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LINEAR FORWARD FRACTIONAL DIFFERENCE EQUATIONS

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ABSTRACT. In this paper, we continue our study of the linear forward fractional difference equation. We define a convolution and obtain a convolution theorem for the R-transform. We then apply a transform method and obtain a variation of parameters formula for the equation $-\Delta^{\nu} y(t) + \lambda y(t + \nu - 1) = h(t + \nu - 2)$, where $1 < \nu \leq 2$. We introduce two discrete Mittag-Leffler type functions and address their convergence.

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1. INTRODUCTION

In this article, we continue our study of the discrete forward fractional difference equation. Currently, there is a considerable amount of interest in fractional calculus; to our knowledge, the primary interest remains in the continuous case and we refer the reader to well-known and excellent references [4], [5], [11], [12], or [13], for example. There has been less activity in the development of discrete fractional calculus. In this article we shall consider a forward fractional operator of Riemann-Liouville type; we refer the reader to our previous work, [1], [2], and [3], some recent work of Goodrich and co-authors [7], [8], [9], and Ferreira [6].

The purpose of this article is to construct a variation of parameters formula for solutions of linear nonhomogeneous forward fractional difference equations. The construction is performed with a transform method. In Section 2, we present sufficient fundamental definitions and calculations so that the article is self-contained. In Section 3, we introduce two discrete Mittag-Leffler type functions and address their convergence. We also develop the variation of parameters formula.

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2. PRELIMINARIES

Let Γ denote the usual special gamma function and recall the notation

$$t^{(\mu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)}.$$

Throughout, we assume that if $t+1-\mu \in \{0,-1,\ldots,-k,\ldots\}$, then $t^{(\mu)} = 0$. We consider the forward fractional sum as defined by Miller and Ross [10]

$$\Delta_a^{-\nu} f(t) = \sum_{s=a}^{t-\nu} \frac{(t-\sigma(s))^{(\nu-1)}}{\Gamma(\nu)} f(s), \tag{2.1}$$

where $\nu \geq 0$, $a \in \mathbb{R}$, and $\sigma(s) = s+1$. Define $\mathbb{N}_{t_0} = \{t_0, t_0+1, t_0+2, \ldots\}$ and note that $\Delta_a^{-\nu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu}$. For the purpose of this study, we shall have need to extend the domain of $\Delta_a^{-\nu}$ to functions defined on $\mathbb{Z}_{t_0} = \{\ldots, t_0-2, t_0-1, t_0, t_0+1, t_0+2, \ldots\}$. Further, we shall consider the Riemann-Liouville fractional difference

$$\Delta_a^{\mu} f(t) = \Delta_a^{m-\nu} f(t) = \Delta^m(\Delta_a^{-\nu} f(t)),$$

where $\mu > 0$, $m-1 < \mu \leq m$, m denotes a positive integer, and $-\nu = \mu - m$.

We recall the power rules

$$\Delta t^{(\mu)} = \mu \Delta t^{(\mu-1)},$$

and, if $\mu \neq -1$ and $\mu+\nu+1$ is not a non-positive integer, then

$$\Delta_{\mu}^{-\nu} t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{(\mu+\nu)}.$$

We also recall the discrete transform (R-transform) defined by

$$R_{t_0}(f(t))(s) = \sum_{t=t_0}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} f(t), \tag{2.2}$$

where f is defined on $\mathbb{N}_{t_0} = \{t_0, t_0+1, t_0+2, \ldots\}$. This is the Laplace transform on the time scale $\mathbb{N}_{t_0}$ and it is not intended to be the commonly known z-transform.

The following two lemmas have been previously obtained in [1] or [2].

