the path h_n^k defined by $h_n^k(t) = \eta_n(\tau_n, f_n(t))$ for $t \in [0,1]$. Clearly $h_n^k \in C([0,1],X)$ and, since $\{0,e\} \cap N_n = \emptyset$, it follows from (d3) that $h_n^k \in \Gamma$. Furthermore, for all $t \in [0,1]$, by (d4),

$$\Phi(h_n^k(t)) = \Phi(\eta_n(\tau_n, f_n(t)) \leq \Phi(f_n(t_n)) \leq M_n$$

and so $\{h_n^k\}$ is an optimal sequence of paths with $u_n^k = \eta_n(\tau_n, f_n(t_n)) \in h_n^k([0,1])$. □

The final property that the sequence $\{u_n^k\}$ lies on an optimal sequence of paths is probably true under the general hypotheses (I) and (II), but proving this seems to require a different form of deformation lemma.

5. THE POSITIVE CONE IN $H_0^1(\Omega)$

We outline a typical situation where Corollary 1.2 has proved useful when dealing with certain types of second order elliptic partial differential equations. It deals with a case where S is a cone, as discussed in Remark 1.4.

Let Ω be an open subset of $\mathbb{R}^N$ and $H = H_0^1(\Omega)$ with the usual norm

$$\|u\|_1 = \{\int_\Omega u^2 + |\nabla u|^2 dx\}^{1/2}.$$

Proposition 5.1. *For all $u \in H$, $|u| \in H$ and $u \mapsto |u|$ is continuous from H into H.*

Proof It is well known that $|u| \in H$ for all $u \in H$ with

$$\nabla |u| = \begin{cases} \nabla u & \text{on } \{x \in \Omega : u(x) > 0\} \\ 0 & \text{on } \{x \in \Omega : u(x) = 0\} \\ -\nabla u & \text{on } \{x \in \Omega : u(x) < 0\} \end{cases}$$

and it follows that $|\nabla |u|| = |\nabla u|$ a.e. on Ω. Suppose that there exists an element $w \in H$ such that $u \mapsto |u|$ is not continuous at w. Then there exist $\delta > 0$ and a sequence $\{u_n\} \subset H$ such that

$$\|u_n - w\|_1 \to 0 \text{ and } \||u_n| - |w|\|_1 \geq \delta \text{ for all } n.$$

Since

$$\int_\Omega (u_n - w)^2 + |\nabla(u_n - w)|^2 dx \to 0$$

we have that

$$\int_\Omega (|u_n| - |w|)^2 dx \leq \int_\Omega |u_n - w|^2 dx \to 0.$$

Furthermore, since $\{u_n\}$ is a bounded sequence in H and we have that

$$\int_\Omega |\nabla |u_n||^2 dx = \int_\Omega |\nabla u_n|^2 dx \leq \|u_n\|_1^2,$$

it follows that, for $i = 1, \ldots, N$, $\{\partial_i |u_n|\}$ is a bounded sequence in $L^2(\Omega)$ and so there exist $w_i \in L^2(\Omega)$ and a subsequence $\{u_{n_k}\}$ such that $\partial_i |u_{n_k}| \rightharpoonup w_i$ weakly in $L^2(\Omega)$. Hence, for any $z \in L^2(\Omega)$,

$$\int_\Omega \partial_i |u_{n_k}| \, z \, dx \to \int_\Omega w_i \, z \, dx.$$

On the other hand, for $z \in C_0^\infty(\Omega)$,

$$\int_\Omega \partial_i |u_{n_k}| \, z \, dx = -\int_\Omega |u_{n_k}| \partial_i z \, dx \to -\int_\Omega |w| \partial_i z \, dx$$

since we have already noted that $|u_n| \to |w|$ strongly in $L^2(\Omega)$. This implies that $\partial_i|w| = w_i$ and hence $\partial_i|u_{n_k}| \rightharpoonup \partial_i|w|$ weakly in $L^2(\Omega)$. But then, for $i = 1, \ldots, N$,

$$\begin{aligned}\int_\Omega \{\partial_i[|u_{n_k}| - |w|]\}^2 dx &= \int_\Omega \{\partial_i|u_{n_k}|\}^2 + \{\partial_i|w|\}^2 - 2\partial_i|u_{n_k}|\partial_i|w|\, dx \\ &= \int_\Omega \{\partial_i u_{n_k}\}^2 + \{\partial_i w\}^2 - 2\partial_i|u_{n_k}|\partial_i|w|\, dx \\ &\to \int_\Omega \{\partial_i w\}^2 + \{\partial_i w\}^2 - 2\partial_i|w|\partial_i|w|\, dx = 0\end{aligned}$$

