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EXISTENCE AND UNIQUENESS RESULTS FOR POSITIVE SOLUTIONS OF A NONLINEAR FRACTIONAL DIFFERENCE EQUATION

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ABSTRACT. In this paper we are concerned with the fractional self-adjoint equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) + q(t+\mu-1)x(t+\mu-1) = h(t), \quad t \in \mathbb{N}_0, \tag{0.1}$$

where $0 < \mu \leq 1$, $p : \mathbb{N}_{\mu-1} \to (0,\infty)$, $q : \mathbb{N}_{\mu-1} \to [0,\infty)$, $h : \mathbb{N}_0 \to \mathbb{R}$. Our sole reason for calling this equation self-adjoint is that when $\mu = 1$ we get the well-studied second order self-adjoint difference equation [16, Chapters 6-9]. We will prove various results concerning the existence and uniqueness of positive solutions of the nonlinear fractional equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) + F(t, x(t+\mu-1)) = 0, \quad t \in \mathbb{N}_0, \tag{0.2}$$

where $0 < \mu \leq 1$, $p : \mathbb{N}_{\mu-1} \to (0,\infty)$, and $F : \mathbb{N}_0 \times \mathbb{R} \to [0,\infty)$ by applying the Contraction Mapping Theorem. We also give examples illustrating our main results.

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1. INTRODUCTION

We will obtain qualitative and quantitative properties of solutions of certain fractional difference equations. We refer the reader to the related results in [1] and [2]. In particular, we would like to compare some of our results to the following theorem.

Theorem 1.1 (Awasthi [2]). *Let $0 < \mu \leq 1$, $p : \mathbb{N}_{\mu-1} \to \mathbb{R}$, $f : \mathbb{N}_0 \to \mathbb{R}$, $q : \mathbb{N}_{\mu-1} \to \mathbb{R}$ and assume*

1. $p(t) > 0$ *and* $q(t) \geq 0$, *for all* $t \in \mathbb{N}_{\mu-1}$
2. $\displaystyle\sum_{\tau=\mu}^{\infty} \frac{1}{p(\tau)} < \infty$
3. $\displaystyle\sum_{\tau=0}^{\infty} f(\tau) < \infty$
4. $\displaystyle\sum_{\tau=\mu}^{\infty} \frac{1}{p(\tau)} \left(\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} q(s+\mu-1) \right) < \infty$

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hold; then the forced self-adjoint fractional difference equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) + q(t+\mu-1)x(t+\mu-1) = f(t), \quad t \in \mathbb{N}_0$$

has a solution x which satisfies $\lim_{t\to\infty} x(t) = 0$.

We now introduce a few basic definitions and lemmas which will be helpful for the reader to understand the material in this paper.

Definition 1.2. If $a \in \mathbb{R}$ and $t \in \mathbb{N}_a := \{a, a+1, \dots\}$, the forward jump operator is denoted by $\sigma(t)$ and is defined by $\sigma(t) = t+1$.

Definition 1.3 ([16])**.** We define the falling function in terms of the gamma function by

$$t^{\underline{\mu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)}$$

whenever the right hand side is defined. Furthermore, we use the standard convention that $t^{\underline{\mu}} = 0$ when $t+1-\mu$ is a nonpositive integer and $t+1$ is not a nonpositive integer.

Definition 1.4 ([13])**.** Let $N \in \mathbb{N}$, $a \in \mathbb{R}$ and $\mu > 0$, we define the μ-th order fractional sum of a function f which is defined on $\mathbb{N}_a := \{a, a+1, \dots\}$ by

$$\Delta_a^{-\mu} f(t) := \begin{cases} \frac{1}{\Gamma(\mu)} \sum_{s=a}^{t-\mu} (t-\sigma(s))^{\underline{\mu-1}} f(s), & N-1 < \mu < N \\ \Delta^N f(t), & \mu = N, \end{cases}$$

where $t \in \{a+\mu, a+\mu+1, \dots\} =: \mathbb{N}_{a+\mu}$. We also define the μ-th order fractional difference with $\mu > 0$ and $N \in \mathbb{N}$ by

$$\Delta_a^{\mu} f(t) := \begin{cases} \frac{1}{\Gamma(-\mu)} \sum_{s=a}^{t+\mu} (t-\sigma(s))^{\underline{-\mu-1}} f(s), & N-1 < \mu < N \\ \Delta^N f(t), & \mu = N, \end{cases}$$

where $t \in \mathbb{N}_{a+N-\mu}$.

Remark 1.5. If $f : \mathbb{N}_a \to \mathbb{R}$ and $c, d \in \mathbb{N}_a$, we use the standard convention that

$$\sum_{t=c}^{d} f(t) = 0$$

whenever $d < c$.

Lemma 1.6 ([13])**.** *If $t \in \mathbb{N}_0$ and $N-1 < \mu \le N$, we have that*

$$\Delta_0^{-\mu}\Delta^{\mu}_{\mu-N}\, y(t) = y(t) + \sum_{i=0}^{N-1} C_i\, t^{\underline{i+\mu-N}},$$

for some constants C_i, $0 \le i \le N-1$.