Lemma 2.1. *For any* $\nu \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$,

$$R_{\nu-1}(t^{(\nu-1)})(s) = \frac{\Gamma(\nu)}{s^{\nu}}.$$

Lemma 2.2. *If* $\mu > 0$ *and* $m-1 < \mu < m$ *where* m *denotes a positive integer and* f *is defined on* $\mathbb{N}_{\mu-m}$, *then*

$$R_0(\Delta_{\mu-m}^{\mu} f)(s) = s^{\mu} R_{\mu-m}(f)(s) - \sum_{k=0}^{m-1} s^{m-k-1}(\Delta^k \Delta_{\mu-m}^{-(\mu-m)} f)|_{t=0}. \tag{2.3}$$

The purpose of this article is to construct a variation of parameters formula for linear nonhomogeneous fractional difference equations. To do so, we define a convolution and obtain a convolution theorem for the discrete transform.

Lemma 2.3. *Assume* $1 < \nu \leq 2$. *Assume* f *is defined on* $\mathbb{N}_{\nu-2}$. *Define* $F(t) = f(t+\nu-1)$. *Then*

$$R_0(F(t)) = (s+1)^{\nu-1} R_{\nu-2}(f(t)) - f(\nu-2).$$

Remark 2.1. We may write $R_0(f(t+\nu-1))$ but we mean $R_0(F(t))$.

Proof. The proof follows from the definition of the R-transform and a substitution $u = t+\nu-1$. In fact,

$$R_0(f(t+\nu-1)) = \sum_{t=0}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} f(t+\nu-1) = (s+1)^{\nu-1} \sum_{u=\nu-1}^{\infty} \left(\frac{1}{s+1}\right)^{u+1} f(u)$$

$$= (s+1)^{\nu-1} \sum_{u=\nu-2}^{\infty} \left(\frac{1}{s+1}\right)^{u+1} f(u) - f(\nu-2).$$

□

Next we introduce a new convolution product of two functions defined on $\mathbb{N}_{\nu-2}$. Define $f *_{\nu-2} g$ by

$$f *_{\nu-2} g(t) = \sum_{s=\nu-2}^{t} f(t-s+\nu-2) g(s), \tag{2.4}$$

where $1 < \nu \leq 2$.

Lemma 2.4.

$$R_{\nu-2}(f *_{\nu-2} g)(s) = (s+1)^{\nu-1} R_{\nu-2}(f)(s) R_{\nu-2}(g)(s).$$

Proof. Interchange the order of summation to obtain

$$\begin{aligned}
R_{\nu-2}(f *_{\nu-2} g)(s) &= \sum_{t=\nu-2}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} \sum_{\tau=\nu-2}^{t} f(t-\tau+\nu-2) g(\tau) \\
&= \sum_{\tau=\nu-2}^{\infty} \sum_{t=\tau}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} f(t-\tau+\nu-2) g(\tau) \\
&= \sum_{\tau=\nu-2}^{\infty} \sum_{u=\nu-2}^{\infty} \left(\frac{1}{s+1}\right)^{u+\tau-\nu+3} f(u) g(\tau) \\
&= (s+1)^{\nu-1} \sum_{u=\nu-2}^{\infty} \left(\frac{1}{s+1}\right)^{u+1} f(u) \sum_{\tau=\nu-2}^{\infty} \left(\frac{1}{s+1}\right)^{\tau+1} g(\tau).
\end{aligned}$$

□

Remark 2.2. In [3], the authors define

$$(h * g)(t) = \sum_{s=0}^{t-\nu} (t - \sigma(s))^{(\nu-1)} g(s),$$

where $h(t) = t^{(\nu-1)}$ and $g(t) = \alpha^t$ and show

$$R_\nu((h * g)(t))(s) = R_{\nu-1}(h(t))(s) R_0(g(t))(s).$$

3. VARIATION OF PARAMETERS

Now in this article, we shall consider a fractional equation of order ν where $\nu \in (1, 2]$. In particular, we shall consider

$$-\Delta^\nu y(t) + \lambda y(t + \nu - 1) = h(t + \nu - 2), \quad t = 0, 1, 2, \ldots. \tag{3.1}$$

We shall use the transform method and construct a variation of parameters formula to solve (3.1).

