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HYERS–ULAM STABILITY OF LINEAR DIFFERENTIAL EQUATIONS WITH VANISHING COEFFICIENTS

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This paper is dedicated to Professor Allan Peterson in honor of his 45th year at the University of Nebraska-Lincoln.

ABSTRACT. We establish the Hyers-Ulam stability of certain first-order linear differential equations where the coefficients are allowed to vanish. We then extend these results to higher-order linear differential equations with vanishing coefficients that can be written with these first-order factors.

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1. INTRODUCTION

In 1940, Ulam [31] posed the following problem concerning the stability of functional equations: give conditions in order for a linear mapping near an approximately linear mapping to exist. The problem for the case of approximately additive mappings was solved by Hyers [10] who proved that the Cauchy equation is stable in Banach spaces, and the result of Hyers was generalized by Rassias [27]. Obloza [24] appears to be the first author who investigated the Hyers-Ulam stability of a differential equation, followed by Alsina and Ger [2].

Since then there has been a significant amount of interest in Hyers-Ulam stability, especially in relation to ordinary differential equations; for example see [8, 9, 11, 12, 13, 14, 16, 17, 18, 20, 21, 29, 32]. In our context, Hyers-Ulam stability on (a, b) means that given $\varepsilon > 0$ and continuous f, whenever an appropriately differentiable function $x : (a, b) \to \mathbb{C}$ satisfies

$$\left|a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \cdots + a_1(t)x'(t) + a_0(t)x(t) - f(t)\right| \le \varepsilon, \quad t \in (a, b)$$

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there exists a solution $y : (a, b) \to \mathbb{C}$ of

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = f(t), \quad t \in (a, b)$$

such that $|y - x| \leq K\varepsilon$ on (a, b) for some constant $K > 0$.

Some contemporary developments in Ulam's-type stability have been reviewed in [7]. Rezaei, Jung, and Rassias [28] recently utilized Laplace transforms to investigate the Hyers-Ulam stability of linear equations with constant coefficients. András and Mészáros [5] used an operator approach to show the stability of linear dynamic equations on time scales with constant coefficients, as well as for certain integral equations.

The purpose of this work is to partially fill gaps left by several recent papers. Abdollahpour and Najati [1] and Jung [15] looked at third-order equations with constant coefficients. Anderson et al [4] considered the Hyers-Ulam stability of second-order linear dynamic equations with nonconstant coefficients, while Tunç and Biçer [30] proved the Hyers-Ulam stability of third and fourth-order Cauchy-Euler differential equations, but not general-order Cauchy-Euler equations. See also Mortici, Rassias, and Jung [23].

We will also build on work by Popa and Raşa [25, 26]. In particular, we note that in [26, Theorem 2.2], Popa and Raşa prove the Hyers-Ulam stability of

$$y'(t) + \lambda(t)y(t) = f(t), \quad t \in I = (a, b), \tag{1.1}$$

where $a, b \in \mathbb{R} \cup \{\pm\infty\}$, assuming that the condition

$$\inf_{t \in I} |\Re\lambda(t)| := m > 0 \tag{1.2}$$

is met, where $\Re z$ is the real part of the complex number z. Building on (1.1), they discuss Hyers-Ulam stability of higher-order linear differential equations with variable coefficients, assuming these equations are made up of first-order factors as in (1.1).

What happens if the coefficient function λ in (1.1) vanishes? For example, in [26, Corollary 3.4], Popa and Raşa discuss the second-order linear equation with variable coefficients

$$y''(t) + a_1(t)y'(t) + a_2(t)y(t) = 0, \quad t \in I \tag{1.3}$$

and state that if r_2 is a solution of the associated Riccati equation

$$y' = y^2 - a_1(t)y + a_2(t), \quad t \in I$$

with $\inf|r_2(t)|>0$ and $\inf|a_1(t)-r_2(t)|>0$ on I, then (1.3) has Hyers-Ulam stability. Note that if we consider the second-order Cauchy-Euler equation with constants $z_k\in\mathbb{C}$, that is if

$$a_1(t):=(z_1+z_2+1)t^{-1},\quad a_2(t):=z_1z_2t^{-2},\quad r_2(t):=z_1t^{-1},\quad I:=[1,\infty),\tag{1.4}$$

