

COMPARISON THEOREMS FOR HIGHER ORDER FORCED NONLINEAR FUNCTIONAL DYNAMIC EQUATIONS

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ABSTRACT. The purpose of this paper is to establish comparison criteria for higher order forced nonlinear dynamic equation with mixed nonlinearities

$$\{r_{n-1}(t) (r_{n-2}(t)(\cdots (r_1(t)x^\Delta(t))^\Delta \cdots)^\Delta)^\Delta\}^\Delta + \sum_{j=0}^N p_j(t)\phi_{\gamma_j}(x(\varphi_j(t))) = g(t),$$

on an above-unbounded time scale \mathbb{T} , where $n \geq 2$. The results improve the main results of a number of recent papers and are established for a time scale \mathbb{T} without assuming certain restrictive conditions on \mathbb{T} .

AMS (MOS) Subject Classification. 34K11, 39A10, 39A99.

1. INTRODUCTION

Following Hilger's landmark paper [22], there have been plenty of references focused on the theory of time scales in order to unify continuous and discrete analysis, where a time scale is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, e.g., $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ for $q > 1$ (which has important applications in quantum theory), $\mathbb{T} = h\mathbb{N}$ with $h > 0$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{H}_n$ the space of the harmonic numbers. For the notions used below we refer to [5, 6] that provides some basic facts on time scale. In this paper, we will establish comparison criteria for the higher order forced nonlinear dynamic equation with mixed nonlinearities of the form

$$\{r_{n-1}(t) (r_{n-2}(t)(\cdots (r_1(t)x^\Delta(t))^\Delta \cdots)^\Delta)^\Delta\}^\Delta + \sum_{j=0}^N p_j(t)\phi_{\gamma_j}(x(\varphi_j(t))) = g(t), \quad (1.1)$$

on an above-unbounded time scale \mathbb{T} , where $n \geq 2$; $\phi_\beta(u) := |u|^{\beta-1}u$, $\beta > 0$; $r_i \in C_{rd}(\mathbb{T}, (0, \infty))$, $i = 1, 2, \dots, n-1$ and $p_j \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, $j = 0, 1, \dots, N$ with $p_j \not\equiv 0$, are real valued, rd-continuous functions; $\varphi_j : \mathbb{T} \rightarrow \mathbb{T}$ is a rd-continuous function such that $\lim_{t \rightarrow \infty} \varphi_j(t) = \infty$, $j = 0, 1, \dots, N$; and $g \in C_{rd}(\mathbb{T}, \mathbb{R})$. Throughout this paper, we let

$$x^{[i]} := r_i (x^{[i-1]})^\Delta, \quad i = 1, 2, \dots, n \text{ with } r_n = 1 \text{ and } x^{[0]} = x,$$

and there exists an oscillatory function $h \in C_{rd}^1[t_0, \infty)_{\mathbb{T}}$ such that $g(t) = h^{[n]}(t) = (h^{[n-1]})^\Delta$ for $t \geq t_0 \in \mathbb{T}$, where $h^{[i]} = r_i (h^{[i-1]})^\Delta$, $i = 1, 2, \dots, n-1$ with $h^{[0]} = h$; and assume that

$$\int_{t_0}^{\infty} \frac{\Delta t}{r_i(t)} = \infty, \quad i = 1, 2, \dots, n-1, \quad (1.2)$$

and

$$\gamma_j < \gamma_0, \quad j = 1, 2, \dots, l; \quad \text{and} \quad \gamma_j > \gamma_0, \quad j = l+1, l+2, \dots, N. \quad (1.3)$$

By a solution of Eq. (1.1) we mean a nontrivial real-valued function $x \in C_{rd}^1[T_x, \infty)_{\mathbb{T}}$ for some $T_x \geq t_0$ such that $x^{[i]} \in C_{rd}^1[T_x, \infty)_{\mathbb{T}}$, $i = 1, 2, \dots, n-1$ and $x(t)$ satisfies Eq. (1.1) on $[T_x, \infty)_{\mathbb{T}}$, where C_{rd} is the space of right-dense continuous functions. An extendable solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is said to be nonoscillatory. There has been an increasing interest in studying the oscillatory behavior of all order dynamic equations on time scales, see, for example [2, 1, 7, 8, 9, 12, 13, 15, 16, 17, 18, 20, 21, 24] and the references contained therein.

Recently, Erbe, Mert, Peterson and Zafer [10] obtained comparison criterion for even order dynamic equation

$$x^{\Delta^n}(t) + p(t) \phi_\gamma(x(\varphi(t))) = g(t), \quad (1.4)$$

where $\varphi \in C_{rd}(\mathbb{T}, \mathbb{T})$ such that $\varphi(t) \leq t$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. The results in [10] apply only to time scales satisfying on unbounded time scale \mathbb{T} where $\sigma(t) = at + b$, where $a \geq 1$, $b \geq 0$ are constants. Hassan [19] extended previous results for even order dynamic equation

$$x^{[n]}(t) + p(t) \phi_\gamma(x(\varphi(t))) = g(t),$$

without assuming certain restrictive conditions on \mathbb{T} . The purpose of this paper is to obtain comparison criteria for the more general forced nonlinear dynamic equation mixed nonlinearities (1.1) where $n \geq 2$ and still without assuming certain restrictive conditions on \mathbb{T} . The results extend the oscillation criteria established in [10, 19].

2. MAIN RESULTS

Before stating our main results, we begin with the following lemmas which will play an important role in the proof of our main results. The first one is cited from [19] and improves the well-known lemma due to Kiguradze.