Define

$$E_1(t, \lambda, \nu) = \sum_{n=0}^{\infty} \lambda^n \frac{(t + n(\nu - 1))^{((n+1)\nu-2)}}{\Gamma((n + 1)\nu - 1)}, \quad t \in \mathbb{N}_{\nu-2}, \tag{3.2}$$

$$E_2(t, \lambda, \nu) = \sum_{n=0}^{\infty} \lambda^n \frac{(t + n(\nu - 1))^{((n+1)\nu-1)}}{\Gamma((n + 1)\nu)}, \quad t \in \mathbb{N}_{\nu-2}. \tag{3.3}$$

We list some properties of the special functions, E_1 and E_2.

Lemma 3.5. *The following are valid:*

1. $E_1(t, \lambda, \nu)$ *and* $E_2(t, \lambda, \nu)$ *converge absolutely if* $|\lambda| \frac{1}{(\nu-1)^{1-\nu}} \frac{1}{\nu^\nu} < 1$.
2. $\Delta_t E_2(t, \lambda, \nu) = E_1(t, \lambda, \nu)$.
3. $E_1(\nu - 2, \lambda, \nu) = 1, \quad E_1(\nu - 1, \lambda, \nu) = \nu - 1 + \lambda;$
   $E_2(\nu - 2, \lambda, \nu) = 0, \quad E_2(\nu - 1, \lambda, \nu) = 1.$
4. $E_1(\nu - k, \lambda, \nu) = E_2(\nu - k, \lambda, \nu) = 0, \quad k = 3, 4, \ldots.$
5. $E_1(t, \lambda, \nu) + E_2(t, \lambda, \nu) = E_2(t + 1, \lambda, \nu)$.
6. $\Delta^\nu_{\nu-2} E_1(t, \lambda, \nu) = \lambda E_1(t + \nu - 1, \lambda, \nu)$.
7. $\Delta^\nu_{\nu-2} E_2(t, \lambda, \nu) = \lambda E_2(t + \nu - 1, \lambda, \nu)$.

Proof. (1) We only prove that $E_1(t, \lambda, \nu)$ is convergent using the ratio test. In the proof, we will use the following property for the Gamma function

$$\lim_{n\to\infty} \frac{\Gamma(n + \alpha)}{\Gamma(n) n^\alpha} = 1$$

where $\alpha \in \mathbb{R}$. Then

$$\lim_{n\to\infty} \frac{\Gamma(t + (n + 1)(\nu - 1) + 1)}{(t + n(\nu - 1) + 1)^{\nu-1} \Gamma(t + n(\nu - 1) + 1)} = 1$$

and

$$\lim_{n\to\infty}\frac{(n\nu-1+\nu)^{\nu}\Gamma((n+1)\nu-1)}{\Gamma((n+2)\nu-1)}=1.$$

Define $a_n=\lambda^n\dfrac{(t+n(\nu-1))^{((n+1)\nu-2)}}{\Gamma((n+1)\nu-1)}$. Then we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\lambda\frac{\Gamma(t+3-n-\nu)}{\Gamma(t+2-n-\nu)}\frac{\Gamma(t+(n+1)(\nu-1)+1)}{\Gamma(t+n(\nu-1)+1)}\frac{\Gamma((n+1)\nu-1)}{\Gamma((n+2)\nu-1)}\right| \\ &= \lim_{n\to\infty}\left|\lambda(t+2-n-\nu)\frac{\Gamma(t+(n+1)(\nu-1)+1)}{\Gamma(t+n(\nu-1)+1)}\frac{\Gamma((n+1)\nu-1)}{\Gamma((n+2)\nu-1)}\right| \\ &= \lim_{n\to\infty}|\lambda(t+2-n-\nu)\frac{(t+n(\nu-1)+1)^{\nu-1}}{(n\nu-1+\nu)^{\nu}} \\ &\quad \frac{\Gamma(t+(n+1)(\nu-1)+1)}{(t+n(\nu-1)+1)^{\nu-1}\Gamma(t+n(\nu-1)+1)}\frac{(n\nu-1+\nu)^{\nu}\Gamma((n+1)\nu-1)}{\Gamma((n+2)\nu-1)}| \\ &= \lim_{n\to\infty}\left|\lambda(t+2-n-\nu)\frac{(t+n(\nu-1)+1)^{\nu-1}}{(n\nu-1+\nu)^{\nu}}\right| = |\lambda|\frac{1}{\nu-1}(\frac{\nu-1}{\nu})^{\nu}<1.\end{aligned}$$

