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OSCILLATION OF CERTAIN DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. Oscillation criteria are established for second-order nonlinear nabla dynamic equations on an isolated time scale $\mathbb{T}$. Our main goal is to establish a relationship between the oscillatory behavior of these equations. We also give two results about the behavior of a second-order self-adjoint equation with mixed derivatives on a time scale that is unbounded above. We use the Riccati transformation technique to obtain our results.

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1. INTRODUCTION

In this paper, we are concerned with the oscillation of the second-order nonlinear nabla functional dynamic equation

$$\big(p(t)y^{\nabla}(t)\big)^{\nabla} + q(t)f(y(\tau(t))) = 0 \tag{1.1}$$

and the second-order nonlinear nabla dynamic equation

$$\big(p(t)y^{\nabla}(t)\big)^{\nabla} + q(t)f(y^{\rho}(t)) = 0 \tag{1.2}$$

on $[t_0, \infty)_{\mathbb{T}}$ where $\mathbb{T}$ is an isolated time scale and positive t_0 belongs to $\mathbb{T}$. Since oscillation of solutions is our primary concern, we assume throughout that all time scales are unbounded above. We assume that p, q, τ, and f satisfy the following Conditions (H):

(i) $p \in C_{ld}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ satisfies $\displaystyle\int_{t_0}^{\infty} \frac{1}{p(t)}\,\nabla t = \infty, \quad t \in \mathbb{T};$

(ii) $q \in C_{ld}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$;

(iii) $\tau \in C_{ld}(\mathbb{T}, \mathbb{T})$ satisfies $\lim_{t\to\infty} \tau(t) = \infty$ and there exists $M > 0$ such that

$$|P(t) - P(\tau(t))| < M \text{ for all } t \in \mathbb{T},$$

where $P(t) = \displaystyle\int_{t_0}^{t} \frac{1}{p(s)}\,\nabla s$;

(iv) $f : \mathbb{R} \to \mathbb{R}$ is continuous, increasing, and $f(-u) = -f(u)$ for $u \in \mathbb{R}$.

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By a solution of (1.1) we mean a nontrivial real-valued function y satisfying (1.1) for $t \in [t_0, \infty)_{\mathbb{T}}$. A solution y of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Also, (1.1) is said to be oscillatory if all its solutions are oscillatory. Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_y, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|y(t)| : t > t_0\} > 0$ for any $t_0 \geq t_y$. Additionally, we give two results about the second-order self-adjoint dynamic equation

$$\left(p(t)y^{\Delta}(t)\right)^{\nabla} + q(t)y(t) = 0 \tag{1.3}$$

on a time scale $\mathbb{T}$ where $\sup \mathbb{T} = \infty$, $p \in C(\mathbb{T}, (0, \infty))$, and $q \in C_{ld}(\mathbb{T}, \mathbb{R})$.

For completeness, we recall the following concepts related to the notion of time scales. The *forward* and *backward jump operators* are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in T : s < t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is called *left-dense* if $t > \inf \mathbb{T}$ and $\rho(t) = t$, *right-dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$, *right-scattered* if $\sigma(t) > t$, *dense* if $\rho(t) = t = \sigma(t)$, and *isolated* if $\rho(t) < t < \sigma(t)$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *left-dense continuous* provided f is continuous at left-dense points in $\mathbb{T}$ and its right-hand limits exist and are finite at right-dense points in $\mathbb{T}$. The set of all left-dense continuous functions is denoted by $C_{ld}(\mathbb{T})$. The *backward graininess function* ν on $\mathbb{T}$ is defined by $\nu(t) := t - \rho(t)$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\rho}(t)$ denotes $f(\rho(t))$. For more on nabla dynamic equations, see Chapter 3 of [2].

In the next section, we establish a relationship between the oscillatory behavior of (1.1) and (1.2). We present two lemmas necessary to prove our first main result. In the last section, we present oscillation criteria for (1.3). We use the Riccati transformation to obtain these results and close with an example.

2. OSCILLATION EQUIVALENCE OF (1.1) AND (1.2)

Throughout this section, we assume $\mathbb{T}$ is isolated. We begin with the following definition.

Definition 2.1. A nonempty closed subset K of a Banach space $\mathbb{X}$ is called a cone if it possess the following properties:

(i) if $\alpha \in \mathbb{R}^+$ and $x \in K$, then $\alpha x \in K$;
(ii) if $x, y \in K$, then $x + y \in K$;
(iii) if $x \in K - \{0\}$, then $-x \notin K$.

Let $\mathbb{X}$ be a Banach space and K be a cone with nonempty interior. Then we define a partial ordering $\leq$ on $\mathbb{X}$ by

$$x \leq y \quad \text{if and only if } y - x \in K.$$

We will use the following theorem [4] in order to prove some of our results.

Theorem 2.2 (Knaster's Fixed-Point Theorem)**.** *Let X be a partially ordered Banach space with ordering $\leq$. Let Ω be a subset of X with the following properties: The infimum of Ω belongs to Ω and every nonempty subset of Ω has a supremum which belongs to Ω. If $S : \Omega \to \Omega$ is an increasing mapping, then S has a fixed point in Ω.*

We continue with the following lemma.

Lemma 2.3. *Assume* (H) *holds. A necessary and sufficient condition for* (1.2) *to be oscillatory is that the inequality*

$$(p(t)y^{\nabla}(t))^{\nabla} + q(t)f(y^{\rho}(t)) \leq 0 \tag{2.1}$$

has no eventually positive solutions.