Lemma 2.1. *If Eq. (1.1) has an eventually positive solution x , then there exists an integer $m \in [0, n]$ with $m + n$ odd such that*

$$m \geq 1 \quad \text{implies} \quad y^{[k]} > 0 \quad \text{for} \quad k = 0, 1, 2, \dots, m-1, \quad (2.1)$$

eventually, and

$$m \leq n \quad \text{implies} \quad (-1)^{m+k} y^{[k]} > 0 \quad \text{for} \quad k = m, m+1, \dots, n, \quad (2.2)$$

eventually, where $y := x - h$.

The second one is cited from [21, 20].

Lemma 2.2. *Assume (1.3) holds. Then there exists an N -tuple $(\eta_1, \eta_2, \dots, \eta_N)$ with $\eta_j > 0$ satisfying*

$$\sum_{j=1}^N \gamma_j \eta_j = \gamma_0 \quad \text{and} \quad \sum_{j=1}^N \eta_j = 1. \quad (2.3)$$

We will use the following notations: $\varphi(t) := \inf \{\varphi_0(t), \varphi_1(t), \dots, \varphi_N(t)\}$; and for any $u, v \in \mathbb{T}$, define the functions $R_i(v, u)$, $i = 0, 1, \dots, m$, $P_i(t)$ and $\bar{P}_i(t)$, $i = 0, \dots, n-1$, by the following recurrence formulas:

$$R_i(v, u) := \begin{cases} \int_u^v R_{i-1}(s, u) / r_{m-i+1}(s) \Delta s, & i = 1, \dots, m, \\ 1, & i = 0; \end{cases}$$

$$P_i(t) := \begin{cases} \int_t^\infty P_{i-1}(s) \Delta s / r_{n-i}(t) & i = 1, \dots, n-1, \\ \sum_{j=0}^N p_j(t), & i = 0; \end{cases}$$

and

$$\bar{P}_i(t) := \begin{cases} \int_t^\infty \bar{P}_{i-1}(s) \Delta s / r_{n-i}(t) & i = 1, \dots, n-1, \\ p(t), & i = 0, \end{cases}$$

where $p(t) := p_0(t) + \prod_{j=1}^N [p_j(t) / \eta_j]^{\eta_j}$ and provided the improper integrals involved are convergent.

Theorem 2.3. *Let φ be a nondecreasing function on $[t_0, \infty)_{\mathbb{T}}$ and there exist two sequences $\{s_n\}$ and $\{\bar{s}_n\}$ tending to infinity such that for all n ,*

$$\begin{aligned} h(s_n) &= \inf \{h(t) : t \geq s_n\}; \\ h(\bar{s}_n) &= \sup \{h(t) : t \geq \bar{s}_n\}. \end{aligned} \quad (2.4)$$

Assume that the first order dynamic equation

$$z^\Delta(t) + K_m(t) \phi_{\gamma_0}(z(\varphi(t))) = 0, \quad (2.5)$$

is oscillatory, where

$$K_m(t) := \bar{P}_{n-m-1}(t) R_m^{\gamma_0}(\varphi(t), T) \quad \text{for} \quad \varphi(t) \in [T, \infty)_{\mathbb{T}}, \quad (2.6)$$

for every an integer number $m \in \{1, \dots, n-1\}$ with $m+n$ is odd and for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$.

- (1) If $n \in 2\mathbb{N}$, then every solution of Eq. (1.1) is oscillatory.
(2) If $n \in 2\mathbb{N} + 1$ and, in addition, $\lim_{t \rightarrow \infty} h(t) = 0$ and

$$\int_{t_0}^{\infty} P_{n-1}(s) \Delta s = \infty, \quad (2.7)$$

then every solution of Eq. (1.1) is either oscillatory or tends to zero monotonically.

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, $x(t) > 0$ and $x(\varphi_j(t)) > 0$, $j = 0, 1, 2, \dots, N$ on $[t_0, \infty)_{\mathbb{T}}$. Define

$$y(t) := x(t) - h(t) \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \quad (2.8)$$

Then

$$y^{[i]}(t) = x^{[i]}(t) - h^{[i]}(t), \quad i = 1, 2, \dots, n.$$

Therefore Eq. (1.1) becomes

$$y^{[n]}(t) + \sum_{j=0}^N p_j(t) \phi_{\gamma_j}(x(\varphi_j(t))) = 0, \quad (2.9)$$

which implies

$$y^{[n]}(t) = - \sum_{j=0}^N p_j(t) \phi_{\gamma_j}(x(\varphi_j(t))) \leq 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

This implies that $y^{[i]}$, $i = 0, 1, \dots, n-1$ are eventually of one sign. Also since h is an oscillatory function, then $y(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$ for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an integer $m \in \{0, \dots, n-1\}$ with $m+n$ is odd such that (2.1) and (2.2) hold for $t \geq t_2 \in [t_1, \infty)_{\mathbb{T}}$.

(I) When $m = 0$. In this case n is odd and

$$(-1)^k y^{[k]} > 0 \quad \text{for } k = 0, 1, \dots, n. \quad (2.10)$$

Since $y^{\Delta} < 0$ eventually and $\lim_{t \rightarrow \infty} h(t) = 0$, then $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} y(t) = l_1 \geq 0$. Then for sufficiently large $t_3 \in [t_2, \infty)_{\mathbb{T}}$, we have $x(\varphi_j(t)) \geq l$ for $t \geq t_3$. It follows that

$$\phi_{\gamma_j}(x(\varphi_j(t))) \geq l^{\gamma_j} \geq L \quad \text{for } t \in [t_3, \infty)_{\mathbb{T}},$$

where $L := \inf_{0 \leq j \leq N} \{l^{\gamma_j}\} > 0$. Then from (1.1), we obtain

$$-(y^{[n-1]}(t))^{\Delta} = \sum_{j=0}^N p_j(t) \phi_{\gamma_j}(x(\varphi_j(t))) \geq L \sum_{j=0}^N p_j(t) = L P_0(t).$$