Lemma 1.7 ([3]). *If we let* $\Delta_s f(t,s) := f(t,s+1) - f(t,s)$, *then*

$$\Delta_s (t-s)^{\underline{\mu}} = -\mu(t-\sigma(s))^{\underline{\mu-1}}.$$

Lemma 1.8 ([14]). *If* $t \in \mathbb{N}_a$ *and* $h : \mathbb{N}_a \to \mathbb{R}$, *then the general solution to the equation*

$$\Delta^{\mu}_{a+\mu-N}\ y(t) = h(t)$$

is given by,

$$y(t) = \sum_{i=0}^{N-1} c_i\ (t-a)^{\underline{i+\mu-N}} + \Delta_a^{-\mu}\ h(t), \quad t \in \mathbb{N}_{a+\mu-N} \tag{1.1}$$

where c_i, $0 \le i \le N-1$, *are arbitrary constants.*

For papers related to the results in this paper and other references to papers on discrete fractional calculus see the papers by Goodrich, [4], [5], [6], [7], [8], [9], [10], [11], and [12].

Definition 1.9 (Contraction Mapping). If (ζ, d) and $(\xi, d^{'})$ are metric spaces then a map $T : \zeta \to \xi$ is called a contraction mapping if there exist a non-negative real number $0 \le k < 1$ such that

$$d^{'}(T(x), T(y)) \le kd(x,y).$$

We will use the following well-known result in this paper.

Theorem 1.10 (Contraction Mapping Theorem). *If* (ζ, d) *is a complete metric space and* $T : \zeta \to \zeta$ *is a contraction mapping then* T *has a unique fixed point in* ζ.

2. EXISTENCE OF SOLUTIONS WITH POSITIVE HORIZONTAL ASYMPTOTES

In this section we will give conditions under which the following nonlinear fractional equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) + F(t, x(t+\mu-1)) = 0, \quad t \in \mathbb{N}_0$$

has a solution with a positive limit as t goes to ∞. The following theorem relates the above fractional equation to a summation equation.

Theorem 2.1. *Let* $p : \mathbb{N}_{\mu-1} \to (0,\infty)$ *and* $F : \mathbb{N}_0 \times \mathbb{R} \to [0,\infty)$. *Let* $M > 0$ *and define*

$$\zeta_M = \Big\{ x : \mathbb{N}_{\mu-1} \to [M,\infty) : \Delta x(t) \le 0, \Delta x(\mu-1) = 0 \Big\}.$$

Suppose for all the functions x *defined on* $\mathbb{N}_{\mu-1}$, *the following series*

$$\sum_{\tau=\mu-1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right]$$

is convergent. Then the fractional equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) + F(t, x(t+\mu-1)) = 0 \tag{2.1}$$

has a positive solution $x \in \zeta_M$ *such that* $\lim_{t\to\infty} x(t) = M$ *if and only if the summation equation*

$$x(t) = M + \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right] \tag{2.2}$$

has a solution x *on* $\mathbb{N}_{\mu-1}$.

Proof. Suppose the fractional equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x(t)) + F(t, x(t+\mu-1)) = 0 \tag{2.3}$$

has a positive solution $x \in \zeta_M$ such that $\lim_{t\to\infty} x(t) = M$. First we let $y(t) = (p\Delta x)(t)$. Then applying the fractional sum operator on both sides of equation (2.3) and using the fractional composition rule given in Lemma 1.6 we get that

$$\begin{aligned} y(t) &= -\Delta_0^{-\mu} F(t, x(t+\mu-1)) + ct^{\underline{\mu-1}} \\ &= -\sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) + ct^{\underline{\mu-1}}. \end{aligned}$$

It follows that

$$\Delta x(t) = -\frac{1}{p(t)} \left[\sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right].$$

Now summing from $\tau = t$ to ∞ we get that

$$M - x(t) = -\sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right] + ct^{\underline{\mu-1}}.$$

Note that $\Delta x(\mu-1) = 0$ implies $c = 0$. Hence,

$$x(t) = M + \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right].$$

Thus x is a solution to the summation equation (2.2).

On the other hand, if the summation equation given by (2.2) has a solution x on $\mathbb{N}_{\mu-1}$, then

$$x(t) = M + \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right]. \tag{2.4}$$

Now by taking the delta difference on both sides of the last equation, we get that

$$\Delta x(t) = -\frac{1}{p(t)} \left[\sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right]. \tag{2.5}$$

Hence,

$$(p\Delta x)(t) = -\left[\Delta_0^{-\mu} F(\cdot, x(\cdot - \mu + 1)\right](t).$$

Taking the fractional difference of both sides of the last equation and using a particular case of composition rules as proved in [13], we get that

$$\begin{aligned}\Delta_{\mu-1}^{\mu}(p\Delta x)(t) &= -\Delta_{\mu-1}^{\mu}\Delta_0^{-\mu}(F(\cdot, x(\cdot + \mu - 1))(t)\\ &= -F(t, x(t + \mu - 1)), \quad t \in \mathbb{N}_0.\end{aligned}$$

This implies that

$$\Delta_{\mu-1}^{\mu}(p\Delta x)(t) + F(t, x(t + \mu - 1)) = 0.$$

Hence x is a solution of the fractional equation (2.1). We also observe that $x(t) \geq M$, since $p(t) > 0$ for all $t \in \mathbb{N}_{\mu-1}$ and $F(t, u) \geq 0$ for all $(t, u) \in \mathbb{N}_0 \times \mathbb{R}$. Moreover, notice that

$$\Delta x(\mu - 1) = -\frac{1}{p(\mu - 1)}\left[\sum_{s=0}^{-1} \frac{(\mu - 1 - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s + \mu - 1))\right] = 0$$

by the convention given in Remark 1.5. From the expression for $\Delta x(t)$ given by equation (2.5), we see that $\Delta x(t) \leq 0$ for all $t \in \mathbb{N}_{\mu-1}$. Thus $x \in \zeta_M$. Furthermore, since the series

$$\sum_{\tau=t}^{\infty} \frac{1}{p(\tau)}\left[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s + \mu - 1))\right]$$

is convergent it follows from equation (2.4) that $\lim_{t\to\infty} x(t) = M$. This completes the proof. $\square$

Remark 2.2. If $\mu = 1$ (non-fractional case) in the preceding theorem, the fractional summation equation reduces to the summation equation

$$x(t) = M + \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)}\left[\sum_{s=0}^{\tau-1} F(s, x(s))\right].$$

The proof of the following lemma is straight forward and left to the reader.