then r_2 is a solution of the associated Riccati equation with $\inf|r_2(t)|=0$ and $\inf|a_1(t)-r_2(t)|=0$ on I, so [26, Corollary 3.4] is silent on this example. In [26, Corollary 3.5], Popa and Raşa discuss the third-order linear equation with variable coefficients

$$y'''(t)+a_1(t)y''(t)+a_2(t)y'(t)+a_3(t)y(t)=0,\quad t\in I\tag{1.5}$$

and state that if r_3 is a solution of the associated Liénard equation

$$y''+(a_1-3y)y'+y^3-a_1y^2+a_2y-a_3=0,\quad t\in I,$$

r_2 is a solution of the associated Riccati equation

$$y'=y^2-(a_1-r_3)y+a_2-2r_3'+r_3^2-a_1r_3,\quad t\in I,$$

and $r_1:=a_1-r_3-r_2$ with $\inf|r_k(t)|>0$ on I for $k=1,2,3$, then (1.5) has Hyers-Ulam stability. If we consider the third-order Cauchy-Euler equation with constants $z_k\in\mathbb{C}$, that is if

$$\begin{aligned} a_1(t)&:=(z_1+z_2+z_3+3)t^{-1},\quad a_3(t):=z_1z_2z_3t^{-3},\\ a_2(t)&:=(z_1+z_2+z_3+z_1z_2+z_1z_3+z_2z_3+1)t^{-2},\end{aligned}\tag{1.6}$$

$$r_3(t):=z_1t^{-1},\quad r_2(t):=(z_2+1)t^{-1},\quad r_1(t):=(z_3+2)t^{-1},\quad I:=[1,\infty),$$

then r_3 is a solution of the Liénard equation, r_2 is a solution of the Riccati equation, and $\inf|r_k(t)|=0$ on I for $k=1,2,3$, so [26, Corollary 3.5] does not apply to this example.

In fact it can be shown that (1.3) and (1.5) as written, with coefficients from (1.4) and (1.6), respectively, are unstable in the Hyers-Ulam sense on $(0,\infty)$. It turns out that the presentation of the original equation, even if homogeneous, may determine the stability. For example, consider a generalization of the second-order Cauchy-Euler equation (1.3) with coefficients from (1.4), namely

$$t^ry''(t)+(z_1+z_2+1)t^{r-1}y'(t)+z_1z_2t^{r-2}y(t)=0,\quad t\in(0,\infty)\tag{1.7}$$

for $r \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}\backslash\{r-2\}$. If $y(t) := c_1 t^{-z_1} + c_2 t^{-z_2}$ and

$$x(t) := y(t) + \frac{\varepsilon t^{2-r}}{(z_1+2-r)(z_2+2-r)},$$

then it is clear y is a general solution of (1.7), x is an approximate solution for any given $\varepsilon > 0$, and $|y - x| \to \infty$ on $(0, \infty)$ for $r \neq 2$, so that (1.7) is unstable if $r \neq 2$. The equation

$$t^2 y''(t) + (z_1 + z_2 + 1) t y'(t) + z_1 z_2 y(t) = 0, \quad t \in (0, \infty),$$

however, is shown below to be Hyers-Ulam stable if and only if $\Re(z_1, z_2) \neq 0$. Indeed, our results in Theorem 3.2 will include general-order Cauchy-Euler equations as a special case.

The discussion will proceed as follows. In Section 2 we will consider singular first-order linear differential equations that can be written in the form (1.1), where (1.2) is not assumed, in other words equations with possibly vanishing coefficients. (Note that Hyers-Ulam stability is independent of the nonhomogeneous term f in (1.1), so in the sequel we consider only homogeneous equations, that is $f \equiv 0$; see Remark 2.3.) In Section 3, we will investigate singular higher-order linear differential equations that may be rewritten in factored form using first-order factors discussed in Section 2; the form we consider incorporates constant coefficients and Cauchy-Euler coefficients, thus in the process extending the results in [1, 4, 15, 25, 26, 30]. Section 4 will cover Hyers-Ulam stability for variable-coefficient first-order equations, and Section 5 will touch briefly on a corresponding result on time scales.