since $\partial_i u_{n_k} \to \partial_i w$ strongly in $L^2(\Omega)$ and $\partial_i|u_{n_k}| \rightharpoonup \partial_i|w|$ weakly in $L^2(\Omega)$. This shows that

$$\int_\Omega |\nabla\{|u_{n_k}| - |w|\}|^2 dx \to 0$$

and consequently

$$\||u_{n_k}| - |w|\|_1^2 = \int_\Omega (|u_{n_k}| - |w|)^2 + |\nabla\{|u_{n_k}| - |w|\}|^2 dx \to 0,$$

contradicting the fact that $\||u_n| - |w|\|_1 \geq \delta$ for all n. This proves that $u \mapsto |u|$ is continuous from H into H at every $u \in H$. □

Let $S = \{u \in H_0^1(\Omega) : u \geq 0 \text{ a.e. on } \Omega\}$.

Suppose that $\Phi \in C^1(H, \mathbb{R})$, that $\Phi(u) = \Phi(|u|)$ for all $u \in H$ and that Φ satisfies (MPG). Replacing e by $|e|$ we can suppose that $e \in S$. Let

$\Gamma = \{f \in C([0,1], H) : f(0) = 0 \text{ and } f(1) = e\}$ and $c = \inf_{f\in\Gamma} \max_{t\in[0,1]} \Phi(f(t))$.

By Proposition 5.1, $|f| \in \Gamma$ whenever $f \in \Gamma$ and so $\Phi(f(t)) = \Phi(|f|(t))$ for all $t \in [0,1]$. Thus S contains an optimal sequence of paths and Corollary 1.2 ensures that, for every $k \in [0,1]$, there exists a sequence $\{u_n^k\} \subset H = H_0^1(\Omega)$ such that

$$\Phi(u_n^k) \to c, (1 + \|u_n^k\|)^k \Phi'(u_n^k) \to 0 \text{ and } \frac{d(u_n^k, S)}{(1 + \|u_n^k\|)^k} \to 0.$$

6. **Note added in proof**

While this paper was being refereed, the issue raised in Section 1.2 concerning the the metrics d and δ has been resolved by P.J. Rabier,

Rabier, P.J.: On the Ekeland-Ghoussoub-Preiss and Stuart criteria for locating Cerami sequences, preprint.

In the notation of our Section 1.2, Rabier's Theorem 1.1 establishes that, for any non-empty subset S of X and for any sequence $\{u_n\} \subset X$,

$$\lim_{n\to\infty} \delta(u_n, S) = 0 \Leftrightarrow \lim_{n\to\infty} \frac{d(u_n, S)}{1 + \|u_n\|} = 0.$$

A similar conclusion holds when S is replaced by an arbitrary sequence $\{S_n\}$ of non-empty subsets of X. Consequently, our conclusion (1.4) with $k = 1$ can be obtained by combining Rabier's result with Theorem 1 from

Ghoussoub, N.: A min-max principle with a relaxed boundary condition, *Proc. AMS*, 117 (1993), 439-447,

which is a more complete version of the result by Ekeland mentioned in Section 1.2 above. It includes localization near an optimal sequence of paths.

In fact, Rabier also deals with a broad class of weights, not just $1 + \|\cdot\|$. His Corollary 3.2 shows that, for any sequence $\{S_n\}$ of non-empty subsets of X,

$$\lim_{n\to\infty} \delta(u_n, S_n) = 0 \Leftrightarrow \lim_{n\to\infty} \frac{d(u_n, S_n)}{\omega(\|u_n\|)} = 0$$

where $\omega \geq 1$ is concave and non-decreasing and δ is the corresponding metric defined by

$$\delta(u, v) = \inf\{\int_0^1 \frac{\|h'(t)\|}{\omega(\|h(t)\|)} dt : h \in C^1([0,1]), X) \text{ with } h(0) = u \text{ and } h(1) = v\}.$$

Together with Ghoussoub's result, this leads to a proof of (1.4) for all $k \in [0, 1]$.

I am grateful to Patrick Rabier for his illuminating correspondance with me on these matters.