Integrating above inequality from t to $v \in [t, \infty)_{\mathbb{T}}$, we get

$$-y^{[n-1]}(v) + y^{[n-1]}(t) \geq L \int_t^v P_0(s) \Delta s.$$

and by (2.2) we see that $y^{[n-1]}(v) > 0$. Hence by taking limits as $v \rightarrow \infty$ we have

$$y^{[n-1]}(t) \geq L \int_t^\infty P_0(s) \Delta s,$$

which implies

$$(y^{[n-2]}(t))^\Delta \geq L \frac{1}{r_{n-1}(t)} \int_t^\infty P_0(s) \Delta s = L P_1(t).$$

Integrating from t to $v \in [t, \infty)_{\mathbb{T}}$ and letting $v \rightarrow \infty$ and using (2.2), we get

$$-y^{[n-2]}(t) \geq L \int_t^\infty P_1(s) \Delta s.$$

Continuing this process $(n-3)$ -times, we get

$$-y^{[1]}(t) \geq L \int_t^\infty P_{n-2}(s) \Delta s,$$

which implies

$$-y^\Delta(t) \geq L \frac{1}{r_1(t)} \int_t^\infty P_{n-2}(s) \Delta s = L P_{n-1}(t).$$

Again integrating above inequality from t_3 to $t \in [t_3, \infty)_{\mathbb{T}}$, we get

$$-y(t) + y(t_2) \geq L \int_{t_3}^t P_{n-1}(s) \Delta s$$

Hence by (2.7), we have $\lim_{t \rightarrow \infty} y(t) = -\infty$, which contradicts the fact that $y > 0$ eventually. This shows that if $m = 0$, then $\lim_{t \rightarrow \infty} x(t) = 0$.

(II) When $m \geq 1$. By the facts that $x(t) > 0$, $y(t) > 0$, $y^\Delta(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$ and from (2.4) it follows that there exists a constant λ , $0 < \lambda < 1$, such that for sufficiently large $t_3 \in [t_2, \infty)_{\mathbb{T}}$,

$$\varphi_j(t) \geq t_1 \quad \text{and} \quad x(t) \geq \lambda y(t) \quad \text{for } t \in [t_3, \infty)_{\mathbb{T}},$$

and so

$$x(\varphi_j(t)) \geq \lambda y(\varphi_j(t)) \quad \text{for } t \in [t_3, \infty)_{\mathbb{T}}.$$

Therefore, Eq. (2.9) becomes for $t \in [t_3, \infty)_{\mathbb{T}}$,

$$\begin{aligned} -y^{[n]}(t) &\geq \sum_{j=0}^N \lambda^{\gamma_j} p_j(t) \phi_{\gamma_j}(y(\varphi_j(t))) \\ &\geq \sum_{j=0}^N \lambda^{\gamma_j} p_j(t) \phi_{\gamma_j}(y(\varphi(t))) \\ &= \phi_{\gamma_0}(y(\varphi(t))) \sum_{j=0}^N \lambda^{\gamma_j} p_j(t) [y(\varphi(t))]^{\gamma_j - \gamma_0}. \end{aligned} \quad (2.11)$$

From (2.3) we have

$$\sum_{j=1}^N \gamma_j \eta_j - \gamma_0 \sum_{j=1}^N \eta_j = 0.$$

Using the Arithmetic-geometric mean inequality, see [4, Page 17], we have

$$\sum_{j=1}^N \eta_j v_j \geq \prod_{j=1}^N v_j^{\eta_j}, \quad \text{for any } v_j \geq 0, \quad j = 1, \dots, N.$$

Then for $t \geq t_3$,

$$\begin{aligned} & \sum_{j=0}^N \lambda^{\gamma_j} p_j(t) [x(\varphi(t))]^{\gamma_j - \gamma_0} \\ &= \lambda^{\gamma_0} p_0(t) + \sum_{j=1}^N \eta_j \frac{\lambda^{\gamma_j} p_j(t)}{\eta_j} [x(\varphi(t))]^{\gamma_j - \gamma_0} \\ &\geq \lambda^{\gamma_0} p_0(t) + \prod_{j=1}^N \left[\frac{\lambda^{\gamma_j} p_j(t)}{\eta_j} \right]^{\eta_j} [x(\varphi(t))]^{\eta_j(\gamma_j - \gamma_0)} \\ &= \lambda^{\gamma_0} p_0(t) + \prod_{j=1}^N \left[\frac{\lambda^{\gamma_j} p_j(t)}{\eta_j} \right]^{\eta_j} = p(t). \end{aligned}$$

This together with (2.11) shows that

$$-y^{[n]}(t) \geq p(t) \phi_{\gamma_0}(y(\varphi(t))) \quad \text{for } t \in [t_3, \infty)_{\mathbb{T}}. \quad (2.12)$$