(6) We prove that $E_1(t,\lambda,\nu)$ solves the linear fractional difference equation

$$\Delta^{\nu}y(t)=\lambda y(t+\nu-1).$$

First note that

$$\Delta_{\nu-2}^{-(2-\nu)}(t+n(\nu-1))^{((n+1)\nu-2)}=\Delta_{\nu-2+n}^{-(2-\nu)}(t+n(\nu-1))^{((n+1)\nu-2)}$$

for $n=0,1,2,\ldots$. Hence we have

$$\begin{aligned}\Delta^{\nu}E_1(t,\lambda,\nu) &= \Delta^2\Delta_{\nu-2}^{-(2-\nu)}E_1(t,\lambda,\nu) \\ &= \Delta^2\Delta_{\nu-2}^{-(2-\nu)}\sum_{n=0}^{\infty}\lambda^n\frac{(t+n(\nu-1))^{((n+1)\nu-2)}}{\Gamma((n+1)\nu-1)} \\ &= \Delta^2\sum_{n=0}^{\infty}\lambda^n\frac{\Delta_{\nu-2}^{-(2-\nu)}(t+n(\nu-1))^{((n+1)\nu-2)}}{\Gamma((n+1)\nu-1)} \\ &= \Delta^2\sum_{n=0}^{\infty}\lambda^n\frac{\Delta_{\nu-2+n}^{-(2-\nu)}(t+n(\nu-1))^{((n+1)\nu-2)}}{\Gamma((n+1)\nu-1)} \\ &= \Delta^2\sum_{n=0}^{\infty}\lambda^n\frac{\Gamma((n+1)\nu-1)}{\Gamma(n\nu+1)\Gamma((n+1)\nu-1)}(t+n(\nu-1))^{(n\nu)} \\ &= \lambda\sum_{n=0}^{\infty}\lambda^n\frac{(t+\nu-1+n(\nu-1))^{((n+1)\nu-2)}}{\Gamma((n+1)\nu-1)} \\ &= \lambda E_1(t+\nu-1,\lambda,\nu),\end{aligned}$$

where we used the power rule in the following way.

$$\Delta_{\nu-2+n}^{-(2-\nu)}(t+n(\nu-1))^{((n+1)\nu-2)}=\sum_{s=\nu-2+n}^{t-2+\nu}\frac{(t-\sigma(s))^{(1-\nu)}}{\Gamma(2-\nu)}(s+n(\nu-1))^{((n+1)\nu-2)}.$$

Let's call $u = s + n(\nu - 1)$. Then we have

$$\sum_{s=\nu-2+n}^{t-2+\nu} \frac{(t-\sigma(s))^{(1-\nu)}}{\Gamma(2-\nu)}(s+n(\nu-1))^{((n+1)\nu-2)}$$
$$= \sum_{u=(n+1)\nu-2}^{t+n(\nu-1)-2+\nu} \frac{(t+n(\nu-1)-\sigma(u))^{(1-\nu)}}{\Gamma(2-\nu)} u^{((n+1)\nu-2)}$$
$$= \Delta_{(n+1)\nu-2}^{-(2-\nu)} \tau^{((n+1)\nu-2)}$$
$$= \frac{\Gamma((n+1)\nu-1)}{\Gamma(n\nu+1)} \tau^{(n\nu)},$$