Lemma 2.3. *Assume $M > 0$ and*

$$\zeta_M = \left\{x : \mathbb{N}_{\mu-1} \to [M, \infty) : \Delta x(t) \leq 0, \Delta x(\mu - 1) = 0\right\},$$

where $F : \mathbb{N}_0 \times \mathbb{R} \to [0, \infty)$. Assume $p : \mathbb{N}_{\mu-1} \to (0, \infty)$ satisfies $\sum_{\tau=\mu-1}^{\infty} \ln(1 + \frac{1}{p(\tau)}) < \infty$ and define $d : \zeta_M \times \zeta_M \to [0, \infty)$ by $d(x, y) = \sup_{t\in\mathbb{N}_{\mu-1}} \frac{|x(t) - y(t)|}{w(t)}$, where $w(t) = e^{-[\sum_{\tau=\mu-1}^{t} ln(1+\frac{1}{p(\tau)})]}$. Note that $0 < L := \lim_{t\to\infty} w(t) \leq 1$. Then the pair (ζ_M, d) is a complete metric space.

Next we prove a theorem regarding the existence and uniqueness of a positive solution of the fractional equation (2.1) tending to M as t goes to ∞ by applying the Contraction Mapping Theorem.

Theorem 2.4. *Assume* $F : N_0 \times \mathbb{R} \to [0,\infty)$ *satisfies a uniform Lipschitz condition with respect to its second variable, i.e. if* $u, v \in \mathbb{R}$ *and* $t \in \mathbb{N}_0$ *then* $\big|F(t,u) - F(t,v)\big| \leq K\big|u - v\big|$ *and assume* $p : \mathbb{N}_{\mu-1} \to (0,\infty)$ *and let* (ζ_M, d) *be the complete metric space as defined in Lemma* 2.3. *Assume the following hypotheses* $(H1)$ *and* $(H2)$:

(H1) The series $\displaystyle\sum_{\tau=\mu}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1))\right]$ *is convergent for each* $x \in \zeta_M$,

(H2) $\frac{1}{\Gamma(\mu+1)} \frac{K}{L} \left[\displaystyle\sum_{\tau=\mu}^{\infty} \frac{\tau^{\underline{\mu}}}{p(t)}\right] = \alpha < 1$

are satisfied. Then there exists a unique positive solution of the fractional equation (2.1). *Moreover* $\lim_{t\to\infty} x(t) = M$.

Proof. Let (ξ_M, d) be the complete metric space as defined in Lemma 2.3. Consider the map T on ζ_M defined by

$$Tx(t) = M + \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1))\right].$$

First we will show that $T : \zeta_M \to \zeta_M$. Note that the above expression for $Tx(t)$ guarantees that $Tx(t) \geq M$ since $F(t,u) \geq 0$ for all $(t,u) \in N_0 \times \mathbb{R}$ and $p(t) > 0$ for all $t \in \mathbb{N}_{\mu-1}$. Next note that

$$\Delta Tx(t) = -\frac{1}{p(t)} \left[\sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}} F(s, x(s+\mu-1))}{\Gamma(\mu)}\right] \leq 0.$$

Also it can be easily verified that $(\Delta Tx)(\mu-1) = 0$ by our convention as mentioned in Remark 2.2. Thus $T : \xi_M \to \xi_M$. Moreover we will show that T is a contraction mapping on ζ_M. Let $t \in \mathbb{N}_{\mu-1}$ be arbitrary, then

$$\begin{aligned}\left|\frac{Tx(t)-Ty(t)}{w(t)}\right| &= \frac{1}{w(t)} \sum_{\tau=t}^{\infty} \frac{1}{p(t)} \Big[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} |F(s, x(s+\mu-1)) \\ &\quad - F(s, y(s+\mu-1)|\Big]. \\ &\leq \frac{K}{w(t)} \sum_{\tau=t}^{\infty} \frac{1}{p(t)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} w(s+\mu-1)\right] d(x,y).\end{aligned}$$

$$d(T(x), T(y)) \leq \frac{K}{L} \sum_{\tau=t}^{\infty} \frac{1}{p(t)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} w(s+\mu-1)\right] d(x,y)$$

$$\le \frac{K}{L}\sum_{\tau=t}^{\infty}\frac{1}{p(t)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}\right]d(x,y)$$
$$=\frac{K}{L}\sum_{\tau=t}^{\infty}\frac{1}{p(t)}\left[\frac{\tau^{\underline{\mu}}}{\Gamma(\mu+1)}\right]d(x,y)$$
$$\le \frac{1}{\Gamma(\mu+1)}\frac{K}{L}\left[\sum_{\tau=\mu-1}^{\infty}\frac{\tau^{\underline{\mu}}}{p(t)}\right]d(x,y)$$
$$=\alpha\, d(x,y).$$

Since $\alpha < 1$, T is a contraction mapping on ζ_M. Then it follows from the Contraction Mapping Theorem that there exists a unique fixed point x of T in ζ_M such that $T(x) = x$ and therefore x is the unique positive solution to the summation equation (2.2). Hence, by Theorem 2.1, x is the unique positive solution to the fractional equation (2.1). Moreover, since the series

$$\sum_{\tau=\mu-1}^{\infty}\frac{1}{p(\tau)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\right]$$

is convergent,

$$\lim_{t\to\infty}x(t) = M + \lim_{t\to\infty}\sum_{\tau=t}^{\infty}\frac{1}{p(\tau)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\right]$$
$$= M.$$

This completes the proof. □

Next we give an example to illustrate the above theorem.