2. HYERS–ULAM STABILITY FOR CERTAIN FIRST-ORDER EQUATIONS

Let us recall the definition of Hyers-Ulam stability, here stated for certain first-order linear differential equations, where the leading coefficient will be allowed to vanish.

Definition 2.1 (Hyers-Ulam stability)**.** Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$, $\varphi : (a, b) \to \mathbb{R}$ be a continuous function, $z \in \mathbb{C}$ be a complex constant, and $\varepsilon > 0$ be given. If whenever a differentiable function $x : (a, b) \to \mathbb{C}$ satisfies

$$|\varphi(t) x'(t) + z x(t)| \leq \varepsilon, \quad t \in (a, b)$$

there exists a solution $y : (a, b) \to \mathbb{C}$ of $\varphi(t)y'(t)+zy(t) = 0$ such that $|y-x| \leq K\varepsilon$ on (a, b) for some constant $K > 0$, then equation $\varphi(t)y'(t) + zy(t) = 0$ has Hyers-Ulam stability (a, b).

Theorem 2.2. *Let $z \in \mathbb{C}$ be a given constant, and let $t_0 \in (a, b) \subseteq (-\infty, \infty)$. Let $\varphi : (a, b) \to \mathbb{R}$ be a non-negative continuous function such that*

(a) $\lim_{t\to a^+} \int_t^{t_0} \frac{1}{\varphi(s)} ds = \infty$, *or*

(b) $\lim_{t\to b^-} \int_{t_0}^{t} \frac{1}{\varphi(s)} ds = \infty$.

Then the differential equation

$$\varphi(t)y'(t) + zy(t) = 0, \quad t \in (a, b), \tag{2.1}$$

is Hyers-Ulam stable on (a, b) if and only if $\Re z \neq 0$. Moreover, if (a) and $\Re z > 0$ hold, or if (b) and $\Re z < 0$ hold, then the solution y used to show the Hyers-Ulam stability of (2.1) is unique. If $0 < \int_a^b \frac{1}{\varphi(s)} ds < \infty$, then (2.1) is Hyers-Ulam stable for all $z \in \mathbb{C}$.

Proof. If $\Re z = 0$, let $\beta \in \mathbb{R}$ and $i = \sqrt{-1}$, and consider the differential equation

$$\varphi(t)y'(t) + i\beta y(t) = 0, \quad t \in (a, b). \tag{2.2}$$

We will show (2.2) is unstable in the sense of Hyers and Ulam. Given any $\varepsilon > 0$, for $t \in (a, b)$ let

$$x(t) = \varepsilon\Phi(t)e^{-i\beta\Phi(t)}, \quad \Phi(t) := \int_{t_0}^{t} \frac{d\tau}{\varphi(\tau)}. \tag{2.3}$$

Then

$$|\varphi(t)x'(t) + i\beta x(t)| = \left|\varepsilon e^{-i\beta\Phi(t)}\right| = \varepsilon$$

for all $t \in (a, b)$. Clearly

$$y(t) = ce^{-i\beta\Phi(t)}$$

where c is a constant, is the only type of solution of (2.2), but

$$|y(t) - x(t)| = \left|ce^{-i\beta\Phi(t)} - \varepsilon\Phi(t)e^{-i\beta\Phi(t)}\right| = |c - \varepsilon\Phi(t)|$$

is unbounded on (a, b) for any choice of c by (a) or (b). Consequently, (2.2) is unstable in the sense of Hyers and Ulam on (a, b).

If $\Re z \neq 0$, suppose $x : (a, b) \to \mathbb{C}$ is an approximate solution of (2.1) such that

$$\varphi(t)x'(t) + zx(t) = q(t), \quad |q(t)| \leq \varepsilon, \quad t \in (a, b),$$

for some integrable perturbation $q:(a,b)\to\mathbb{C}$ and some $\varepsilon>0$. Pick $y:(a,b)\to\mathbb{C}$ to be a solution of (2.1) in the following way: Let

$$A=\begin{cases} a & \text{if } \ (a), \Re z>0 \ \text{ hold,} \\ \infty & \text{if } \ (b), \Re z<0 \ \text{ hold,} \\ t_0 & \text{if } \ \text{otherwise,} \end{cases} \tag{2.4}$$