where $\tau = t + n(\nu - 1)$. □

Remark 3.3. We want to point out that the functions E_1 and E_2 define finite sums in the domain we choose to work here. Indeed,

$$E_1(\nu-2+k,\lambda,\nu) = \sum_{n=0}^{k} \lambda^n \frac{(t+n(\nu-1))^{((n+1)\nu-2)}}{\Gamma((n+1)\nu-1)}\Big|_{t=\nu-2+k} \text{ for } k = 0,1,2,\ldots,$$

$$E_2(\nu-2+k,\lambda,\nu) = \sum_{n=0}^{k} \lambda^n \frac{(t+n(\nu-1))^{((n+1)\nu-1)}}{\Gamma((n+1)\nu)}\Big|_{t=\nu-2+k} \text{ for } k = 0,1,2,\ldots.$$

Hence, we do not require $|\lambda|\frac{1}{(\nu-1)^{1-\nu}}\frac{1}{\nu^\nu} < 1$ to construct a solution of (3.1).

Remark 3.4. In the paper [2], we considered a fractional equation of order ν where $\nu \in (0, 1]$. In particular, we employed the method of successive approximations and constructed the solution of the initial value problem

$$\Delta^\nu y(t) = \lambda y(t+\nu-1), \quad t = 0,1,2,\ldots, \tag{3.4}$$

$$\Delta^{\nu-1} y(t)|_{t=0} = a_0; \tag{3.5}$$

we obtained a formal solution in terms of the so-called discrete Mittag Leffler function,

$$y(t) = a_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma((n+1)\nu)} (t+n(\nu-1))^{((n+1)\nu-1)}, \quad t \in \mathbb{N}_{\nu-1}. \tag{3.6}$$

To address convergence of (3.6), apply the ratio test, precisely as in the proof of Lemma 3.5 and obtain that (3.6) converges if

$$\lambda \frac{1}{(1-\nu)^{1-\nu}} \frac{1}{\nu^\nu} < 1. \tag{3.7}$$

Note that $1 \le \frac{1}{(1-\nu)^{1-\nu}}\frac{1}{\nu^\nu} \le 2$ for $0 < \nu \le 1$, and so, (3.7) makes sense. Again, if $t \in \mathbb{N}_{\nu-1}$, the sum in (3.6) is a finite sum.

Theorem 3.1. *Assume $\lambda \in \mathbb{R}$. The fractional difference equation (3.1) has the general solution*

$$y(t) = AE_1(t,\lambda,\nu) + BE_2(t,\lambda,\nu) - (E_2 *_{\nu-2} h)(t-1), \quad t \in \mathbb{N}_{\nu-2}, \tag{3.8}$$

where A and B are constants.

Proof. Apply R_0 to each side of (3.1) and employ Lemma 2.1 and Lemma 2.3 to obtain

$$-(s^{\nu}R_{\nu-2}(y(t))(s) - s\Delta^{\nu-2}y|_{t=0} - \Delta^{\nu-1}y|_{t=0}) + \lambda(s+1)^{\nu-1}R_{\nu-2}(y(t))(s) \\ - \lambda y(\nu-2) = (s+1)^{\nu-2}R_{\nu-2}(h(t))(s).$$

Rearrange terms and obtain

$$(s^{\nu} - \lambda(s+1)^{\nu-1})R_{\nu-2}(y(t))(s) = sA + B - (s+1)^{\nu-2}R_{\nu-2}(h(t))(s),$$

where $A = \Delta^{\nu-2}y|_{t=0}$ and $B = \Delta^{\nu-1}y|_{t=0} - \lambda y(\nu-2)$. This implies that

$$R_{\nu-2}(y(t))(s) = \frac{A}{s^{\nu-1}(1-\lambda s^{-\nu}(s+1)^{\nu-1})} + \frac{B-(s+1)^{\nu-2}R_{\nu-2}(h(t))(s)}{s^{\nu}(1-\lambda s^{-\nu}(s+1)^{\nu-1})}.$$