Integrating Eq. (2.12) from $t \geq t_3$ to $v \in [t, \infty)_{\mathbb{T}}$ and then using the facts that y is strictly increasing and φ is a nondecreasing function, we get

$$\begin{aligned} -y^{[n-1]}(v) + y^{[n-1]}(t) &\geq \int_t^v p(s) \phi_{\gamma_0}(y(\varphi(s))) \Delta s \\ &\geq \phi_{\gamma_0}(y(\varphi(t))) \int_t^v p(s) \Delta s, \end{aligned}$$

and by (2.2) we see that $y^{[n-1]}(v) > 0$. Hence by taking limits as $v \rightarrow \infty$ we have

$$\begin{aligned} y^{[n-1]}(t) &\geq \phi_{\gamma_0}(y(\varphi(t))) \int_t^{\infty} p(s) \Delta s \\ &= \phi_{\gamma_0}(y(\varphi(t))) \int_t^{\infty} \bar{P}_0(s) \Delta s, \end{aligned}$$

which implies for $t \geq t_3$

$$\begin{aligned} [y^{[n-2]}(t)]^{\Delta} &\geq \phi_{\gamma_0}(y(\varphi(t))) \frac{1}{r_{n-1}(t)} \int_t^{\infty} p(s) \Delta s \\ &= \phi_{\gamma_0}(y(\varphi(t))) \bar{P}_1(t). \end{aligned} \quad (2.13)$$

Integrating above inequality (2.13) from $t \geq t_3$ to $v \in [t, \infty)_{\mathbb{T}}$ and letting $v \rightarrow \infty$ and using (2.1) and (2.2), we get

$$-y^{[n-2]}(t) \geq \phi_{\gamma_0}(y(\varphi(t))) \int_t^{\infty} \bar{P}_1(s) \Delta s,$$

Proceeding as above, we obtain

$$\begin{aligned}
y^{[n-3]}(t) &\geq \phi_{\gamma_0}(y(\varphi(t))) \int_t^\infty \bar{P}_2(s) \Delta s; \\
-y^{[n-4]}(t) &\geq \phi_{\gamma_0}(y(\varphi(t))) \int_t^\infty \bar{P}_3(s) \Delta s; \\
&\vdots \\
-y^{[n-(n-m-1)]}(t) &\geq \phi_{\gamma_0}(y(\varphi(t))) \int_t^\infty \bar{P}_{n-m-2}(s) \Delta s.
\end{aligned}$$

Therefore

$$-y^{[m+1]}(t) \geq \phi_{\gamma_0}(y(\varphi(t))) \int_t^\infty \bar{P}_{n-m-2}(s) \Delta s. \quad (2.14)$$

Also, from (2.1) and (2.2), we get

$$\begin{aligned}
y^{[m-1]}(t) &= y^{[m-1]}(t_3) + \int_{t_3}^t \frac{y^{[m]}(s)}{r_m(s)} \Delta s \\
&\geq y^{[m]}(t) \int_{t_3}^t \frac{\Delta s}{r_m(s)} = y^{[m]}(t) R_1(t, t_3).
\end{aligned}$$

It follows that

$$(y^{[m-2]}(t))^\Delta \geq y^{[m]}(t) \frac{R_1(t, t_3)}{r_{m-1}(t)}.$$

Then for $t \in [t_3, \infty)_{\mathbb{T}}$,

$$\begin{aligned}
y^{[m-2]}(t) &\geq y^{[m-2]}(t) - y^{[m-2]}(t_3) \\
&\geq \int_{t_3}^t y^{[m]}(s) \frac{R_1(s, t_3)}{r_{m-1}(s)} \Delta s \\
&\geq y^{[m]}(t) \int_{t_3}^t \frac{R_1(s, t_3)}{r_{m-1}(s)} \Delta s \\
&= y^{[m]}(t) R_2(t, t_3).
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
y^{[m-3]}(t) &\geq y^{[m]}(t) R_3(t, t_3); \\
y^{[m-4]}(t) &\geq y^{[m]}(t) R_4(t, t_3); \\
&\vdots \\
y^{[m-m]}(t) &\geq y^{[m]}(t) R_m(t, t_3).
\end{aligned} \quad (2.15)$$

It implies that

$$y(t) \geq y^{[m]}(t) R_m(t, t_3) \quad \text{for } t \in [t_3, \infty)_{\mathbb{T}}.$$

Then for $\varphi(t) \in [t_3, \infty)_{\mathbb{T}}$

$$y(\varphi(t)) \geq y^{[m]}(\varphi(t)) R_m(\varphi(t), t_3). \quad (2.16)$$

From (2.14) and (2.16), we get

$$-y^{[m+1]}(t) \geq \phi_{\gamma_0}(y^{[m]}(\varphi(t))) R_m^{\gamma_0}(\varphi(t), t_3) \int_t^\infty \bar{P}_{n-m-2}(s) \Delta s,$$

or

$$\begin{aligned} -[y^{[m]}(t)]^\Delta &\geq \frac{1}{r_{m+1}(t)} \int_t^\infty \bar{P}_{n-m-2}(s) \Delta s R_m^{\gamma_0}(\varphi(t), t_3) \phi_{\gamma_0}(y^{[m]}(\varphi(t))) \\ &= \bar{P}_{n-m-1}(t) R_m^{\gamma_0}(\varphi(t), t_3) \phi_{\gamma_0}(y^{[m]}(\varphi(t))) \\ &= K_m(t) \phi_{\gamma_0}(y^{[m]}(\varphi(t))). \end{aligned}$$

Let $z(t) := y^{[m]}(t) > 0$, we get

$$-z^\Delta(t) \geq K_m(t) \phi_{\gamma_0}(z(\varphi(t))),$$

or

$$z^\Delta(t) + K_m(t) \phi_{\gamma_0}(z(\varphi(t))) \leq 0. \quad (2.17)$$

By Corollary 2.3.5 in [3], equation (2.5) has an eventually positive solution which is a contradiction. This completes the proof. \square

In the following theorem we use the following notation:

$$\bar{R}_i(v, u) := \begin{cases} \int_u^v \bar{R}_{i-1}(v, s) / r_{n-i}(s) \Delta s, & i = 1, \dots, n-1, \\ 1, & i = 0, \end{cases}$$

for any $u, v \in \mathbb{T}$.