and for Φ given in (2.3), take

$$y(t)=ce^{-z\Phi(t)}, \quad x(t)=y(t)+e^{-z\Phi(t)}\int_A^t \frac{q(s)}{\varphi(s)}e^{z\Phi(s)}ds, \quad c=x(t_0)+\int_{t_0}^A \frac{q(s)}{\varphi(s)}e^{z\Phi(s)}ds, \tag{2.5}$$

and see that

$$|y(t)-x(t)|=\left|e^{-z\Phi(t)}\int_A^t \frac{q(s)}{\varphi(s)}e^{z\Phi(s)}ds\right| \le \varepsilon \begin{cases} \int_a^t \frac{1}{\varphi(s)}e^{\Re z(\Phi(s)-\Phi(t))}ds \\ \int_t^\infty \frac{1}{\varphi(s)}e^{\Re z(\Phi(s)-\Phi(t))}ds \\ \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} \frac{1}{\varphi(s)}e^{\Re z(\Phi(s)-\Phi(t))}ds \end{cases} = \varepsilon\frac{e^{\Re z}}{|\Re z|}.$$

Therefore (2.1) is Hyers-Ulam stable on (a,b) if $\Re z\neq 0$.

To see uniqueness, suppose $x:(a,b)\to\mathbb{C}$ is an approximate solution of (2.1) such that

$$|\varphi(t)x'(t)+zx(t)|\le\varepsilon \ \text{ for all } \ t\in(a,b)$$

for some $\varepsilon>0$. Suppose further that $y_1,y_2:(a,b)\to\mathbb{C}$ are two solutions of (2.1) such that $|y_j(t)-x(t)|\le\varepsilon K_j$ for all $t\in(a,b)$, for $j=1,2$. Then we have for constants c_j that

$$y_j(t)=c_je^{-z\Phi(t)},$$

where Φ is defined in (2.3), so that

$$e^{-\Phi(t)\Re z}|c_1-c_2|=|y_1(t)-y_2(t)|\le|y_1(t)-x(t)|+|x(t)-y_2(t)|\le\varepsilon(K_1+K_2);$$

letting $t\to a^+$ for $\Re z>0$, or $t\to b^-$ for $\Re z<0$, yields $\infty<\varepsilon(K_1+K_2)$, a contradiction. The uniqueness of the solution y is proven for these cases.

If $0<\int_a^b \frac{1}{\varphi(s)}ds<\infty$, then Hyers-Ulam stability follows easily for all $z\in\mathbb{C}$. □

Remark 2.3. Consider the nonhomogeneous version of (2.1), namely

$$\varphi(t)y'(t)+zy(t)=f(t), \quad t\in(a,b), \tag{2.6}$$

for some continuous function $f : (a, b) \to \mathbb{R}$. We can easily modify the proof of the previous theorem as follows. If $x : (a, b) \to \mathbb{C}$ is an approximate solution of (2.6) such that

$$\varphi(t)x'(t) + zx(t) - f(t) = q(t), \quad |q(t)| \le \varepsilon, \quad t \in (a, b),$$

for some integrable perturbation $q : (a, b) \to \mathbb{C}$ and some $\varepsilon > 0$, pick $y : (a, b) \to \mathbb{C}$ to be a solution of (2.6) in the following way: Let Φ be as in (2.3), choose A as in (2.4), and take

$$\begin{aligned} y(t) &= ce^{-z\Phi(t)} + e^{-z\Phi(t)} \int_A^t \frac{f(s)}{\varphi(s)} e^{z\Phi(s)} ds, \quad x(t) = y(t) + e^{-z\Phi(t)} \int_A^t \frac{q(s)}{\varphi(s)} e^{z\Phi(s)} ds, \\ c &= x(t_0) + \int_{t_0}^A \frac{f(s) + q(s)}{\varphi(s)} e^{z\Phi(s)} ds. \end{aligned} \tag{2.7}$$

As the key calculations are based on $|y(t) - x(t)|$, we see from (2.5) and (2.7) that nothing is changed due to f, as its terms subtract off.