Proof. Assume (2.1) has no eventually positive solutions. Then neither does (1.2), and so it is oscillatory. If y is an eventually negative solution of (1.2), then let $x = -y$. Then x is eventually positive and

$$(px^{\nabla})^{\nabla} + qf(x^{\rho}) = -(py^{\nabla})^{\nabla} - qf(y^{\rho}) = -\left[(px^{\nabla})^{\nabla} + qf(x^{\rho})\right] = 0,$$

and so x is an eventually positive solution of (2.1), which is a contradiction. Hence (1.2) is oscillatory.

Suppose that (1.2) is oscillatory, and by way of contradiction, assume that (2.1) has an eventually positive solution y. So there exists $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t) > 0$ for $t \in [T_0, \infty)_{\mathbb{T}}$. Furthermore, there exists $T_1 \in [T_0, \infty)_{\mathbb{T}}$ such that $y(\rho(t)) > 0$ for all $t \in [T_1, \infty)_{\mathbb{T}}$. Using the sign condition of f in (H), we have $f(y^{\rho}(t)) > 0$ on $[T_1, \infty)_{\mathbb{T}}$. The $[p(t)y^{\nabla}(t)]^{\nabla} \leq 0$ on $[T_1, \infty)_{\mathbb{T}}$, and so $p(t)y^{\nabla}(t)$ is decreasing on $[T_1, \infty)_{\mathbb{T}}$.

We claim that $y^{\nabla}(t) > 0$ on $[T_1, \infty)_{\mathbb{T}}$. If not, then for some $t_1 \in [T_1, \infty)_{\mathbb{T}}$, we have $y^{\nabla}(t_1) \leq 0$. It follows that $p(t)y^{\nabla}(t) \leq 0$ on $[t_1, \infty)_{\mathbb{T}}$. Now, if $y^{\nabla}(t_2) < 0$ for some $t_2 \in [t_1, \infty)_{\mathbb{T}}$, then

$$\begin{aligned} y(t) - y(t_2) &= \int_{t_2}^{t} y^{\nabla}(s)\,\nabla s \\ &\leq p(t_2)y^{\nabla}(t_2)\int_{t_2}^{t} \frac{\nabla s}{p(s)} \\ &\to -\infty \quad \text{as t} \to \infty, \end{aligned}$$

which is a contradiction to our assumption that $y(t) > 0$ $[T_0,\infty)_{\mathbb{T}}$. Hence it follows that $y^{\nabla}(t) \equiv 0$ on $[t_1,\infty)_{\mathbb{T}}$, and so $(p(t)y^{\nabla}(t))^{\nabla} \equiv 0$ and $q(t)f(y^{\rho}(t)) > 0$, which is contradictory. Consequently, there exists $T_2 \in [T_1,\infty)_{\mathbb{T}}$ such that

$$y(t) > 0, \quad y^{\nabla}(t) > 0, \qquad \text{and} \qquad (p(t)y^{\nabla}(t))^{\nabla} \le 0$$

on $[T_2,\infty)_{\mathbb{T}}$.

Now integrating (2.1) from t to s yields

$$p(s)y^{\nabla}(s) - p(t)y^{\nabla}(t) + \int_t^s q(u)f(y^{\rho}(u))\,\nabla u \le 0$$

for $(s,t) \in [t,\infty)_{\mathbb{T}}\times[T_2,\infty)_{\mathbb{T}}$. Since $p(t)y^{\nabla}(t)$ is decreasing on $[T_1,\infty)_{\mathbb{T}}$, $\lim\limits_{t\to\infty} p(t)y^{\Delta}(t) = L \ge 0$ exists. Letting $s \to \infty$ in the above we obtain

$$y^{\nabla}(t) \ge \frac{L}{p(t)} + \frac{1}{p(t)}\int_t^{\infty} q(u)f(y^{\rho}(u))\,\nabla u \ge \frac{1}{p(t)}\int_t^{\infty} q(u)f(y^{\rho}(u))\,\nabla u. \tag{2.2}$$

Now integrating (2.2) from T_2 to t yields

$$y(t) \ge y(T) + \int_{T_2}^t \frac{1}{p(s)}\int_s^{\infty} q(u)f(y^{\rho}(u))\,\nabla u\,\nabla s, \quad t \in [T_2,\infty)_{\mathbb{T}}. \tag{2.3}$$

Let $\mathbb{X}$ be the set of all continuous functions on $[t_0,\infty)_{\mathbb{T}}$ satisfying $\lim\limits_{t\to\infty} y(t) = \infty$, where $\|\cdot\|$ is defined by $\|y\| = \sup\{|u(t)| : t_0 \le t < \infty\}$. Then $\mathbb{X}$ is a Banach space. Now, define the set

$$\Omega := \left\{\omega \in C([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+) : 0 \le \omega(t) \le 1, \text{ for } t \ge t_0\right\},$$

which is endowed with the usual pointwise ordering $\le$: $\omega_1 \le \omega_2 \Leftrightarrow \omega_1(t) \le \omega_2(t)$ for $t \ge t_0$.