Example 2.5. As an example let us choose $F(t, x) = K|x| - M$ for $t \in \mathbb{N}_0$. Then F clearly satisfies a uniform Lipschitz condition with respect to the second variable x with Lipschitz constant K. Let $p : \mathbb{N}_{\mu-1} \to (0,\infty)$ be defined by

$$p(t) = \begin{cases} 4, & t = \mu - 1 \\ \frac{K2^{(t-\mu+3)}t^{\underline{\mu}}}{\Gamma(\mu+1)L}, & t \in \mathbb{N}_{\mu}. \end{cases} \tag{2.6}$$

We claim all the hypotheses in Theorem 2.4 hold. First notice that

$$\sum_{\tau=\mu-1}^{\infty}\frac{1}{p(t)} = \frac{1}{4} + \sum_{\tau=\mu}^{\infty}\frac{1}{p(t)}$$
$$= \frac{1}{4} + \frac{L}{K}\sum_{\tau=\mu}^{\infty}\frac{\Gamma(\mu+1)}{2^{(t-\mu+3)}t^{\underline{\mu}}}$$
$$\le \frac{1}{4} + \frac{L}{K}\sum_{\tau=\mu}^{\infty}\frac{1}{2^{(t-\mu+3)}}$$

$$
\begin{aligned}
&= \frac{1}{4} + \frac{L}{4K} \\
&< \infty.
\end{aligned}
$$

Since $\ln\left(1 + \frac{1}{p(t)}\right) \leq \frac{1}{p(t)}$, for $t \in \mathbb{N}_\mu$, we have by the comparison theorem that

$$
\sum_{\tau=\mu-1}^{\infty} \ln\left(1 + \frac{1}{p(t)}\right) < \infty.
$$

Next we will show that $(H1)$ is satisfied. Let $x \in \zeta_M$ be arbitrary but fixed. The following calculations show $(H1)$ is satisfied for each such $x \in \zeta_M$:

$$
\begin{aligned}
&\sum_{\tau=\mu-1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s + \mu - 1))\right] \\
&= \sum_{\tau=\mu}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s + \mu - 1))\right] \\
&\leq K|x(\mu - 1)| \sum_{\tau=\mu}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}\right] \\
&= \frac{1}{4} + Kx(\mu - 1) \sum_{\tau=\mu}^{\infty} \frac{1}{p(\tau)} \frac{\tau^{\underline{\mu}}}{\Gamma(\mu + 1)} \\
&= \frac{1}{4} + Kx(\mu - 1)(\frac{1}{4}) \\
&< \infty.
\end{aligned}
$$

Next we will show that the second hypothesis (H2) is also satisfied. Notice that

$$
\begin{aligned}
\frac{1}{\Gamma(\mu + 1)} \frac{K}{L} \left[\sum_{\tau=\mu-1}^{\infty} \frac{\tau^{\underline{\mu}}}{p(t)}\right] &= \frac{1}{\Gamma(\mu + 1)} \left[\sum_{\tau=\mu}^{\infty} \frac{\tau^{\underline{\mu}}\Gamma(\mu + 1)}{\tau^{\underline{\mu}} 2^{(\tau-\mu+3)}}\right] \\
&= \sum_{\tau=\mu}^{\infty} \frac{1}{2^{(\tau-\mu+3)}} \\
&= \frac{1}{4} \\
&< 1.
\end{aligned}
$$

Thus the second hypothesis (H2) is also satisfied. Hence, Theorem 2.4 implies that with the above defined functions F, p with their respective domains

$$
\Delta^{\mu}_{\mu-1}(p\Delta x(t)) + F(t, x(t + \mu - 1)) = 0 \tag{2.7}
$$

has a unique positive solution that converges to M as t goes to ∞.

3. MORE RESULTS CONCERNING SOLUTIONS WITH POSITIVE LIMITS AT INFINITY

In this section, we will give conditions under which the fractional equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) + F(t, x(t+\mu-1)) = 0$$

has a solution with a positive limit at ∞. Our methods of proofs in this section are to use weighted norms and the Contraction Mapping Theorem. First we give a relationship between the existence of solutions of the above fractional equation and solutions of a summation equation.