Theorem 2.4. *Assume that (2.4) holds and the first order dynamic equation*

$$z^\Delta(t) + H_m(t) \phi_{\gamma_0}(z(t)) = 0, \quad (2.18)$$

is oscillatory, where for $\tau(t) \in [T, \infty)_{\mathbb{T}}$,

$$H_m(t) := p(t) [\bar{R}_{n-m-1}(t, \tau(t)) R_m(\tau(t), T)]^{\gamma_0} \text{ with } \tau(t) := \inf \{t, \varphi(t)\}, \quad (2.19)$$

for every an integer number $m \in \{1, \dots, n-1\}$ with $m+n$ is odd and for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$.

- (1) *If $n \in 2\mathbb{N}$, then every solution of Eq. (1.1) is oscillatory.*
- (2) *If $n \in 2\mathbb{N} + 1$ and, in addition, $\lim_{t \rightarrow \infty} h(t) = 0$ and (2.7) holds, then every solution of Eq. (1.1) is either oscillatory or tends to zero monotonically.*

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, $x(t) > 0$ and $x(\varphi_j(t)) > 0$, $j = 0, 1, 2, \dots, N$ on $[t_0, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 2.3, there exists an integer $m \in \{0, \dots, n-1\}$ with $m+n$ is odd such that (2.1) and (2.2) hold for $t \geq t_2 \in [t_1, \infty)_{\mathbb{T}}$, for sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) When $m = 0$. As shown in the proof of Theorem 2.3, We show that if $\lim_{t \rightarrow \infty} h(t) = 0$ and (2.7) holds, then $\lim_{t \rightarrow \infty} x(t) = 0$.

(II) When $m \geq 1$. As seen in the proof of Theorem 2.3, we obtain for $t \in [t_3, \infty)_{\mathbb{T}}$, for some $t_3 \in [t_2, \infty)_{\mathbb{T}}$,

$$-y^{[m]}(t) \geq p(t) \phi_{\gamma_0}(y(\varphi(t))). \quad (2.20)$$

By the fact that $y^{[n-1]}$ is nonincreasing on $[t_3, \infty)_{\mathbb{T}}$, we get for $v \geq u \geq t_3$,

$$y^{[n-1]}(u) \geq y^{[n-1]}(v) = y^{[n-1]}(v) \bar{R}_0(v, u),$$

which implies

$$(y^{[n-2]}(u))^{\Delta} \geq y^{[n-1]}(v) \frac{\bar{R}_0(v, u)}{r_{n-1}(u)}.$$

Replacing u by s and integrating with respect to s from $u \geq t_3$ to $v \in [u, \infty)_{\mathbb{T}}$ and using (2.2), we get

$$\begin{aligned} -y^{[n-2]}(u) &\geq y^{[n-2]}(v) - y^{[n-2]}(u) \\ &= y^{[n-1]}(v) \int_u^v \frac{\bar{R}_0(v, s)}{r_{n-1}(s)} \Delta s \\ &= y^{[n-1]}(v) \bar{R}_1(v, u), \end{aligned}$$

which yields

$$-(y^{[n-3]}(u))^{\Delta} \geq y^{[n-1]}(v) \frac{\bar{R}_1(v, u)}{r_{n-2}(u)}.$$

Again replacing u by s and integrating with respect to s from u to v , we get

$$\begin{aligned} y^{[n-3]}(u) &\geq -y^{[n-3]}(v) + y^{[n-3]}(u) \\ &= y^{[n-1]}(v) \int_u^v \frac{\bar{R}_1(v, s)}{r_{n-2}(s)} \Delta s \\ &= y^{[n-1]}(v) \bar{R}_2(v, u). \end{aligned}$$

Proceeding this process, we obtain

$$\begin{aligned} -y^{[n-4]}(u) &\geq y^{[n-1]}(v) \bar{R}_3(v, u); \\ y^{[n-5]}(u) &\geq y^{[n-1]}(v) \bar{R}_4(v, u); \\ &\vdots \\ y^{[n-(n-m)]}(u) &\geq y^{[n-1]}(v) \bar{R}_{n-m-1}(v, u). \end{aligned}$$

Therefore

$$y^{[m]}(u) \geq y^{[n-1]}(v) \bar{R}_{n-m-1}(v, u).$$

Setting $v = t$ and $u = \tau(t)$ gives

$$y^{[m]}(\tau(t)) \geq y^{[n-1]}(t) \bar{R}_{n-m-1}(t, \tau(t)) \quad \text{for } \tau(t) \in [t_3, \infty)_{\mathbb{T}}. \quad (2.21)$$

By (2.15) with t is replaced by $\tau(t)$, we have for $\tau(t) \in [t_3, \infty)_{\mathbb{T}}$,

$$y(\tau(t)) \geq y^{[m]}(\tau(t)) R_m(\tau(t), t_3). \quad (2.22)$$

Pick $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $\tau(t) \in [t_3, \infty)_{\mathbb{T}}$ for $t \geq t_4$. Substituting (2.21) into (2.22), we get for $t \in [t_4, \infty)_{\mathbb{T}}$,

$$y(\tau(t)) \geq y^{[n-1]}(t) \bar{R}_{n-m-1}(t, \tau(t)) R_m(\tau(t), t_3). \quad (2.23)$$

Form (2.20) and (2.23) and using the fact that y is strictly increasing, we have

$$\begin{aligned} - (y^{[n-1]}(t))^\Delta &\geq p(t)\phi_{\gamma_0}(y(\varphi(t))) \geq p(t)\phi_{\gamma_0}(y(\tau(t))) \\ &\geq p(t) [\bar{R}_{n-m-1}(t, \tau(t))R_m(\tau(t), t_3)]^{\gamma_0} \phi_{\gamma_0}(y^{[n-1]}(t)) \\ &= H_m(t) \phi_{\gamma_0}(y^{[n-1]}(t)). \end{aligned}$$