One can show that any nonempty subset A of Ω has a supremum which belongs to Ω and $\inf\Omega \in \Omega$. Define a mapping S on Ω by

$$(S\omega)(t) = \begin{cases} 1, & \text{if } t_0 \le t \le T_2, \\ \frac{1}{y(t)}\left(y(T_2) + \int_{T_2}^t \frac{1}{p(s)}\int_s^{\infty} q(u)f(y^{\rho}(u)\omega^{\rho}(u))\,\nabla u\,\nabla s\right), & \text{if } t \ge T_2. \end{cases}$$

We claim that $S\Omega \subset \Omega$ and S is nondecreasing. For any $\omega \in \Omega$, $(S\omega)(t)$ is certainly continuous and for $t \in [T_2,\infty)_{\mathbb{T}}$

$$q(t)f(y^{\rho}(t)\omega^{\rho}(t)) \le q(t)f(y^{\rho}(t)),$$

Thus, from (2.2), $0 \le (S\omega)(t) \le 1$ for $t \in [T_2,\infty)_{\mathbb{T}}$, and so $S(\omega) \in \Omega$. Moreover, the monotonicity of f yields $(S\omega_1)(t) \le (S\omega_2)(t)$ provided $\omega_1 \le \omega_2$, $\omega_1,\omega_2 \in \Omega$. Therefore, by Knaster's Fixed Point Theorem, there exists $\tilde{\omega} \in \Omega$ such that $S\tilde{\omega} = \tilde{\omega}$. Hence, for $t \in [T_2,\infty)_{\mathbb{T}}$,

$$\tilde{\omega}(t) = \frac{1}{y(t)}\left(y(T_2) + \int_{T_2}^t \frac{1}{p(s)}\int_s^{\infty} q(u)f(y^{\rho}(u)\tilde{\omega}^{\rho}(u))\,\nabla u\,\nabla s\right),$$

On $[T_2,\infty)_{\mathbb{T}}$, define $z(t) := \tilde{\omega}(t)y(t)$. Then $z(t)$ is positive, left-dense continuous, and

$$z(t) = y(T_2) + \int_{T_2}^{t} \frac{1}{p(s)} \int_{s}^{\infty} q(v) f(z^{\rho}(u))\, \Delta u\, \Delta s$$

on $[T_2,\infty)_{\mathbb{T}}$. As $z^{\nabla}(t) = \dfrac{1}{p(t)} \displaystyle\int_t^{\infty} q(s)f(z^{\rho}(s))\,\nabla s$ and $(p(t)z^{\nabla}(t))^{\nabla} = -q(t)f(z^{\rho}(t))$, $(p(t)z^{\nabla}(t))^{\nabla} + q(t)f(z^{\rho}(t)) = 0$ has a positive solution, which is a contradiction to the assumption that all solutions of (1.2) are oscillatory. With this, the proof is complete. $\square$

In a similar manner, we can prove

Lemma 2.4. *Assume that* (H) *holds. Then, every solution of the second -order nonlinear functional dynamic equation* $(p(t)y^{\nabla}(t))^{\nabla} + q(t)f(y(\tau(t))) = 0$ *oscillates if and only if the inequality*

$$(p(t)y^{\nabla}(t))^{\nabla} + q(t)f(y(\tau(t))) \le 0$$

has no eventually positive solutions.

Now we state our main result of this section which is an extension of Theorem 2.1 of [10].

Theorem 2.5. *Assume* (H) *holds and* $\nu(t)/p(t)$ *is bounded. Additionally, assume that for all* $t \in [t_0,\infty)_{\mathbb{T}}$, $\tau(t) \le \rho(t)$ *or* $\tau(t) \ge \rho(t)$. *Then the oscillation of the second-order nonlinear nabla functional dynamic*

$$\big(p(t)y^{\nabla}(t)\big)^{\nabla} + q(t)f(y(\tau(t))) = 0 \tag{1.1}$$

is equivalent to the oscillation of the second-order nonlinear nabla dynamic equation

$$\big(p(t)y^{\nabla}(t)\big)^{\nabla} + q(t)f(y^{\rho}(t)) = 0. \tag{1.2}$$

Proof. The boundedness of ν/p gives the existence of $N > 0$ such that $\dfrac{\nu(t)}{p(t)} \le N$ for all $t \in \mathbb{T}$. Let $K := M + N$, where M satisfies property (iii) of (H).

Assume that (1.2) is oscillatory and, suppose to the contrary, that y is a nonoscillatory solution of (1.1). Without loss of generality, we assume there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that for all $t \in [t_1,\infty)_{\mathbb{T}}$

$$y(t) > 0, \quad y(\tau(t)) > 0, \quad p(t)y^{\nabla}(t) > 0, \quad \text{and} \quad (p(t)y^{\nabla}(t))^{\nabla} \le 0$$

as in the proof of Lemma 2.3. We also conclude that $\lim\limits_{t\to\infty} p(t)y^{\nabla}(t) = L \ge 0$ exists. We now distinguish two cases.

(I) Assume $\rho(t) \le \tau(t)$ for all $t \in \mathbb{T}$. The monotonicity of y and f yields

$$(p(t)y^{\nabla}(t))^{\nabla} + q(t)f(y^{\rho}(t)) \le (p(t)y^{\nabla}(t))^{\nabla} + q(t)f(y(\tau(t))) = 0,$$

and so (2.1) has an eventually positive solution. By Lemma 2.3, (1.2) has a nonoscillatory solution, which is a contradiction.

(II) Assume next that $\tau(t) \le \rho(t)$ for all $t \in \mathbb{T}$.