Next we show that

$$\frac{1}{s^{\nu-1}(1-\lambda s^{-\nu}(s+1)^{\nu-1})} = R_{\nu-2}\left(\sum_{n=0}^{\infty}\lambda^n\frac{(t+n(\nu-1))^{((n+1)\nu-2)}}{\Gamma((n+1)\nu-1)}\right).$$

In fact, we have the following series expansion

$$\frac{1}{s^{\nu-1}(1-\lambda s^{-\nu}(s+1)^{\nu-1})} = \frac{1}{s^{\nu-1}} + \frac{\lambda(s+1)^{\nu-1}}{s^{2\nu-1}} + \frac{\lambda^2(s+1)^{2\nu-2}}{s^{3\nu-1}} \\ + \cdots + \frac{\lambda^n(s+1)^{n\nu-n}}{s^{(n+1)\nu-1}} + \cdots.$$

Employ Lemma 2.1 to see that

$$\sum_{n=0}^{\infty}\lambda^n\frac{(s+1)^{n\nu-n}}{s^{(n+1)\nu-1}} = \sum_{n=0}^{\infty}(s+1)^{n\nu-n}\lambda^n R_{(n+1)\nu-2}\left(\frac{t^{((n+1)\nu-2)}}{\Gamma((n+1)\nu-1)}\right)(s) \\ = \sum_{n=0}^{\infty}\lambda^n R_{\nu-2+n}\left(\frac{(t+n(\nu-1))^{((n+1)\nu-2)}}{\Gamma((n+1)\nu-1)}\right)(s).$$

Due to the convention, $t^{(\mu)} = 0$ if $t+1-\mu \in \{0,-1,\ldots,-k,\ldots\}$, it is the case that

$$(t+n(\nu-1))^{(n+1)\nu-2)}|_{t=\nu-2+l} = 0, \text{ if } l = 0,\ldots,n-1,$$

and,

$$\sum_{n=0}^{\infty}\lambda^n\frac{(s+1)^{n\nu-n}}{s^{(n+1)\nu-1}} = \sum_{n=0}^{\infty}\lambda^n R_{\nu-2}\left(\frac{(t+n(\nu-1))^{((n+1)\nu-2)}}{\Gamma((n+1)\nu-1)}\right)(s) \\ = R_{\nu-2}\left(\sum_{n=0}^{\infty}\lambda^n\frac{(t+n(\nu-1))^{((n+1)\nu-2)}}{\Gamma((n+1)\nu-1)}\right)(s).$$

In a similar way, we show that

$$\frac{1}{s^{\nu}(1-\lambda s^{-\nu}(s+1)^{\nu-1})} = R_{\nu-2}\left(\sum_{n=0}^{\infty}\lambda^n\frac{(t+n(\nu-1))^{((n+1)\nu-1)}}{\Gamma((n+1)\nu)}\right)(s).$$

Thus, with the help of Lemma 2.4, we have

$$R_{\nu-2}y(t) = AR_{\nu-2}\left(\sum_{n=0}^{\infty}\lambda^n\frac{(t+n(\nu-1))^{((n+1)\nu-2)}}{\Gamma((n+1)\nu-1)}\right)(s)$$
$$+ BR_{\nu-2}\left(\sum_{n=0}^{\infty}\lambda^n\frac{(t+n(\nu-1))^{((n+1)\nu-1)}}{\Gamma((n+1)\nu)}\right)(s)$$
$$-(s+1)^{-1}R_{\nu-2}\left\{\sum_{n=0}^{\infty}\lambda^n\frac{(t+n(\nu-1))^{((n+1)\nu-1)}}{\Gamma((n+1)\nu)} *_{\nu-2} h(t)\right\}(s).$$