Corollary 2.4. *Let $z \in \mathbb{C}$ and $\gamma \in \mathbb{R}$ be given constants. The singular differential equation with vanishing coefficient*

$$t^\gamma y'(t) + zy(t) = 0, \quad t \in (0, \infty), \tag{2.8}$$

is Hyers-Ulam stable on $(0, \infty)$ if and only if $\Re z \ne 0$. Moreover, if $\Re z < 0$ and $\gamma \le 1$, or if $\Re z > 0$ and $\gamma \ge 1$, then the solution y used to show the Hyers-Ulam stability of (2.8) *is unique.*

Proof. In Theorem 2.2, take $a = 0$, $b = \infty$, $\varphi(t) = t^\gamma$. □

3. FACTORING

In this section we show how the results in the previous section can be incorporated into an investigation of Hyers-Ulam stability for certain higher-order singular linear differential equations with nonconstant coefficients.

Let D be the differential operator defined by $Dy = y'$ for differentiable functions $y : (a, b) \to \mathbb{C}$, and I the identity operator given by $Iy = y$. For a given function $\varphi : (a, b) \to \mathbb{R}$, let $(\varphi D)^0 y = Iy = y$, $(\varphi D)y = \varphi y'$, and for positive integers n, let $(\varphi D)^n y = (\varphi D)^{n-1} \varphi Dy$. We consider the higher-order singular linear differential equation

$$\sum_{k=0}^{n} \alpha_{n-k} (\varphi D)^k y(t) = 0 \tag{3.1}$$

for some real constants α_m, where $\alpha_0 = 1$ for convenience. Note that if $\varphi \equiv 1$, then this is the nth-order linear constant coefficient differential equation

$$\sum_{k=0}^{n} \alpha_{n-k} y^{(k)}(t) = 0,$$

while if $\varphi(t) = t$, this is a nested form of the nth-order linear Cauchy-Euler differential equation

$$\sum_{k=0}^{n} \alpha_{n-k} (tD)^k y(t) = 0.$$

Remark 3.1. Our first task will be to factor (3.1) for the analysis to follow. To accomplish the factorization of (3.1), we use the substitutions (see also [3, Remark 2.2] and [22])

$$\alpha_1 = \sum_i z_i, \quad \alpha_2 = \sum_{i<j} z_i z_j, \quad \alpha_3 = \sum_{i<j<k} z_i z_j z_k, \quad \alpha_4 = \sum_{i<j<k<\ell} z_i z_j z_k z_\ell,$$

$$\cdots \quad \alpha_m = \sum_{i_1<i_2<\cdots<i_m} z_{i_1} z_{i_2} \cdots z_{i_m}, \quad \cdots \quad \alpha_n = z_1 z_2 z_3 \cdots z_n,$$

where $z_i \in \mathbb{C}$ for $i = 1, 2, \cdots, n$. Then we have the factorization of (3.1) given by

$$\sum_{k=0}^{n} \alpha_{n-k} (\varphi D)^k y(t) = \prod_{k=1}^{n} (\varphi D + z_k I)\, y(t) = 0, \qquad n \in \mathbb{N}; \tag{3.2}$$

see [3, Lemma 2.3] and [30, Section 2] for more on this type of substitution and factorization. The following result for (3.1) is the main result in this section.

Theorem 3.2 (Hyers-Ulam Stability). *Let $t_0 \in (a, b) \subseteq (-\infty, \infty)$, and let $\varphi : (a, b) \to \mathbb{R}$ be a non-negative continuous function such that*

(a) $\displaystyle\lim_{t\to a^+} \int_t^{t_0} \frac{1}{\varphi(s)} ds = \infty$, *or*

(b) $\displaystyle\lim_{t\to b^-} \int_{t_0}^{t} \frac{1}{\varphi(s)} ds = \infty$.

For positive integer n, consider (3.1) *with real constants α_m for $m = 0, 1, \ldots, n$, where $\alpha_0 = 1$. Let the substitutions $z_1, \ldots, z_n$ be as given in Remark 3.1. Then* (3.1) *has Hyers-Ulam stability on (a, b) if and only if $\Re z_k \neq 0$ for each $k = 1, 2, \ldots, n$. If $\int_a^b \frac{1}{\varphi(s)} ds < \infty$, then* (3.1) *is Hyers-Ulam stable on (a, b) for all $z \in \mathbb{C}$.*

Proof. By Remark 3.1 and (3.2) we have that (3.1) can be written in factored form as

$$\sum_{k=0}^{n} \alpha_{n-k} (\varphi D)^k y(t) = \prod_{k=1}^{n} (\varphi D + z_k I)\, y(t) = 0.$$