(a) Suppose $L > 0$. Then there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $p(t)y^{\nabla}(t) \le L+1$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. Furthermore, since τ diverges, there is a $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $\tau(t) \ge t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. Therefore, if t belongs to $[t_3, \infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} y^{\rho}(t) - y(\tau(t)) &= \int_{\tau(t)}^{\rho(t)} \frac{p(s)y^{\nabla}(s)}{p(s)} \nabla s \\ &\le (L+1) \int_{\tau(t)}^{\rho(t)} \frac{\nabla s}{p(s)} \\ &= (L+1)[P(\rho(t)) - P(\tau(t))] \\ &\le (L+1) \left[\left| - \int_{\rho(t)}^{t} \frac{\nabla s}{p(s)} \right| + |P(t) - P(\tau(t))| \right] \\ &= (L+1) \left[\frac{\nu(t)}{p(t)} + M \right] \\ &\le (L+1)[N+M]. \end{aligned}$$

Consequently, for all $t \in [t_3, \infty)_{\mathbb{T}}$, $y(\tau(t)) \ge y^{\rho}(t) - (L+1)K$. Now let $z(t) = y(t) - (L+1)K$. Since $p(t)y^{\nabla}(t) \ge L$, for all t sufficiently large, by integrating the previous inequality, we conclude that $z(t) > 0$ for large enough t. Hence, for all sufficiently large t,

$$z(t) > 0, \quad z^{\rho}(t) \le y(\tau(t)), \quad \text{and} \quad (p(t)z^{\nabla}(t))^{\nabla} + q(t)f(z^{\rho}(t)) \le 0.$$

We have that (2.1) has an eventually positive solution. By Lemma 2.3, we conclude (1.2) is nonoscillatory, which is a contradiction.

(b) Assume $L = 0$. Since y is positive and nondecreasing on $[t_1, \infty)_{\mathbb{T}}$, there exists $\epsilon_0 > 0$ and $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $y(t) > M\epsilon_0$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. Furthermore, there exists $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $p(t)y^{\nabla}(t) \le \epsilon_0$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. Now, if $t \in [t_3, \infty)_{\mathbb{T}}$, we have

$$y(t) - y(\tau(t)) \le \epsilon_0 \int_{\tau(t)}^{t} \frac{\nabla s}{p(s)} \le \epsilon_0 \left[\frac{\nu(t)}{p(t)} + M \right],$$

and so, since y is nondecreasing, for all $t \in [t_3, \infty)_{\mathbb{T}}$

$$y(\tau(t)) \ge y^{\rho}(t) - \epsilon_0 M.$$

Again, we set $z(t) := y(t) - \epsilon_0 M$. Then for sufficiently large t

$$z(t) > 0, \quad z^{\rho}(t) \le y(\tau(t)), \quad \text{and} \quad (p(t)z^{\nabla}(t))^{\nabla} + q(t)f(z^{\rho}(t)) \le 0.$$

Hence, (2.1) has an eventually positive solution. Again, we conclude by Lemma 2.3, $(p(t)y^{\nabla}(t))^{\nabla} + q(t)f(y^{\rho}(t)) = 0$ is nonoscillatory, which is a contradiction.

Conversely, assume that (1.1) is oscillatory. By way of contradiction, suppose y is a nonoscillatory solution of (1.2). As we saw in the proof of Lemma 2.3, we can assume there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that for all $t \in [t_1, \infty)_{\mathbb{T}}$

$$y(t) > 0, \quad y(\tau(t)) > 0, \quad p(t)y^{\nabla}(t) > 0, \quad \text{and} \quad (p(t)y^{\nabla}(t))^{\nabla} \leq 0.$$

Again, the monotonicity and boundedness of $p(t)y^{\nabla}(t)$ lead the existence of its non-negative limit L, and we distinguish two cases.

(I) Assume $\tau(t) \leq \rho(t) \leq t$ for all t. The monotonicity of y and f yields

$$(p(t)y^{\nabla}(t))^{\nabla} + q(t)f(y(\tau(t))) \leq (p(t)y^{\nabla}(t))^{\nabla} + q(t)f(y^{\rho}(t)) = 0,$$

and so y is an eventually positive solution of (1.1). By Lemma 2.4, (1.1) has a nonoscillatory solution, which is a contradiction.

(II) Suppose $\tau(t) \geq t \geq \rho(t)$ for all t.

(a) Suppose $L > 0$. Then there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $p(t)y^{\nabla}(t) \leq L + 1$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. The unboundedness of τ yields the existence of $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $\tau(t) \geq t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. Consequently, for all $t \in [t_3, \infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} y(\tau(t)) - y^{\rho}(t) &\leq (L+1)\int_{\rho(t)}^{\tau(t)} \frac{\nabla s}{p(s)} \\ &= (L+1)[P(\tau(t)) - P(\rho(t))] \\ &= (L+1)\left[\int_{\rho(t)}^{t} \frac{\nabla s}{p(s)} + \int_{t}^{\tau(t)} \frac{\nabla s}{p(s)}\right] \\ &\leq (L+1)[N+M], \end{aligned}$$

which leads to

$$y^{\rho}(t) \geq y(\tau(t)) - (L+1)K$$

for all $t \in [t_3, \infty)_{\mathbb{T}}$. As we have done previously, let $z(t) = y(t) - (L+1)K$. Then for sufficiently large t, we have

$$z(t) > 0, \ z(\tau(t)) \leq y^{\rho}(t), \ \text{and} \ (p(t)z^{\rho}(t))^{\rho} + q(t)f(z(\tau(t))) \leq 0.$$

This leads to a contradiction as in part (I) above.