We argue that

$$(s+1)^{-1}R_{\nu-2}\left\{\sum_{n=0}^{\infty}\lambda^n\frac{(t+n(\nu-1))^{((n+1)\nu-1)}}{\Gamma((n+1)\nu)} *_{\nu-2} h(t)\right\}(s)$$
$$= R_{\nu-2}((E_2 *_{\nu-2} h)(t-1))(s)$$

and to do so, recall the domain $\mathbb{Z}_{\nu-2}$.

$$(s+1)^{-1}R_{\nu-2}\{\sum_{n=0}^{\infty}\lambda^n\frac{(t+n(\nu-1))^{((n+1)\nu-1)}}{\Gamma((n+1)\nu)} *_{\nu-2} h(t)\}(s)$$
$$= (s+1)^{-1}\sum_{t=\nu-2}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}(E_2 *_{\nu-2} h)(t)$$
$$= \sum_{t=\nu-1}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}(E_2 *_{\nu-2} h)(t-1).$$

Note that

$$(E_2 *_{\nu-2} h)(t-1)|_{t=\nu-2} = -E_2(\nu-3)h(\nu-2) - E_2(\nu-2)h(\nu-3) = 0.$$

Thus,

$$\sum_{t=\nu-1}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}(E_2 *_{\nu-2} h)(t-1) = \sum_{t=\nu-2}^{\infty}(\frac{1}{s+1})^{t+1}(E_2 *_{\nu-2} h)(t-1)$$
$$= R_{\nu-2}((E_2 *_{\nu-2} h)(t-1))(s).$$

Apply to each side the inverse of $R_{\nu-2}$ to obtain (3.8). □

Remark 3.5. A direct substitution gives that $-(E_2 *_{\nu-2} h)(t-1)$ is a particular solution of the equation

$$-\Delta^\nu y(t) + \lambda y(t+\nu-1) = h(t+\nu-2).$$

Proof. We show that

$$\Delta^\nu \sum_{s=\nu-2}^{t-1} E_2(t-s+\nu-3)h(s) = \lambda \sum_{s=\nu-2}^{t+\nu-2} E_2(t+\nu-1-s+\nu-3)h(s) + h(t+\nu-2),$$

where we drop λ and ν when we write $E_2(t,\lambda,\nu)$.

Using the definition of the fractional difference operator we have

$$\begin{aligned}
\Delta^\nu \sum_{s=\nu-2}^{t-1} E_2(t-s+\nu-3)h(s) &= \Delta^2\Delta^{-(2-\nu)} \sum_{s=\nu-2}^{t-1} E_2(t-s+\nu-3)h(s) \\
&= \Delta^2 \sum_{u=\nu-2}^{t+\nu-2} \frac{(t-\sigma(u))^{(1-\nu)}}{\Gamma(2-\nu)} \sum_{s=\nu-2}^{u-1} E_2(u-s+\nu-3)h(s) \\
&= \Delta^2 \sum_{u=\nu-2}^{t+\nu-2} \frac{(t-\sigma(u))^{(1-\nu)}}{\Gamma(2-\nu)} \sum_{s=\nu-2}^{u-1} E_2(u-s+\nu-3)h(s) \\
&= I.
\end{aligned}$$

Next we interchange the order of sums and obtain

$$I = \Delta^2 \sum_{s=\nu-2}^{t+\nu-3} \sum_{u=s+1}^{t+\nu-2} \frac{(t-\sigma(u))^{(1-\nu)}}{\Gamma(2-\nu)} E_2(u-s+\nu-3)h(s).$$

Next we apply the following rule to the above expression

$$\Delta \sum_{s=\nu-2}^{t+\nu-3} f(t,s) = \sum_{s=\nu-2}^{t+\nu-3} \Delta f(t,s) + f(t+1, t+\nu-2). \tag{3.9}$$