Theorem 3.1. *Assume $p : \mathbb{N}_{\mu-1} \to (0,\infty)$ and $F : \mathbb{N}_0 \times \mathbb{R} \to [0,\infty)$. Let $M > 0$ and define*

$$\zeta_M = \Big\{x : \mathbb{N}_{\mu-1} \to [M,\infty) : \Delta x(t) \le 0, \Delta x(\mu-1) = 0\Big\}.$$

Let $P(\tau, t) := \sum_{u=t}^{\tau} \frac{1}{p(u)}$, where $t \in \mathbb{N}_{\mu-1}$. Suppose for all the functions x defined on $\mathbb{N}_{\mu-1}$, the following two series

$$\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\left[\sum_{s=0}^{\tau-\mu+1} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)\right], \tag{3.1}$$

$$\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\left[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1))\right]$$

are convergent and moreover, the later series satisfies the condition that

$$\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\left[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1))\right] \le 0. \tag{3.2}$$

Then the fractional equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) + F(t, x(t+\mu-1)) = 0$$

has a positive solution $x \in \zeta_M$ such that $\lim_{t\to\infty} x(t) = M$ if and only if the summation equation

$$x(t) = M - \sum_{\tau=t}^{\infty} P(\tau,t)\left[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1))\right] \tag{3.3}$$

has a solution $x \in \mathbb{N}_{\mu-1}$.

Proof. Suppose the fractional equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) + F(t, x(t+\mu-1)) = 0 \tag{3.4}$$

has a positive solution $x \in \zeta_M$ such that $\lim_{t\to\infty} x(t) = M$. We let $y(t) = (p\Delta x)(t)$ in equation (3.4). Then by applying the fractional sum operator on both sides of equation (3.4) and using the fractional composition rule given in Lemma 1.6 we get that

$$\begin{aligned} y(t) &= -\Delta_0^{-\mu} F(t, x(t+\mu-1)) + ct^{\underline{\mu-1}} \\ &= -\sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) + ct^{\underline{\mu-1}}. \end{aligned}$$

This implies that

$$\Delta x(t) = -\frac{1}{p(t)} \left[\sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right]. \tag{3.5}$$

Now summing from $\tau = t$ to ∞ we get that

$$M - x(t) = -\sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right].$$

Therefore,

$$x(t) = M + \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right].$$

Now by using the definition of $P(\tau, t)$ as defined in the statement of this theorem we can rewrite the preceding equation as

$$x(t) = M + \sum_{\tau=t}^{\infty} [\Delta_\tau (P(\tau-1, t))] \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right],$$

and then by applying the summation by parts formula and the convergence of the series in (3.1) we get that

$$\begin{aligned} x(t) =& M + P(\tau-1, t) \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right] \Big|_{\tau=t}^{\infty} \\ & - \sum_{\tau=t}^{\infty} P(\tau, t) \Big[\sum_{s=0}^{\tau-\mu} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \\ & + F(\tau-\mu+1, x(\tau)) \Big] \\ =& M - \sum_{\tau=t}^{\infty} P(\tau, t) \left[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right]. \end{aligned}$$

On the other hand, if the summation equation

$$y(t) = M - \sum_{\tau=t}^{\infty} P(\tau, t) \left[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, y(s+\mu-1)) \right]$$

has a solution x on $\mathbb{N}_{\mu-1}$, then

$$x(t) = M - \sum_{\tau=t}^{\infty} P(\tau, t) \left[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right].$$

Note that since the series

$$\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\left[\sum_{s=0}^{\tau-\mu+1}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}F(s,x(s+\mu-1)\right]$$

is convergent and $\left(\sum_{u=t}^{\tau-1}\frac{1}{p(u)}\right)|_{\tau=t}=0$, we have that

$$\left[\left(\sum_{u=t}^{\tau-1}\frac{1}{p(u)}\right)\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\right]|_{\tau=t}^{\infty}=0$$

and hence the expression for $x(t)$ as mentioned above can be rewritten as

$$\begin{aligned} x(t) &= M+\big(\sum_{u=t}^{\tau-1}\frac{1}{p(u)}\big)\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}F(s,x(s+\mu-1))|_{\tau=t}^{\infty} \\ &\quad -\sum_{\tau=t}^{\infty}P(\tau,t)\left[\sum_{s=0}^{\tau-\mu+1}\frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\right] \\ &= M+\sum_{\tau=t}^{\infty}\left[\Delta_\tau P(\tau-1,t)\right]\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\right] \\ &= M+\sum_{\tau=t}^{\infty}\frac{1}{p(\tau)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\right]. \end{aligned}$$

Now by taking the delta difference of both sides of the last equation, we get that

$$\Delta x(t)=-\frac{1}{p(t)}\left[\sum_{s=0}^{t-\mu}\frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\right]. \tag{3.6}$$

Hence,

$$(p\Delta x)(t)=-\left[\Delta_0^{-\mu}F(.,x(.-\mu+1)\right](t).$$

Taking the fractional difference of both sides of the last equation, we get that

$$\begin{aligned} \Delta_{\mu-1}^{\mu}(p\Delta x)(t) &= -\Delta_{\mu-1}^{\mu}\Delta_0^{-\mu}(F(.,x(.+\mu-1))(t) \\ &= -F(t,x(t+\mu-1)), \quad t\in\mathbb{N}_0. \end{aligned}$$

Therefore,

$$\Delta_{\mu-1}^{\mu}(p\Delta x)(t)+F(t,x(t+\mu-1))=0, \quad t\in N_0.$$

Hence, x is the solution of equation (3.4). Moreover it is not hard to see from the expression for Δx as given by equation (3.6), that $\Delta x(t)\le 0$ and $\Delta x(\mu-1)=0$. Also, since the series

$$\sum_{\tau=\mu-1}^{\infty}P(\tau,\mu-1)\big[\sum_{s=0}^{\tau-\mu+1}\frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\big]$$

is convergent we have that $\lim_{t\to\infty} x(t) = M$. Furthermore, since

$$\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\Big[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1))\Big] \leq 0, \tag{3.7}$$

we have that

$$M - \sum_{\tau=t}^{\infty} P(\tau,t)\Big[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, y(s+\mu-1))\Big] \geq M,$$

i.e. $x(t) \geq M$. Hence, we conclude that the fractional equation (3.4) has a positive solution $x \in \zeta_M$. □

Next we will prove the following theorem which is an application of the Contraction Mapping Theorem.