(b) Assume $L = 0$. Since $y^{\nabla}(t)$ and $y(t)$ are both positive, there is an $\epsilon_0 > 0$ and a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $y(t) > M\epsilon_0$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. Corresponding to this ϵ_0, there exists $t_3 \in [t_1, \infty)_{\mathbb{T}}$ such that $p(t)y^{\nabla}(t) \leq \epsilon_0$ for all $t \in$

$[t_3,\infty)_{\mathbb{T}}$. In the same manner as above, we set $z(t) = y(t) - \epsilon_0 K$ and obtain for sufficiently large t,

$$z(t) > 0,\ z(\tau(t)) \le y^{\rho}(t),\ \text{and } (p(t)z^{\nabla}(t))^{\nabla} + q(t)f(z^{\rho}(t)) \le 0,$$

which again leads to a contradiction.

This completes the proof. □

Remark 2.6. Under the assumptions Theorem 2.5, we see that oscillatory behavior of the more difficult functional equation can be established by considering the nabla dynamic equation that only involves the backward jump operator ρ.

We now turn our attention to an example of Theorem 2.5. Recall

Theorem 2.7 ([3, Theorem 8.48 (ii)]). *If $\mathbb{T}$ consists of only isolated points and $a < b$, then*

$$\int_a^b f(t)\nabla t = \sum_{t\in(a,b]} f(t)\nu(t).$$

We now we have an example.

Example 2.8. Let $\mathbb{T} = 3^{\mathbb{N}}$ and set

$$p(t) = t, \quad q(t) = 3 + (-1)^{\frac{\ln t}{\ln 3}}, \quad \tau(t) = 3t, \quad f(u) = \sqrt[3]{u}.$$

We claim that $p, q, f,$ and τ satisfy the appropriate conditions of (H). First we show that (i) holds. If $b = 3^m$ and $t_0 = 3^n$, $m, n \in \mathbb{N}$, we have

$$\begin{aligned} \int_{t_0}^{\infty} \frac{1}{p(t)}\nabla t &= \lim_{b\to\infty}\int_{t_0}^{b} \frac{1}{p(t)}\nabla t \\ &= \lim_{m\to\infty}\sum_{t=\sigma(3^n)}^{3^m} \frac{\nu(t)}{p(t)} \\ &= \frac{2}{3}\lim_{m\to\infty}\sum_{t=3^{n+1}}^{3^m} 1 \\ &= \infty \end{aligned}$$

because $\nu(t) = \dfrac{2}{3}t$. Note that for any $t \in \mathbb{T}$,

$$q(t) = 3 + (-1)^n = \begin{cases} 2, & n \text{ odd} \\ 4, & n \text{ even}, \end{cases}$$

and so (ii) holds. It is clear that τ belongs to $C_{ld}(\mathbb{T},\mathbb{T})$ and diverges. Also, since,

$$\begin{aligned}|P(t)-P(\tau(t))| &= \left|\int_{t_0}^{t}\frac{1}{p(s)}\nabla s-\int_{t_0}^{3t}\frac{1}{p(s)}\nabla s\right| \\ &= \left|-\int_{t}^{3t}\frac{1}{s}\nabla s\right| \\ &= \frac{2}{3}\sum_{s=3t}^{3t}1 \\ &= \frac{2}{3},\end{aligned}$$

(iii) holds. Finally, $f(u)=\sqrt[3]{u}$ satisfies (iv). Therefore, the oscillation of

$$\left(ty^{\nabla}(t)\right)^{\nabla}+\left[3+(-1)^{\frac{\ln t}{\ln 3}}\right]y^{\frac{1}{3}}\left(\frac{t}{3}\right)=0$$

and

$$\left(ty^{\nabla}(t)\right)^{\nabla}+\left[3+(-1)^{\frac{\ln t}{\ln 3}}\right]y^{\frac{1}{3}}\left(3t\right)=0$$

is equivalent since $\nu(t)/p(t)=2/3$.

We conclude this section with comparing

$$(p(t)y^{\nabla})^{\nabla}+q(t)f(y(\tau(t))) \tag{1.1}$$

to

$$(p(t)y^{\nabla}(t))^{\nabla}+Q(t)g(y(\eta(t)))=0, \tag{2.4}$$

on an isolated time scale $\mathbb{T}$ where $Q, g,$ and η satisfy the appropriate conditions (H) and ν/p is bounded. From Theorem 2.5 we see that the oscillation of (2.4) is equivalent to that of

$$(p(t)y^{\nabla}(t))^{\nabla}+Q(t)g(y^{\rho}(t))=0. \tag{2.5}$$

We have the following result.

Theorem 2.9. *Assume ν/p is bounded on an isolated time scale $\mathbb{T}$. Further assume that $Q(t)\le q(t)$ for all large t and $|g(u)|\le|f(u)|$ for $|u|>0$. Then, the oscillation of* (2.4) *implies that of* (1.1).

Proof. Without loss of generality, suppose to the contrary that (1.1) has an eventually positive solution. From Theorem 2.5, (1.2) also has an eventually positive solution – call it $y(t)$. Then

$$(p(t)y^{\nabla}(t))^{\nabla}+Q(t)g(y^{\rho}(t)))\le(p(t)y^{\nabla}(t))^{\nabla}+q(t)f(y^{\rho}(t)))=0,$$

which implies (2.5) has an eventually positive solution. Therefore, (2.4) also has an eventually positive solution, which is a contradiction to that fact that this equation is oscillatory. $\square$

3. OSCILLATION OF (1.3)

In this section we give two theorems about the oscillatory behavior of the second-order self-adjoint dynamic equation

$$\left(p(t)y^{\Delta}(t)\right)^{\nabla} + q(t)y(t) = 0 \tag{1.3}$$

on a time scale $[t_0, \infty)_{\mathbb{T}}$ where positive $t_0 \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, $p \in C(\mathbb{T}, (0, \infty))$ and $q \in C_{ld}(\mathbb{T}, \mathbb{R})$. These are Theorems 3.4 and 3.6. To obtain these results, we need the following lemmas:

Lemma 3.1 ([2, Theorem 4.59]). *The self-adjoint equation* (1.3) *has a positive solution on* $\mathbb{T}$ *if and only if the Riccati equation*

$$z^{\nabla}(t) + q(t) + \frac{\left(z^{\rho}(t)\right)^2}{p^{\rho}(t) + \nu(t)z^{\rho}(t)} = 0 \tag{3.1}$$

has a solution z *on* $\mathbb{T}^{\kappa}$.