Assume $\Re z_k \neq 0$. Now suppose there exists a function x such that

$$\left|\prod_{k=1}^{n} (\varphi D + z_k I)\, x(t)\right| \leq \varepsilon \tag{3.3}$$

for some $\varepsilon > 0$, for all $t \in (a,b)$. Define the new functions $y_n \equiv 0$, $x_0 := x$ and

$$x_k := (\varphi D + z_k I)\, x_{k-1}, \quad k = 1, \ldots, n. \tag{3.4}$$

Then

$$x_k(t) = \varphi(t) x'_{k-1}(t) + z_k x_{k-1}(t), \quad k = 1, \ldots, n, \quad t \in (a,b),$$

and by (3.3) and (3.4) (recall $y_n = 0$ and $x_0 = x$), we have

$$|y_n(t) - x_n(t)| = |x_n(t)| \leq \varepsilon, \quad t \in (a,b) \tag{3.5}$$

by construction. Hyers-Ulam stability of (3.4) on (a,b) with $k = n$ implies there exists a solution y_{n-1} of

$$(\varphi D + z_n I)\, y_{n-1}(t) = y_n(t)$$

such that $|y_{n-1}(t) - x_{n-1}(t)| \leq K_n \varepsilon$; substituting from (3.4) with $k = n-1$ this inequality becomes

$$|y_{n-1}(t) - \varphi(t) x'_{n-2}(t) - z_{n-1} x_{n-2}(t)| \leq K_n \varepsilon,$$

and so on. We proceed by iterating the proof of Theorem 2.2, and using Remark 2.3. Let $t_k \in (a,b)$ and let

$$A_k = \begin{cases} a & \text{if } (a), \Re z_k > 0 \text{ hold}, \\ \infty & \text{if } (b), \Re z_k < 0 \text{ hold}, \\ t_k & \text{if otherwise}, \end{cases} \tag{3.6}$$

for $k = 1, \ldots, n$. Let x_{k-1} solve (3.4) with initial condition at t_k, and y_{k-1} solve

$$(\varphi D + z_k I)\, y_{k-1} = y_k, \quad k = 1, \ldots, n. \tag{3.7}$$

If

$$\Phi_k(t) := \int_{t_k}^{t} \frac{d\tau}{\varphi(\tau)},$$

then take

$$\begin{aligned} y_{k-1}(t) &= c_{k-1}e^{-z_k\Phi_k(t)} + e^{-z_k\Phi_k(t)}\int_{A_k}^{t}\frac{y_k(s)}{\varphi(s)}e^{z_k\Phi_k(s)}ds, \\ x_{k-1}(t) &= y_{k-1}(t) + e^{-z_k\Phi_k(t)}\int_{A_k}^{t}\frac{x_k(s)}{\varphi(s)}e^{z_k\Phi_k(s)}ds, \\ c_{k-1} &= x_{k-1}(t_k) + \int_{t_k}^{A_k}\frac{y_k(s)+x_k(s)}{\varphi(s)}e^{z_k\Phi_k(s)}ds. \end{aligned} \tag{3.8}$$

Using (3.4) and (3.5) we have on (a, b) that

$$|y_{k-1}(t) - x_{k-1}(t)| \le \varepsilon \prod_{j=k}^{n}\frac{e^{\Re z_j}}{|\Re z_j|},$$

starting with $k = n$ and proceeding down to $k = 1$. In particular (recall $x_0 = x$, the original approximate solution), we arrive at

$$|y_0(t) - x_0(t)| = |y_0(t) - x(t)| \le \varepsilon \prod_{j=1}^{n}\frac{e^{\Re z_j}}{|\Re z_j|}, \quad t \in (a, b).$$

From (3.7), and using the fact that $y_n = 0$, we see that y_0 is a solution of the factored form of (3.1). Thus (3.1) has Hyers-Ulam stability on (a, b). □

4. HYERS-ULAM STABILITY FOR VARIABLE-COEFFICIENT FIRST-ORDER EQUATIONS

In this section we return explicitly to the first-order form of (1.1), namely

$$y'(t) + \lambda(t)y(t) = f(t), \quad t \in I = (a, b),$$

where $a, b \in \mathbb{R} \cup \{\pm\infty\}$. Here we drop the assumption that condition (1.2) holds, that is we allow

$$\inf_{t\in I} |\Re\lambda(t)| = 0.$$

Then we have the following general result.