Theorem 3.2. *Assume* $F : N_0 \times \mathbb{R} \to [0,\infty)$ *satisfies a uniform Lipschitz condition with respect to its second variable, i.e. if* $u, v \in \mathbb{R}$ *and* $t \in \mathbb{N}_0$ *then* $\big|F(t,u) - F(t,v)\big| \leq K\big|u-v\big|$ *and assume* $p : \mathbb{N}_{\mu-1} \to (0,\infty)$, $P(\tau,t) := \sum_{u=t}^{\tau} \frac{1}{p(u)}$ *and let* (ζ_M, d) *be the complete metric space as defined in Lemma* 2.3. *Assume the following hypotheses* $(H1)$, $(H2)$, $(H3)$ *and* $(H4)$ *are satisfied:*

(H1) The series $\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\left[\sum_{s=0}^{\tau-\mu+1} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)\right]$ *is convergent for each* $x \in \zeta_M$.

(H2) The series $\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\left[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}} F(s, x(s+\mu-1)}{\Gamma(\mu)}\right]$ *is convergent for each* $x \in \zeta_M$.

(H3) $\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\left[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}} F(s, x(s+\mu-1)}{\Gamma(\mu)}\right] \leq 0$ *for each* $x \in \zeta_M$.

(H4) $\frac{K}{L} \sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\left[\sum_{s=0}^{\tau-\mu+1} \frac{|\mu-1||(\tau-\sigma(s))^{\underline{\mu-2}}|w(s+\mu-1)}{\Gamma(\mu)}\right] = \alpha < 1,$

where in hypothesis $(H4)$ *we defined* $L =: \lim_{t\to\infty} w(t) > 0$ *as mentioned in Lemma* 2.3. *Then there exist a unique positive solution* x *of* (3.4) *such that* $\lim_{t\to\infty} x(t) = M$.

Proof. Let (ζ_M, d) be the complete metric space as defined in Lemma 2.3 and consider the map T on ζ_M defined by

$$Tx(t) = M - \sum_{\tau=t}^{\infty} P(\tau,t)\Big[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1))\Big].$$

First, we will show that $T:\zeta_M \to \zeta_M$. Let $x \in \zeta_M$ be arbitrary. Then similar to the derivation of (3.5) in Theorem 3.1 we have

$$\Delta Tx(t) = -\frac{1}{p(t)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}F(s,x(s+\mu-1))}{\Gamma(\mu)}\right].$$

Also since $p(t)>0$ for all $t \in \mathbb{N}_{\mu-1}$ and $F(t,u)\geq 0$ for all $(t,u)\in\mathbb{N}_0\times\mathbb{R}$, we have that

$$\Delta Tx(t) = -\frac{1}{p(t)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}F(s,x(s+\mu-1))}{\Gamma(\mu)}\right]\leq 0.$$

Also it is not hard to see that

$$\Delta Tx(\mu-1)=0.$$

Moreover, by using hypotheses $(H2)$ and $(H3)$ we conclude that $T(x)(t)\geq M$ for $t\in\mathbb{N}_{\mu-1}$. Hence, we proved that $T:\zeta_M\to\zeta_M$. Next we will show that T is a contraction mapping on ζ_M. Let $t\in\mathbb{N}_{\mu-1}$ be arbitrary. Then notice that

$$\left|\frac{Tx(t)-Ty(t)}{w(t)}\right| \leq \frac{1}{w(t)}\sum_{\tau=t}^{\infty}P(\tau,t)\Big[\sum_{s=0}^{\tau-\mu+1}\frac{|(\mu-1)||(\tau-\sigma(s))^{\underline{\mu-2}}|}{\Gamma(\mu)}|F(s,x(s+\mu-1))$$
$$-F(s,y(s+\mu-1)|\Big].$$
$$\leq \frac{K}{w(t)}\left[\sum_{\tau=t}^{\infty}P(\tau,t)\left(\sum_{s=0}^{\tau-\mu+1}\frac{|(\mu-1)||(\tau-\sigma(s))^{\underline{\mu-2}}|}{\Gamma(\mu)}w(s+\mu-1)\right)\right]d(x,y).$$

Thus

$$\begin{aligned}
&d(T(x),T(y))\\
&\leq K/L\left[\sum_{\tau=\mu-1}^{\infty}P(\tau,\mu-1)\left(\sum_{s=0}^{\tau-\mu+1}\frac{|(\mu-1)||(\tau-\sigma(s))^{\underline{\mu-2}}|}{\Gamma(\mu)}w(s+\mu-1)\right)\right]d(x,y)\\
&=\alpha d(x,y).
\end{aligned}$$

Since $\alpha<1$, by the hypothesis $(H4)$, T is a contraction mapping on ζ_M. Hence, by the Contraction Mapping Theorem there exist a unique positive fixed point x of T in ζ_M such that $T(x)=x$. Therefore, Theorem 3.1 guarantees that x is the unique positive solution to the summation equation

$$x(t)=M-\sum_{\tau=t}^{\infty}P(\tau,t)\Big[\sum_{s=0}^{\tau-\mu+1}\frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\Big].$$

Moreover, by using hypothesis $(H2)$, we observe

$$\begin{aligned}
\lim_{t\to\infty}x(t)&=M-\lim_{t\to\infty}\sum_{\tau=t}^{\infty}P(\tau,t)\Big[\sum_{s=0}^{\tau-\mu+1}\frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\Big]\\
&=M.
\end{aligned}$$

This completes the proof. □

Next we provide an interesting example to illustrate the above theorem.