Lemma 3.2 ([2, Theorem 4.55]). *Let* $a \in \mathbb{T}$ *and let* $\omega := \sup \mathbb{T}$. *If* $\omega < \infty$, *then we assume* $\rho(\omega) = \omega$. *If* (1.3) *is nonoscillatory on* $[a, \omega)$, *then there is a solution* u, *called a recessive solution at* ω, *such that* u *is positive on* $[t_0, \omega)$ *for some* $t_0 \in \mathbb{T}$, *and if* v *is any second linearly independent solution, called a dominant solution at* ω, *the following hold:*

(i) $\lim_{t\to\omega^-} \frac{u(t)}{v(t)} = 0$,
(ii) $\int_{t_0}^{\omega} \frac{\Delta t}{p(t)u(t)u^{\sigma}(t)} = \infty$,
(iii) $\int_{b}^{\omega} \frac{\Delta t}{p(t)v(t)v^{\sigma}(t)} < \infty$ $b < \omega$ *is sufficiently close, and*
(iv) $\frac{p(t)v^{\Delta}(t)}{v(t)} > \frac{p(t)u^{\Delta}(t)}{u(t)}$ *for* $t < \omega$ *sufficiently close.*

Lemma 3.3 ([2, Theorem 4.63]). *Assume* z *is a solution of* (3.1) *on* $[\rho(a), b]_{\mathbb{T}}$. *Let* u *be a continuous function on* $[\rho(a), \sigma(b)]_{\mathbb{T}}$ *whose nabla-derivative is piecewise left-dense continuous with* $u^{\rho}(a) = u^{\sigma}(b) = 0$. *Then we have for all* $t \in [a, b]_{\mathbb{T}}$,

$$\begin{aligned}(zu^2)^{\nabla}(t) &= p^{\rho}(t)[u^{\nabla}(t)]^2 - q(t)u^2(t) \\ &\quad - \left\{ \frac{z^{\rho}(t)u(t)}{\sqrt{p^{\rho}(t) + \nu(t)z^{\rho}(t)}} - \sqrt{p^{\rho}(t) + \nu(t)z^{\rho}(t)}\, u^{\nabla}(t) \right\}^2.\end{aligned}$$

By a solution of (1.3), we mean a nontrivial real-valued function y on $[t_0, \infty)_{\mathbb{T}}$ which satisfies (1.3) and the properties $y^{\Delta} \in C([t_0, \infty)_{\mathbb{T}}^{\kappa}, \mathbb{R})$ and $\left(py^{\Delta}\right)^{\nabla} \in C_{ld}([t_0, \infty)_{\mathbb{T}}^{\kappa} \cap ([t_0, \infty)_{\mathbb{T}})_{\kappa}, \mathbb{R})$. Throughout, we impose the following condition: For some $a \in [t_0, \infty)_{\mathbb{T}}$,

$$\int_a^{\infty} \frac{1}{p(s)}\, \Delta s \quad \text{and} \quad \int_a^{\infty} q(s)\, \nabla s < \infty. \tag{3.2}$$

Before we begin, note that if y is an eventually positive solution of (1.3), then y^Δ is eventually positive as well. Next we give the following definitions.

$$\begin{aligned} A_0(t) &= \int_t^\infty q(s)\,\nabla s, \\ A_1(t) &= A_0(t) + \int_t^\infty \frac{\left(A_0^\rho(s)\right)^2}{p^\rho(s)+\nu(s)A_0^\rho(s)},\nabla s, \\ &\vdots \\ A_n(t) &= A_0(t) + \int_t^\infty \frac{\left(A_{n-1}^\rho(s)\right)^2}{p^\rho(s)+\nu(s)A_{n-1}^\rho(s)},\nabla s, \end{aligned}$$

if the integrals on the right-hand side exist. Now we present our first result of this section which is an extension of Theorem 3.1 of [10].

Theorem 3.4. *Assume one of the following two conditions holds:*

(i) there exists some positive integer m such that A_n is well defined for $n = 0, 1, 2, \ldots,\ m-1$, and

$$\lim_{t\to\infty} \int_a^t \frac{\left(A_{m-1}^\rho(s)\right)^2}{p^\rho(s)+\nu(s)A_{m-1}^\rho(s)},\nabla s = \infty.$$