Theorem 4.1. *Let $t_0 \in (a, b) \subseteq (-\infty, \infty)$, and let $\lambda : (a, b) \to \mathbb{C}$ be a continuous function. Then the differential equation*

$$y'(t) + \lambda(t)y(t) = 0, \quad t \in (a, b), \tag{4.1}$$

is Hyers-Ulam stable on (a, b) if

$$K := \sup_{t\in(a,b)} \int_{\min\{t,t_0\}}^{\max\{t,t_0\}} e^{\Re\Lambda(s)-\Re\Lambda(t)}ds < \infty, \tag{4.2}$$

where $\Lambda(t) := \int \lambda(t)dt$.

Proof. Let x be an approximate solution of (4.1), that is x satisfies

$$|x'(t) + \lambda(t)x(t)| \leq \varepsilon.$$

Then a solution y of (4.1) can be chosen so that

$$|y(t) - x(t)| \leq \varepsilon \int_{\min\{t,t_0\}}^{\max\{t,t_0\}} e^{\Re\Lambda(s) - \Re\Lambda(t)} ds$$

using elementary methods. The result follows if (4.2) holds. □

Example 4.2. Let $t_0 \in (a, b)$ and $\lambda(t) := zt^r$, where $z \in \mathbb{C}$ and $r \in \mathbb{R}$, and consider the differential equation

$$y'(t) + zt^r y(t) = 0, \quad t \in (a, b). \tag{4.3}$$

If $(a, b) = (0, 1)$ and $r > 0$, then (4.3) is Hyers-Ulam stable on $(0, 1)$, since

$$K := \sup_{t\in(0,1)} \int_{\min\{t,t_0\}}^{\max\{t,t_0\}} e^{(s^{r+1} - t^{r+1})\Re z/(r+1)} ds < \infty$$

for any $z \in \mathbb{C}$. Clearly $\lambda(t) \to 0$ as $t \to 0^+$, violating condition (1.2). If $(a, b) = (0, \infty)$ and $r = -1$, then (4.3) is unstable on $(0, \infty)$ for any $z \in \mathbb{C}$. Note that in this case, for $t_0 \in (0, \infty)$,

$$\sup_{t\in(a,b)} \int_{\min\{t,t_0\}}^{\max\{t,t_0\}} e^{\Re\Lambda(s) - \Re\Lambda(t)} ds = \sup_{t\in(0,\infty)} \begin{cases} \left|\frac{1}{1+\Re z}\right| \left|t - t_0 \left(\frac{t_0}{t}\right)^{\Re z}\right| & : \Re z \neq -1, \\ t|\ln(t/t_0)| & : \Re z = -1, \end{cases} = \infty.$$

5. TIME SCALES

In this final section we extend Theorem 4.1 to time scales $\mathbb{T}$ in the sense that we allow domains other than $\mathbb{T} = \mathbb{R}$, such as the discrete time scale $\mathbb{T} = \mathbb{Z}$ and so on. Notice again that we do not assume any coefficient function is bounded away from 0, only that an analogous time-scale integral converges.

Theorem 5.1. *Let* $\mathbb{T}$ *be a time scale,* $\varphi, \psi \in \mathrm{C_{rd}}(a, b)_{\mathbb{T}}$*, and assume*

$$\varphi(t) + \mu(t)\psi(t) \neq 0 \quad \textit{and} \quad \int_a^b \left| e_{\frac{\psi}{\varphi}}(t, \sigma(s)) \frac{1}{\varphi(s)} \right| \Delta s < \infty$$

for all $t \in (a, b)_{\mathbb{T}}$*. Then the first-order dynamic equation*

$$\varphi(t)y^{\Delta}(t) - \psi(t)y(t) = 0 \tag{5.1}$$

has Hyers-Ulam stability on $(a,b)_{\mathbb{T}}$.