Example 3.3. Let $P(\tau,t)=\sum_{u=t}^{\tau}\frac{1}{p(u)}=\arctan[(\frac{e}{3})^{-(\tau-t)}]$, for fixed $t\in N_{\mu-1}$ so that $P(\tau,\mu-1)=\sum_{u=\mu-1}^{\tau}\frac{1}{p(u)}$=$\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]$. Note that

$$\Delta_\tau P(\tau,\mu-1)=\Delta_\tau\sum_{u=\mu-1}^{\tau}\frac{1}{p(u)}=\Delta_\tau\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]$$
$$=\arctan[\frac{(3/e-1)}{(\frac{e}{3})^{(\tau-\mu+1)}+(\frac{e}{3})^{-(\tau-\mu+2)}}].$$

Thus,

$$\frac{1}{p(\tau+1)}=\arctan[\frac{(3/e-1)}{(\frac{e}{3})^{(\tau-\mu+1)}+(\frac{e}{3})^{-(\tau-\mu+2)}}]$$

and hence

$$p(\tau)=\frac{1}{\arctan[\frac{(3/e-1)}{(\frac{e}{3})^{(\tau-\mu)}+(\frac{e}{3})^{-(\tau-\mu+1)}}]}.$$

But note that

$$\frac{(3/e-1)}{(\frac{e}{3})^{(\tau-\mu)}+(\frac{e}{3})^{-(\tau-\mu+1)}}>0,$$

which implies that

$$\arctan[\frac{(3/e-1)}{(\frac{e}{3})^{(\tau-\mu)}+(\frac{e}{3})^{-(\tau-\mu+1)}}]>0$$

and hence $p(\tau)>0$. Next we will show that all four hypotheses of the previous theorem are satisfied. First of all it is important to mention that the series

$$\sum_{\tau=\mu-1}^{\infty}P(\tau,\mu-1)=\sum_{\tau=\mu-1}^{\infty}\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]$$

and

$$\sum_{\tau=\mu}^{\infty}(\tau-\mu+2)\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]$$

converge by the ratio test. We will use the convergence of these series in the following. We let $S=\sum_{\tau=\mu}^{\infty}(\tau-\mu+2)\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]$ and observe that $0<S<\infty$. Moreover, let

$$F(t,x):=\begin{cases}\frac{\frac{L}{2S}x}{1+x} & \text{if} \quad t=0\\ 0 & \text{if} \quad t\neq 0.\end{cases}$$

Clearly F satisfies a uniform Lipschitz condition with respect to the second variable with Lipschitz constant $K=\frac{L}{2S}$. Here L is chosen as it is defined in Lemma 2.3. Next we will show that the hypotheses of Theorem 3.2 are satisfied.

Claim (H1) holds:

$$\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\left[\sum_{s=0}^{\tau-\mu+1}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}F(s,x(s+\mu-1)\right]$$
$$=\sum_{\tau=\mu-1}^{\infty}\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]\left[\frac{(\tau-1)^{\underline{\mu-1}}}{\Gamma(\mu)}\frac{\frac{L}{2S}.x(\mu-1)}{1+x(\mu-1)}\right].$$

Notice that

$$\left[\frac{(\tau-1)^{\underline{\mu-1}}}{\Gamma(\mu)}\frac{\frac{L}{2}x(\mu-1)}{1+x(\mu-1)}\right]<1.$$

Therefore,

$$\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]\left[\frac{(\tau-1)^{\underline{\mu-1}}}{\Gamma(\mu)}\frac{\frac{L}{2S}.x(\mu-1)}{1+x(\mu-1)}\right]<\frac{1}{S}\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}],$$

and the series

$$\sum_{\tau=\mu-1}^{\infty}\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]$$

is convergent by the ratio test, which implies that the series in (H1) is convergent by the comparison test.

Next in order to show that (H2) is satisfied we show the series in (H2) converges absolutely.

Claim (H2) holds:

$$\sum_{\tau=\mu-1}^{\infty}\left|P(\tau,\mu-1)\left[\sum_{s=0}^{\tau-\mu+1}\frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}F(s,x(s+\mu-1)}{\Gamma(\mu)}\right]\right|$$
$$=\sum_{\tau=\mu-1}^{\infty}\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]\left|\left[\sum_{s=0}^{\tau-\mu+1}\frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}F(s,x(s+\mu-1)}{\Gamma(\mu)}\right]\right|$$
$$\leq\sum_{\tau=\mu-1}^{\infty}\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]\left|\frac{(\mu-1)(\tau-1)^{\underline{\mu-2}}F(0,x(\mu-1))}{\Gamma(\mu)}\right|$$
$$=\sum_{\tau=\mu-1}^{\infty}\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]\left|\frac{(\mu-1)(\tau-1)^{\underline{\mu-2}}}{\Gamma(\mu)}\frac{\frac{L}{2S}.x(\mu-1)}{1+x(\mu-1)}\right|.$$

Again since

$$\left|\frac{(\mu-1)(\tau-1)^{\underline{\mu-2}}}{\Gamma(\mu)}\frac{\frac{L}{2}x(\mu-1)}{1+x(\mu-1)}\right|<1,$$

and the series

$$\sum_{\tau=\mu-1}^{\infty}\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]$$

is convergent by the ratio test, we have by the comparison test that the series in (H2) converges.