Let $z(t) := y^{[n-1]}(t) > 0$, we get

$$-z^\Delta(t) \geq H_m(t) \phi_{\gamma_0}(z(t)),$$

or

$$z^\Delta(t) + H_m(t) \phi_{\gamma_0}(z(t)) \leq 0.$$

Again, by Corollary 2.3.5 in [3], equation (2.18) has an eventually positive solution which is a contradiction. This completes the proof. \square

Theorem 2.5. *Let φ be a nondecreasing function on $[t_0, \infty)_{\mathbb{T}}$. Assume that (2.4) holds and the second order dynamic equation*

$$[r_m(t)z^\Delta(t)]^\Delta + Q_m(t)\phi_{\gamma_0}(z(\varphi(t))) = 0, \quad (2.24)$$

is oscillatory, where

$$Q_m(t) := \bar{P}_{n-m-1}(t) [R_m(\varphi(t), T)/R_1(\varphi(t), T)]^{\gamma_0} \quad \text{for } \varphi(t) \in (T, \infty)_{\mathbb{T}}, \quad (2.25)$$

for every an integer number $m \in \{1, \dots, n-1\}$ with $m+n$ is odd and for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$.

- (1) If $n \in 2\mathbb{N}$, then every solution of Eq. (1.1) is oscillatory.
- (2) If $n \in 2\mathbb{N} + 1$ and, in addition, $\lim_{t \rightarrow \infty} h(t) = 0$ and (2.7) holds, then every solution of Eq. (1.1) is either oscillatory or tends to zero monotonically.

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, $x(t) > 0$ and $x(\varphi_j(t)) > 0$, $j = 0, 1, 2, \dots, N$ on $[t_0, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 2.3, there exists an integer $m \in \{0, \dots, n-1\}$ with $m+n$ is odd such that (2.1) and (2.2) hold for $t \geq t_2 \in [t_1, \infty)_{\mathbb{T}}$, for sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) When $m = 0$. As shown in the proof of Theorem 2.3, We show that if $\lim_{t \rightarrow \infty} h(t) = 0$ and (2.7) holds, then $\lim_{t \rightarrow \infty} x(t) = 0$.

(II) When $m \geq 1$. As seen in the proof of Theorem 2.3, we obtain for $t \in [t_3, \infty)_{\mathbb{T}}$, for some $t_3 \in [t_2, \infty)_{\mathbb{T}}$,

$$-y^{[m+1]}(t) \geq \phi_{\gamma_0}(y(\varphi(t))) \int_t^\infty \bar{P}_{n-m-2}(s)\Delta s. \quad (2.26)$$

and

$$y^{[m-1]}(t) \geq y^{[m]}(t) R_1(t, t_3). \quad (2.27)$$

Note that

$$\begin{aligned} \left[\frac{y^{[m-1]}(t)}{R_1(t, t_3)} \right]^\Delta &= \frac{R_1(t, t_3) [y^{[m-1]}(t)]^\Delta - \frac{1}{r_1(t)} y^{[m-1]}(t)}{R_1(t, t_3) R_1(\sigma(t), t_3)} \\ &= \frac{1}{r_m(t) R_1(t, t_3) R_1(\sigma(t), t_3)} (R_1(t, t_3) y^{[m]}(t) - y^{[m-1]}(t)), \end{aligned}$$

we have

$$\left[\frac{y^{[m-1]}(t)}{R_1(t, t_3)} \right]^\Delta \leq 0 \quad \text{on } (t_3, \infty)_{\mathbb{T}}. \quad (2.28)$$

Since for $t \in (t_3, \infty)_{\mathbb{T}}$,

$$y^{[m-1]}(t) = \frac{y^{[m-1]}(t)}{R_1(t, t_3)} R_1(t, t_3),$$

we have

$$(y^{[m-2]}(t))^\Delta = \frac{y^{[m-1]}(t) R_1(t, t_3)}{R_1(t, t_3) r_{m-1}(t)}.$$

Then by (2.28), we have for $t \in (t_3, \infty)_{\mathbb{T}}$

$$\begin{aligned} y^{[m-2]}(t) &\geq y^{[m-2]}(t) - y^{[m-2]}(t_3) \\ &= \int_{t_3}^t \frac{y^{[m-1]}(s)}{R_1(s, t_3)} \frac{R_1(s, t_3)}{r_{m-1}(s)} \Delta s \\ &\geq \frac{y^{[m-1]}(t)}{R_1(t, t_3)} \int_{t_3}^t \frac{R_1(s, t_3)}{r_{m-1}(s)} \Delta s \\ &= \frac{y^{[m-1]}(t)}{R_1(t, t_3)} R_2(t, t_3), \end{aligned}$$

which implies

$$[y^{[m-3]}(t)]^\Delta \geq \frac{y^{[m-1]}(t) R_2(t, t_3)}{R_1(t, t_3) r_{m-2}(t)}.$$

Then

$$\begin{aligned} y^{[m-3]}(t) &\geq y^{[m-3]}(t) - y^{[m-3]}(t_3) = \int_{t_3}^t (y^{[m-3]}(s))^\Delta \Delta s \\ &= \int_{t_3}^t \frac{y^{[m-1]}(s)}{R_1(s, t_3)} \frac{R_2(s, t_3)}{r_{m-2}(s)} \Delta s \\ &\geq \frac{y^{[m-1]}(t)}{R_1(t, t_3)} \int_{t_3}^t \frac{R_2(s, t_3)}{r_{m-2}(s)} \Delta s \\ &= \frac{y^{[m-1]}(t)}{R_1(t, t_3)} R_3(t, t_3). \end{aligned}$$