(ii) A_n is well defined for $n = 0, 1, 2, \ldots,$ and there exists $t^* \in [t_0,\infty)_{\mathbb{T}}$ such that

$$\lim_{n\to\infty} A_n(t^*) = \infty.$$

Then (1.3) *is oscillatory.*

Proof. If not, without loss of generality, we assume there exists (1.3) has an eventually positive solution y, specifically, there exists $T \in [t_0,\infty)_{\mathbb{T}}$ such that

$$y(t) > 0 \quad \text{and} \quad y^\Delta(t) > 0$$

on $[T,\infty)_{\mathbb{T}}$. Define the function z on $[T,\infty)_{\mathbb{T}}$ by

$$z(t) = \frac{p(t)y^\Delta(t)}{y(t)}. \tag{3.3}$$

Then $z(t) > 0$, and since $y^\Delta(\rho(t)) = y^\nabla(t)$,

$$p^\rho(t) + \nu(t)z^\rho(t) = \frac{p^\rho(t)\left(y^\rho(t)+\nu(t)y^\nabla(t)\right)}{y^\rho(t)} > 0,$$

for all $t \in [T,\infty)_{\mathbb{T}}$. From (3.3), we obtain z is a solution (3.1) on $[T,\infty)_{\mathbb{T}}$. Since $z(t) > 0$, integration of (3.1) from T to t yields

$$\int_T^t \frac{\left(z^\rho(s)\right)^2}{p^\rho(s)+\nu(s)z^\rho(s)}\nabla s < z(T) - \int_T^t q(s)\,\nabla s < z(T)$$

on $[T,\infty)_{\mathbb{T}}$. Consequently,

$$\lim_{t\to\infty} \int_T^t \frac{\left(z^\rho(s)\right)^2}{p^\rho(s)+\nu(s)z^\rho(s)}\nabla s < \infty.$$

Now integrating (3.1) from t to s we obtain

$$\begin{aligned} z(t) &= z(s) + \int_t^s q(u)\,\nabla u + \int_t^s \frac{(z^\rho(u))^2}{p^\rho(u)+\nu(s)z^\rho(u)}\nabla u \\ &> \int_t^s q(u)\,\nabla u + \int_t^s \frac{(z^\rho(u))^2}{p^\rho(u)+\nu(s)z^\rho(u)}\nabla u \end{aligned}$$

for $(t,s) \in [T,\infty)_{\mathbb{T}} \times [t,\infty)_{\mathbb{T}}$. Therefore,

$$z(t) \geq \int_t^\infty q(u)\,\nabla u + \int_t^\infty \frac{(z^\rho(u))^2}{p^\rho(u)+\nu(s)z^\rho(u)}\nabla u. \tag{3.4}$$

Now assume Condition (i) holds. We first assume that $m = 1$. From (3.4) we obtain that $z(t) \geq A_0(t)$ for all $t \in [T,\infty)_{\mathbb{T}}$. Since $F(u) = \dfrac{u^2}{c_1 + c_2 u}$ is increasing for $u > 0$ and nonnegative constants c_1, c_2, it follows that

$$\int_t^\infty \frac{(A_0^\rho(u))^2}{p^\rho(u)+\nu(u)A_0^\rho(u)}\,\nabla u \leq \frac{(z^\rho(u))^2}{p^\rho(u)+\nu(u)A_0^\rho(u)}\,\nabla u < \infty.$$

This contradicts (i). If $m > 1$, (3.4) gives $z(t) \geq A_1(t)$ for all $t \in [T,\infty)_{\mathbb{T}}$. Repeating this procedure, we obtain $z(t) \geq A_{m-1}(t)$ for all $t \in [T,\infty)_{\mathbb{T}}$ and

$$\int_t^\infty \frac{\left(A_{m-1}^\rho(u)\right)^2}{p^\rho(u)+\nu(u)A_{m-1}^\rho(u)}\,\nabla u \leq \frac{(z^\rho(u))^2}{p^\rho(u)+\nu(u)A_{m-1}^\rho(u)}\,\nabla u < \infty,$$

which contradicts Condition (i).

Assume that Condition (ii) holds. Similar to the above proof, we obtain $A_n(t) \leq z(t)$ for $n \in \mathbb{N}_0$. Then, as $y(t) > 0$,

$$\lim_{n\to\infty} A_n(t^*) \leq z(t^*) < \infty,$$

which gives a contradiction to Condition (ii). The proof is complete. □

Remark 3.5. If $\mathbb{T} = \mathbb{R}$ and $p(t) = 1$ for all t, then Theorem 3.4 is the same as Yan's result for second-order linear differential equations [9].

Our next result is

Theorem 3.6. *Assume $I = [a,\infty)_{\mathbb{T}}$. If $\displaystyle\int_a^\infty \frac{\nabla t}{p(t)} = \infty$ and there is a $t_0 \in I$ and a $u \in C^1_{ld}[t_0,\infty)_{\mathbb{T}}$ such that $u(t) > 0$ on $[t_0,\infty)_{\mathbb{T}}$ and*

$$\int_{t_0}^\infty \{q(t)u^2(t) - p^\rho(t)[u^\nabla(t)]^2\}\nabla t = \infty,$$

then (1.3) *is oscillatory on I.*

Proof. By way of contradiction, assume (1.3) is nonoscillatory on I. By Lemma 3.2, there is a dominant solution y of (1.3) at ∞ such that

$$\int_{t_1}^\infty \frac{\Delta t}{p(t)y(t)y^\sigma(t)} < \infty$$

for sufficiently large $t_1 \in I$. Here we may assume $y(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. With $T :=\max\{t_0, t_1\}$, by Lemma 3.1, (3.1) has a solution z on $[T, \infty)_{\mathbb{T}}$. Now Lemma 3.3 yields

$$(zu^2)^{\nabla}(t) \leq p^{\rho}(t)[u^{\nabla}(t)]^2 - q(t)u^2(t)$$

for $t \in [T, \infty)_{\mathbb{T}}$. Integrating from T to t, we obtain

$$z(t)u^2(t) \leq z(T)u^2(T) - \int_T^t \left\{q(s)u^2(s) - p^{\rho}(s)[u^{\nabla}(s)]^2\right\} \nabla s$$