Hence, we have

$$\begin{aligned}
I &= \Delta \sum_{s=\nu-2}^{t+\nu-3} \Delta \sum_{u=s+1}^{t+\nu-2} \frac{(t-\sigma(u))^{(1-\nu)}}{\Gamma(2-\nu)} E_2(u-s+\nu-3)h(s) \\
&+ \sum_{u=s+1}^{t+\nu-2} \frac{(t-\sigma(u))^{(1-\nu)}}{\Gamma(2-\nu)} E_2(u-s+\nu-3)h(s)|_{t\to t+1, s\to t+\nu-2} \\
&= \Delta \sum_{s=\nu-2}^{t+\nu-3} \Delta \sum_{u=s+1}^{t+\nu-2} \frac{(t-\sigma(u))^{(1-\nu)}}{\Gamma(2-\nu)} E_2(u-s+\nu-3)h(s) \\
&+ \sum_{u=t+\nu-1}^{t+\nu-1} \frac{(t-u)^{(1-\nu)}}{\Gamma(2-\nu)} E_2(u-t-1)h(t+\nu-2) \\
&= \Delta \sum_{s=\nu-2}^{t+\nu-3} \Delta \sum_{u=s+1}^{t+\nu-2} \frac{(t-\sigma(u))^{(1-\nu)}}{\Gamma(2-\nu)} E_2(u-s+\nu-3)h(s),
\end{aligned}$$

since $E_2(\nu-2,\lambda,\nu)=0$.

Using the formula in (3.9) again, we have

$$\begin{aligned}
I = &\sum_{s=\nu-2}^{t+\nu-3} \Delta^2 \sum_{u=s+1}^{t+\nu-2} \frac{(t-\sigma(u))^{(1-\nu)}}{\Gamma(2-\nu)} E_2(u-s+\nu-3)h(s) \\
&+ \Delta \sum_{u=s+1}^{t+\nu-2} \frac{(t-\sigma(u))^{(1-\nu)}}{\Gamma(2-\nu)} E_2(u-s+\nu-3)h(s)|_{t\to t+1, s\to t+\nu-2}
\end{aligned}$$

$$= \sum_{s=\nu-2}^{t+\nu-3} \Delta^2 \sum_{u=s+1}^{t+\nu-2} \frac{(t-\sigma(u))^{(1-\nu)}}{\Gamma(2-\nu)} E_2(u-s+\nu-3)h(s)$$

$$+ \sum_{u=t+\nu-1}^{t+\nu} E_2(\nu-1)h(t-\nu-2)$$

$$= \sum_{s=\nu-2}^{t+\nu-3} \Delta^2 \sum_{u=s+1}^{t+\nu-2} \frac{(t-\sigma(u))^{(1-\nu)}}{\Gamma(2-\nu)} E_2(u-s+\nu-3)h(s) + h(t-\nu-2),$$

$E_2(\nu-1,\lambda,\nu)=1$.

Next we use the substitution $u-s+\nu-3=\tau$, we obtain

$$\sum_{u=s+1}^{t+\nu-2} \frac{(t-\sigma(u))^{(1-\nu)}}{\Gamma(2-\nu)} E_2(u-s+\nu-3)$$

$$= \sum_{\tau=\nu-2}^{t-(2-\nu)-s+\nu-3} \frac{(t-(\tau+s-\nu+3+1)^{(1-\nu)}}{\Gamma(2-\nu)} E_2(\tau)$$

$$= \Delta_{\nu-2}^{-(2-\nu)} E_2(t-s+\nu-3).$$

Thus,

$$I = \sum_{s=\nu-2}^{t+\nu-3} \Delta^2 \Delta_{\nu-2}^{-(2-\nu)} E_2(t-s+\nu-3)h(s) + h(t+\nu-2)$$

$$= \lambda \sum_{s=\nu-2}^{t+\nu-2} E_2(t+\nu-1-s+\nu-3)h(s) + h(t+\nu-2),$$

where we used Lemma 3.5 (7). □

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