Proof. Given $\varepsilon > 0$, suppose there exists an approximate solution $x \in \mathrm{C}^{\Delta}_{\mathrm{rd}}(a,b)_{\mathbb{T}}$ that satisfies

$$\left|\varphi(t)x^{\Delta}(t) - \psi(t)x(t)\right| \le \varepsilon, \quad t \in (a,b)_{\mathbb{T}}.$$

Set

$$q(t) := \varphi(t)x^{\Delta}(t) - \psi(t)x(t), \quad t \in (a,b)_{\mathbb{T}};$$

by the variation of constants formula [6, Theorem 2.77] we have for any $t_0 \in (a,b)_{\mathbb{T}}$ that x is given by

$$x(t) = e_{\frac{\psi}{\varphi}}(t,t_0)x(t_0) + \int_{t_0}^{t} e_{\frac{\psi}{\varphi}}(t,\sigma(s))\frac{q(s)}{\varphi(s)}\Delta s;$$

note that the time scales exponential function $e_{\frac{\psi}{\varphi}}$ exists by the assumption $\varphi(t) + \mu(t)\psi(t) \neq 0$ given in the statement of the theorem. Pick $y \in \mathrm{C}^{\Delta}_{\mathrm{rd}}(a,b)_{\mathbb{T}}$ to be the unique solution of the initial value problem

$$\varphi y^{\Delta} - \psi y = 0, \quad y(t_0) = x(t_0).$$

Then

$$y(t) = e_{\frac{\psi}{\varphi}}(t,t_0)x(t_0), \quad t \in (a,b)_{\mathbb{T}},$$

and

$$|x(t) - y(t)| = \left|\int_{t_0}^{t} e_{\frac{\psi}{\varphi}}(t,\sigma(s))\frac{q(s)}{\varphi(s)}\Delta s\right| \le \varepsilon \int_{a}^{b} \left|e_{\frac{\psi}{\varphi}}(t,\sigma(s))\frac{1}{\varphi(s)}\right| \Delta s.$$

Therefore (5.1) has Hyers-Ulam stability if the condition

$$\int_{a}^{b} \left|e_{\frac{\psi}{\varphi}}(t,\sigma(s))\frac{1}{\varphi(s)}\right| \Delta s < \infty$$

is satisfied. □

Example 5.2. Let $\mathbb{T} = \mathbb{Z}$, the set of integers. In (5.1) take $\varphi \equiv 1$ and $\psi(t) = \frac{-t}{t+1}$, and thus consider the difference equation

$$\Delta y(t) + \frac{t}{t+1}y(t) = 0, \quad t \in \{0,1,2,3,\dots\}, \tag{5.2}$$

where $\Delta y(t) := y(t+1) - y(t)$. Given $\varepsilon > 0$, suppose that some function x is an approximate solution of (5.2), that is to say suppose x satisfies

$$\Delta x(t) + \frac{t}{t+1}x(t) = q(t), \quad |q(t)| \le \varepsilon, \quad t \in \{0,1,2,3,\dots\}.$$

Then x has the form

$$x(t) = \frac{x_0 + \sum_{k=0}^{t-1}(k+1)!\, q(k)}{t!}, \quad t \in \{0,1,2,3,\dots\}.$$

If we take y to be the function given by

$$y(t) = \frac{x_0}{t!}, \quad t \in \{0,1,2,3,\dots\},$$

then y is a solution of (5.2), and for any $t \in \{0,1,2,3,\dots\}$ we have

$$|y(t)-x(t)| = \frac{1}{t!}\left|\sum_{k=0}^{t-1}(k+1)!\, q(k)\right| \le \frac{1}{t!}\sum_{k=0}^{t-1}(k+1)!\ |q(k)| \le \frac{\varepsilon}{t!}\sum_{k=0}^{t-1}(k+1)! \le \frac{3}{2}\varepsilon.$$

Thus, (5.2) is Hyers-Ulam stable. However, note that

$$\int_a^b \left| e_{\frac{\psi}{\varphi}}(t,\sigma(s))\frac{1}{\varphi(s)}\right| \Delta s = \sum_{s=0}^{\infty} |e_{\psi}(t,s+1)| = \sum_{s=0}^{\infty} \frac{(s+1)!}{t!},$$

which is unbounded for any $t \in \{0,1,2,3,\dots\}$. This example and Example 4.2 illustrate why there are no known necessary and sufficient conditions for Hyers-Ulam stability stated in Theorems 4.1 and 5.1.

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