ACKNOWLEDGMENTS

It is with great pleasure that I thank the Wuhan Institute for Physics and Mathematics, Chinese Academy of Sciences, Wuhan, China and OxPDE, Oxford University, England for their hospitality and support. Much of this work was carried out during these visits.

REFERENCES

[1] Ambrosetti, A. and Malchiodi, A.: *Nonlinear Analysis and Semilinear Elliptic Problems*, Cambridge University Press, Cambridge 2007

[2] Ambrosetti, A. and Rabinowitz, P. H.: Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349-381

[3] Brézis, H. and Nirenberg, L.: Remark on finding critical points, *Comm. Pure Appl. Math.*, **44** (1991), 939-963

[4] Cerami, G.: Un critero di esistenza per i punti critici su varietà illimitate, *Ren. Acad. Sci. Let. Ist. Lombardo*, **112** (1978), 332-336

[5] Cerami, G.: Sull' esistenza di autovalori per un problema al contorno non lineare, *Ann. di Mat.*, **24** (1980), 161-179

[6] Bartolo, P., Benci, V. and Fortunato, D.: Abstract critical point theorems and application to some nonlinear problems with strong resonance at infinity, *Nonlin. Anal. TMA*, **7** (1983), 981-1012

[7] Corvellec, J.-M.: Quantitative deformation theorems and critical point theory, *Pacific J. Math.*, **187** (1999), 263-279

[8] Ekeland, I.: *Convexity Methods in Hamiltonian Mechanics*, Springer, Berkin 1990

[9] Ekeland, I. and Ghoussoub, N.: Selected new aspects of the calculus of variations in the large, *Bull. AMS*, **39** (2002), 207-265

[10] El Amrouss, A.R. and Tsouli, N.: A generalization of Ekeland's variational principle and applications, *Electr. J. Diff. Eqts*, **14** (2006), 173-180

[11] Ghoussoub, N. and Preiss, D.: A general mountain pass principle locating and classifying for critical points, *Ann. I.H.P.*, **6** (1989), 321-330

[12] Ghoussoub, N.: *Duality and Perturbation Methods in Critical Point Theory, Cambridge Tracts in Math.*, C.U.P., Cambridge 1993

[13] Jabri, Y.: *The Mountain Pass Theorem*, Cambridge Univ. Press, Cambridge 2003

[14] Jeanjean, L.: On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on $\mathbb{R}^N$, *Proc. Royal Soc. Edinburgh*, **129A** (1999), 787-809

[15] Jeanjean, L. and Tanaka, K.: A positive solution for an asymptotically linear elliptic problem on $\mathbb{R}^N$, *ESAIM Control Opt. Calc. Var.*, **7** (2002),597-614

[16] Jeanjean L. and Toland J.F.: Bounded Palais-Smale Mountain-Pass sequences, *C. R. Acad. Sci. Paris*, **t. 327** (1998), 23-28

[17] Kuzin, I. and Pohozaev, S.: *Entire Solutions of Semilinear Elliptic Equations*, Birkhäuser, Basel 1997

[18] Pucci, P. and Serrin, J.: A mountain pass theorem, *J.D.E.*, **60** (1985), 142-149

[19] Rabinowitz, P.H.: *Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. in Math. No. 65*, AMS, Providence 1986

[20] Schechter, M.: The use of Cerami sequences in critical point theory, *Abstr. Appl. Anal.*, **2007** (2007)

[21] Struwe, M.: *Variational Methods*, Springer, Berlin, 1996

[22] Stuart, C.A. and Zhou, H.-S.: Applying the mountain pass theorem to an asymptotically linear elliptic equation on $\mathbb{R}^N$, *Comm. P.D. E.*, **24** (1999), 1731-1758

[23] Stuart, C.A. and Zhou, H-S.: Axi-symmetric TE-modes in a self-focusing dielectric, *SIAM J. Math. Anal.*, **37** (2005), 218-237

[24] Stuart, C.A. and Zhou, H.-S.: Existence of guided cylindrical TM-modes in an inhomogeneous self-focusing dielectric, to appear in *Math. Models Meth. Appl. Sci.*

[25] Zhou, H-S.: An application of a mountain pass theorem, *Acta Math. Sinica*, **18** (2002), 27-36

[26] Willem, M.: *Minimax Theorems*, Birkhäuser, Basel 1996