Proceeding as above process, we get

$$\begin{aligned} y^{[m-4]}(t) &\geq \frac{y^{[m-1]}(t)}{R_1(t, t_3)} R_4(t, t_3); \\ y^{[m-5]}(t) &\geq \frac{y^{[m-1]}(t)}{R_1(t, t_3)} R_5(t, t_3); \\ &\vdots \\ y^{[m-m]}(t) &\geq \frac{y^{[m-1]}(t)}{R_1(t, t_3)} R_m(t, t_3). \end{aligned}$$

It implies that

$$y(t) \geq \frac{y^{[m-1]}(t)}{R_1(t, t_3)} R_m(t, t_3) \quad \text{for } t \in (t_3, \infty)_{\mathbb{T}}.$$

Then for $\varphi(t) \in (t_3, \infty)_{\mathbb{T}}$

$$y(\varphi(t)) \geq \frac{y^{[m-1]}(\varphi(t))}{R_1(\varphi(t), t_3)} R_m(\varphi(t), t_3). \quad (2.29)$$

From (2.14) and (2.29), we get

$$-y^{[m+1]}(t) \geq \phi_{\gamma_0} (y^{[m-1]}(\varphi(t))) \int_t^\infty \bar{P}_{n-m-2}(s) \Delta s \left[\frac{R_m(\varphi(t), t_3)}{R_1(\varphi(t), t_3)} \right]^{\gamma_0}, \quad (2.30)$$

or

$$\begin{aligned} &- \left[r_m(t) (y^{[m-1]}(t))^\Delta \right]^\Delta \\ &\geq \frac{1}{r_{m+1}(t)} \int_t^\infty \bar{P}_{n-m-2}(s) \Delta s \left[\frac{R_m(\varphi(t), t_3)}{R_1(\varphi(t), t_3)} \right]^{\gamma_0} \phi_{\gamma_0} (y^{[m-1]}(\varphi(t))) \\ &= \bar{P}_{n-m-1}(t) \left[\frac{R_m(\varphi(t), t_3)}{R_1(\varphi(t), t_3)} \right]^{\gamma_0} \phi_{\gamma_0} (y^{[m-1]}(\varphi(t))) \\ &= Q_m(t) \phi_{\gamma_0} (y^{[m-1]}(\varphi(t))). \end{aligned} \quad (2.31)$$

Let $z(t) := y^{[m-1]}(t) > 0$, we get

$$- \left[r_m(t) z^\Delta(t) \right]^\Delta \geq Q_m(t) \phi_{\gamma_0} (z(\varphi(t))),$$

or

$$\left[r_m(t) z^\Delta(t) \right]^\Delta + Q_m(t) \phi_{\gamma_0} (z(\varphi(t))) \leq 0.$$

By [19, Lemma 2.2], we get that (2.24) has an eventually positive solution which is a contradiction. This completes the proof. \square

1. The conclusion of Theorems 2.3–2.5 remains intact if assumption (2.7) is replaced by one of the following conditions

$$\int_{t_0}^\infty P_1(t) \Delta t = \infty, \int_{t_0}^\infty P_2(t) \Delta t = \infty, \dots \text{ or } \int_{t_0}^\infty P_{n-2}(t) \Delta t = \infty.$$

2. It is easy to show that if either

$$\int_{t_0}^{\infty} P_1(t) \Delta t = \infty \text{ or } \int_{t_0}^{\infty} P_2(t) \Delta t = \infty.$$

then $m = n - 1$ in Theorems 2.3–2.5.

3. APPLICATIONS

In this section we establish some oscillation criteria for equation (1.1) by using Theorems 2.3–2.5.

Theorem 3.1. *Let $0 < \gamma_0 < 1$ and $\varphi(t) \in [t, \infty)_{\mathbb{T}}$ be nondecreasing function on $[t_0, \infty)_{\mathbb{T}}$. Assume that (2.4) holds and*

$$\int_T^{\infty} K_m(s) \Delta s = \infty, \quad (3.1)$$

where K_m is defined by (2.6), for every an integer number $m \in \{1, \dots, n - 1\}$ with $m + n$ is odd and for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$.

- (1) *If $n \in 2\mathbb{N}$, then every solution of Eq. (1.1) is oscillatory.*
- (2) *If $n \in 2\mathbb{N} + 1$ and, in addition, $\lim_{t \rightarrow \infty} h(t) = 0$ and (2.7) holds, then every solution of Eq. (1.1) is either oscillatory or tends to zero monotonically.*

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, $x(t) > 0$ and $x(\varphi_j(t)) > 0$, $j = 0, 1, 2, \dots, N$ on $[t_0, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 2.3, there exists an integer $m \in \{0, \dots, n - 1\}$ with $m + n$ is odd such that (2.1) and (2.2) hold for $t \geq t_2 \in [t_1, \infty)_{\mathbb{T}}$, for sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) When $m = 0$. As shown in the proof of Theorem 2.3, We show that if $\lim_{t \rightarrow \infty} h(t) = 0$ and (2.7) holds, then $\lim_{t \rightarrow \infty} x(t) = 0$.