Claim (H3) holds: Note that

$$\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\left[\sum_{s=0}^{\tau-\mu+1}\frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}F(s,x(s+\mu-1)}{\Gamma(\mu)}\right]$$

$$=\sum_{\tau=\mu-1}^{\infty}\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]\left[\frac{(\mu-1)(\tau-1)^{\underline{\mu-2}}\left(\frac{\frac{L}{2S}.x(\mu-1)}{1+x(\mu-1)}\right)}{\Gamma(\mu)}\right]$$

$$=\frac{Lx(\mu-1)}{2S(1+x(\mu-1))}\sum_{\tau=\mu-1}^{\infty}\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]\left[\frac{(\mu-1)(\tau-1)^{\underline{\mu-2}}}{\Gamma(\mu)}\right].$$

It follows that

$$=(\frac{\frac{L}{2S}.x(\mu-1)}{1+x(\mu-1)})\left[\frac{\pi}{4}+\sum_{\tau=\mu}^{\infty}\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]\left[\frac{(\mu-1)(\tau-1)^{\underline{\mu-2}}}{\Gamma(\mu)}\right]\right]$$

$$\leq(\frac{\frac{L}{2S}.x(\mu-1)}{1+x(\mu-1)})\left[\frac{\pi}{4}+\frac{\pi}{4}\sum_{\tau=\mu}^{\infty}\left[\frac{\Delta_\tau(\tau-1)^{\underline{\mu-1}}}{\Gamma(\mu)}\right]\right]$$

$$=(\frac{\frac{L}{2S}.x(\mu-1)}{1+x(\mu-1)})\left[\frac{\pi}{4}+\frac{\pi}{4}\frac{(\tau-1)^{\underline{\mu-1}}}{\Gamma(\mu)}|_{\tau=\mu}^{\tau=\infty}\right]$$

$$=(\frac{\frac{L}{2S}.x(\mu-1)}{1+x(\mu-1)})\left[\frac{\pi}{4}+\frac{\pi}{4}[0-1]\right]$$

$$=(\frac{\frac{L}{2S}.x(\mu-1)}{1+x(\mu-1)})\left[\frac{\pi}{4}-\frac{\pi}{4}\right]$$

$$=0.$$

Thus,

$$\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\left[\sum_{s=0}^{\tau-\mu+1}\frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}F(s,x(s+\mu-1)}{\Gamma(\mu)}\right]\leq 0.$$

Hence (H3) is satisfied.

Claim (H4) holds:

$$\frac{K}{L}\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\left[\sum_{s=0}^{\tau-\mu+1}\frac{|(\mu-1)|\left|(\tau-\sigma(s))^{\underline{\mu-2}}\right|w(s+\mu-1)}{\Gamma(\mu)}\right]$$

$$\leq\frac{L}{2SL}\sum_{\tau=\mu-1}^{\infty}\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]\left[\sum_{s=0}^{\tau-\mu+1}\frac{|(\mu-1)||(\tau-1)^{\underline{\mu-2}}|.1}{\Gamma(\mu)}\right]$$

$$=\frac{1}{2S}\sum_{\tau=\mu-1}^{\infty}\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]\left[\sum_{s=0}^{\tau-\mu+1}\frac{|(\mu-1)||(\tau-1)^{\underline{\mu-2}}|.1}{\Gamma(\mu)}\right]$$

$$=\frac{1}{2S}\sum_{\tau=\mu-1}^{\infty}\arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]\left[\sum_{s=0}^{\tau-\mu+1}\frac{|(\mu-1)||(\tau-1)^{\underline{\mu-2}}|.1}{\Gamma(\mu)}\right]$$

$$= \frac{1}{2S} \sum_{\tau=\mu-1}^{\infty} \arctan[(\frac{e}{3})^{-(\tau-\mu+1)}] \left[\sum_{s=0}^{\tau-\mu+1} \frac{|(\mu-1)||(\tau-1)^{\underline{\mu-1}}|.1}{(\tau-\mu+1)\Gamma(\mu)} \right]$$

$$\leq \frac{1}{2S} \sum_{\tau=\mu-1}^{\infty} \arctan[(\frac{e}{3})^{-(\tau-\mu+1)}] \left[\sum_{s=0}^{\tau-\mu+1} \frac{|(\mu-1)||(\tau-1)^{\underline{\mu-1}}|.1}{\Gamma(\mu)} \right]$$

$$< \frac{1}{2S} \sum_{\tau=\mu-1}^{\infty} \arctan[(\frac{e}{3})^{-(\tau-\mu+1)}] \left[\sum_{s=0}^{\tau-\mu+1} 1 \right]$$

$$= \frac{1}{2S} \sum_{\tau=\mu-1}^{\infty} (\tau-\mu+2) \arctan[(\frac{e}{3})^{-(\tau-\mu+1)}]$$

$$= \frac{1}{2S} S$$

$$= \frac{1}{2} < 1.$$

Hence all four hypotheses in Theorem 3.2 are satisfied therefore we have that if we define F, P, K and L as in this example, then there exists a unique solution to the fractional equation (3.4) with a given positive horizontal asymptote.

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