which implies

$$\lim_{t\to\infty} z(t)u^2(t) = -\infty.$$

Consequently, there is a $S \in [T, \infty)_{\mathbb{T}}$ such that for $t \in [S, \infty)_{\mathbb{T}}$

$$z(t) = \frac{p(t)y^{\Delta}(t)}{y(t)} < 0.$$

This implies y is decreasing on $[S, \infty)_{\mathbb{T}}$, and so

$$\begin{aligned} \int_S^\infty \frac{1}{p(u)}\,\Delta u &= y(S)y^{\sigma}(S)\int_S^\infty \frac{1}{p(u)y(S)y^{\sigma}(S)}\,\Delta u \\ &\leq y(S)y^{\sigma}(S)\int_S^\infty \frac{1}{p(u)y(u)y^{\sigma}(u)}\,\Delta u \\ &< \infty, \end{aligned}$$

which is a contradiction to divergent nature of $\int_S^\infty \frac{1}{p(t)}\,\Delta t$. □

We conclude with an example of Theorem 3.6.

Example 3.7. Let $\mathbb{T} = q^{\mathbb{N}}$, $q > 1$ and set

$$p(t) = \frac{1}{t^2}, \quad Q(t) = \frac{\ln t}{t^2}, \quad u(t) = t.$$

Here we let $a = t_0 = 1$. Immediately we see that u belongs to $C^1_{ld}([1, \infty)_{\mathbb{T}}, (0, \infty))$. First we show that $\int_1^\infty Q(s)\nabla s < \infty$. To do this, we will use part (vi) of Theorem 8.47 in [3]. In that result, we let

$$f(t) = \ln t \quad \text{and} \quad g(t) = -\frac{1}{qt}.$$

Then

$$f^{\nabla}(t) = \left\{\int_0^1 \frac{1}{\frac{t}{q} + h\frac{t}{q}(q-1)\cdot 1}\,dh\right\}\cdot 1 \leq \int_0^1 \frac{q}{t}\,dh = \frac{q}{t}$$

by Theorem 4.8 of [2] and the Pötzsche Chain Rule [3, Theorem 1.90], and

$$g^{\nabla}(t) = -\frac{1}{q}\left(\frac{-1}{t\frac{t}{q}}\right) = \frac{1}{t^2}.$$

Consequently,

$$\begin{aligned}\int_1^\infty Q(t)\nabla t &= \lim_{b\to\infty}\int_1^b \ln t\left(-\frac{1}{qt}\right)^\nabla \nabla t\\ &= \lim_{b\to\infty}\left[-\frac{\ln(b)}{qb}+\frac{\ln(1)}{q}-\int_1^b(\ln t)^\nabla\left(-\frac{1}{q\frac{t}{q}}\right)\nabla t\right]\\ &= \lim_{b\to\infty}\left[\int_1^b\frac{(\ln t)^\nabla}{t}\nabla t\right]\\ &\le \lim_{b\to\infty}\left[\int_1^b\frac{q}{t^2}\nabla t\right]\\ &= q\lim_{n\to\infty}\sum_{t=q}^{q^n}\frac{1}{t^2}\frac{(q-1)t}{q}\\ &= (q-1)\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{q}\right)^k\\ &= 1<\infty.\end{aligned}$$

Next

$$\begin{aligned}\int_1^\infty\frac{1}{p(t)}\nabla t &= \lim_{b\to\infty}\int_1^b t^2\,\nabla t\\ &= \lim_{m\to\infty}\sum_{t=\sigma(1)}^{q^m}\frac{t(q-1)}{q}t^2\\ &= \frac{q-1}{q}\lim_{m\to\infty}\sum_{k=1}^{m}\left(q^3\right)^k\\ &= \infty.\end{aligned}$$

Finally

$$\begin{aligned}\int_1^\infty\left[q(t)u^2(t)-p^\rho(t)[u^\nabla(t)]^2\right]\nabla t &= \int_1^\infty\left(\ln t-\frac{q^2}{t^2}\right)\nabla t\\ &= \frac{q-1}{q}\lim_{n\to\infty}\sum_{k=1}^{n}\left[q^k\left(k\ln q-q^2\left(\frac{1}{q^2}\right)^k\right)\right]\\ &= \infty\end{aligned}$$

because

$$\lim_{n\to\infty}\sum_{k=1}^{n}q^kq^2\left(\frac{1}{q^2}\right)^k=q^2\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{q}\right)^k<\infty.$$

4. CONCLUSION

In this article, we studied the oscillatory behavior of the functional nabla dynamic equation

$$\left(p(t)y^\nabla(t)\right)^\nabla+q(t)f(y(\tau(t)))=0$$

on an isolated time scale $\mathbb{T}$. We showed this equation's oscillatory behavior is equivalent to that of

$$\left(p(t)y^{\nabla}(t)\right)^{\nabla} + q(t)f(y^{\rho}(t)) = 0.$$

This equivalence was obtained by establishing a relationship between the oscillatory solutions of the functional nabla dynamic equation and $\left(p(t)y^{\nabla}(t)\right)^{\nabla} + q(t)f(y(\tau(t))) \leq 0$ as well as one between the oscillatory solutions of the nabla dynamic equation and its corresponding inequality. On any time scale $\mathbb{T}$ we considered the second-order self-adjoint equation with mixed derivatives

$$\left(p(t)y^{\Delta}(t)\right)^{\nabla} + q(t)y(t) = 0$$

and established two sufficient conditions for the oscillation using the Riccati transformation technique.

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