(II) When $m \geq 1$. As seen in the proof of Theorem 2.3, we obtain that the first order dynamic equation

$$z^{\Delta}(t) + K_m(t) \phi_{\gamma_0}(z(\varphi(t))) = 0. \quad (3.2)$$

has an eventually positive solution $z(t)$ for $t \in [t_3, \infty)_{\mathbb{T}}$, for some $t_3 \in [t_2, \infty)_{\mathbb{T}}$. By the fact that z is nonincreasing on $[t_3, \infty)_{\mathbb{T}}$ and $\varphi(t) \leq t$ we get from (3.2) that

$$K_m(t) \leq -\frac{z^{\Delta}(t)}{[z(t)]^{\gamma_0}} \quad \text{for } t \in [t_3, \infty)_{\mathbb{T}}.$$

Integrating this inequality from t_3 to t , we get

$$\int_{t_3}^t K_m(s) \Delta s \leq -\int_{t_3}^t \frac{z^{\Delta}(s)}{[z(s)]^{\gamma_0}} \Delta s.$$

Define

$$F(z(s)) := \int_{z(t_3)}^{z(s)} \frac{du}{u^{\gamma_0}},$$

and so

$$(F(z(s)))^\Delta = \int_0^1 F'(z_h(s)) dh z^\Delta(s) = \int_0^1 \frac{1}{[z_h(s)]^{\gamma_0}} dh z^\Delta(s).$$

where $z_h(s) := (1-h)z(s) + hz^\sigma(s) > 0$, for $0 \leq h \leq 1$, $t \in [t_3, \infty)_{\mathbb{T}}$. Since z is nonincreasing on $[t_3, \infty)_{\mathbb{T}}$, we have

$$z_h(s) = (1-h)z(s) + hz^\sigma(s) \leq z(s).$$

Therefore

$$(F(z(s)))^\Delta = \int_0^1 \frac{1}{[z_h(s)]^{\gamma_0}} dh z^\Delta(s) \leq \frac{z^\Delta(s)}{[z(s)]^{\gamma_0}}.$$

Hence it follows that

$$\begin{aligned} \int_{t_3}^t K_m(s) \Delta s &\leq - \int_{t_3}^t \frac{z^\Delta(s)}{[z(s)]^{\gamma_0}} \Delta s \leq F(z(t_3)) - F(z(t)) \\ &= \frac{[z(t_3)]^{1-\gamma_0}}{1-\gamma_0} - \frac{[z(t)]^{1-\gamma_0}}{1-\gamma_0} \\ &\leq \frac{[z(t_3)]^{1-\gamma_0}}{1-\gamma_0}, \end{aligned}$$

which contradicts (3.1). \square

Theorem 3.2. *Let $0 < \gamma_0 < 1$. Assume that (2.4) holds and*

$$\int_T^\infty H_m(s) \Delta s = \infty \quad (3.3)$$

where H_m is defined by (2.19), for every an integer number $m \in \{1, \dots, n-1\}$ with $m+n$ is odd and for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$.

- (1) *If $n \in 2\mathbb{N}$, then every solution of Eq. (1.1) is oscillatory.*
- (2) *If $n \in 2\mathbb{N} + 1$ and, in addition, $\lim_{t \rightarrow \infty} h(t) = 0$ and (2.7) holds, then every solution of Eq. (1.1) is either oscillatory or tends to zero monotonically.*

Proof. The proof is similar to the proof of Theorem 3.1 with $K_m(t)$ is replaced by $H_m(t)$ and hence can be omitted. \square

Theorem 3.3. *Let φ be nondecreasing function on $[t_0, \infty)_{\mathbb{T}}$. Assume that (2.4) holds and*

$$\int_T^\infty Q_m(s) \Delta s = \infty \quad (3.4)$$

where Q_m is defined by (2.25), for every an integer number $m \in \{1, \dots, n-1\}$ with $m+n$ is odd and for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$.

- (1) *If $n \in 2\mathbb{N}$, then every solution of Eq. (1.1) is oscillatory.*
- (2) *If $n \in 2\mathbb{N} + 1$ and, in addition, $\lim_{t \rightarrow \infty} h(t) = 0$ and (2.7) holds, then every solution of Eq. (1.1) is either oscillatory or tends to zero eventually.*

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, $x(t) > 0$ and $x(\varphi_j(t)) > 0$, $j = 0, 1, 2, \dots, N$ on $[t_0, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 2.3, there exists an integer $m \in \{0, \dots, n-1\}$ with $m+n$ is odd such that (2.1) and (2.2) hold for $t \geq t_2 \in [t_1, \infty)_{\mathbb{T}}$, for sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) When $m = 0$. As shown in the proof of Theorem 2.3, We show that if $\lim_{t \rightarrow \infty} h(t) = 0$ and (2.7) holds, then $\lim_{t \rightarrow \infty} x(t) = 0$.

(II) When $m \geq 1$. As seen in the proof of Theorem 2.3, we obtain that the second order dynamic equation

$$[r_m(t)z^\Delta(t)]^\Delta + Q_m(t)\phi_{\gamma_0}(z(\varphi(t))) = 0. \quad (3.5)$$

has an eventually positive solution $z(t)$ for $t \in [t_3, \infty)_{\mathbb{T}}$, for some $t_3 \in [t_2, \infty)_{\mathbb{T}}$. It is easy to see that

$$[r_m(t)z^\Delta(t)]^\Delta \leq 0, \quad z^\Delta(t) > 0 \quad \text{for } t \in [t_3, \infty)_{\mathbb{T}}.$$

Integrating the inequality (3.5) from t_3 to $v \in [t, \infty)_{\mathbb{T}}$ and letting $v \rightarrow \infty$ and using the fact z is increasing on $[t_3, \infty)_{\mathbb{T}}$, we get

$$r_m(t_3)z^\Delta(t_3) \geq \phi_{\gamma_0}(z(\varphi(t_3))) \int_{t_3}^{\infty} Q_m(s)\Delta s,$$

this contradicts the assumption (3.4). This completes the proof. \square

Remark 3.4. For further oscillation criteria for equation (1.1), see [1, 2, 8, 9, 12, 13, 15, 17, 24].

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